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The $ER(2)$-cohomology of $\prod^n \mathbb{CP}^\infty$ and $BU(n)$ ☆

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ABSTRACT

We continue the development of the computability of the second real Johnson-Wilson theory. As $ER(2)$ is not complex orientable, this gives some difficulty even with basic spaces. In this paper we compute the second real Johnson-Wilson theory for products of infinite complex projective spaces and for the classifying spaces for the unitary groups.

1. Introduction

The $p = 2$ Johnson-Wilson theory, [4, Remark 5.13], $E(n)$, has coefficients

$$E(n)^* \cong \mathbb{Z}_{(2)}[v_1, v_2, \ldots, v_n^{\pm 1}]$$

with the degree of v_k equal to $-2(2^k - 1)$. There is a $\mathbb{Z}/(2)$ action on $E(n)$ coming from complex conjugation. The real Johnson-Wilson theory, $ER(n)$, is the homotopy fixed points of $E(n)$. This was initially studied by Hu and Kriz in [2]. Since then the theories have been studied intensively and applied to the problem of non-immersions of real projective space ([1,5–14]).

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The first theory, $ER(1)$, is just $KO_{(2)}$, and it was a decades long process of computing the details of the KO-(co)homology of $\mathbb{CP}^\infty$, finally ending in [16]. The second theory, $ER(2)$, is, by [3], closely related to $TMF_0(3)$ (the same after a suitable completion). The second author computed all $ER(n)^*(\mathbb{CP}^\infty)$ in complete detail, [14]. This is already much more than has been done with $TMF_0(3)$.

The fibre of the restriction, $ER(n) \longrightarrow E(n)$ is $\Sigma^{2(2^n-1)^2-1}ER(n)$ from [8]. This gives a Bockstein spectral sequence from $E(n)^*(X)$ to $ER(n)^*(X)$. In this paper we are concerned with $ER(2)$, so we have the map $x : \Sigma^{17}ER(2) \to ER(2)$. This map has $2x = 0 = x^7$. The resulting Bockstein spectral sequence just measures x^i-torsion. We use the untruncated version, see Remark 2.4. That just means that d_1 detects all of the x^1-torsion generators and E_2 is what is left after you throw them all away. In our cases, we only have d_1, d_3, and d_7, so $E_2 = E_3$. When we compute d_3, it gives us all the x^3-torsion, but then we throw it all away to get our $E_4 = E_5 = E_6 = E_7$. Our d_7 gives the x^7-torsion and leaves us with $E_8 = 0$.

Our goal here is to give a computation of this Bockstein spectral sequence for $X = \prod^n \mathbb{CP}^\infty$ and $BU(n)$, computing $ER(2)^*(-)$ from $E(2)^*(-)$. The computation is accomplished by going through an auxiliary spectral sequence to compute d_1. Once that is done, d_3 and d_7 follow.

Our actual computations are carried out with $\wedge^n\mathbb{CP}^\infty$ and $MU(n)$ because the product and $BU(n)$ can be recovered from the stable splittings, (e.g. $BU(n) = MU(n) \vee BU(n-1)$, [15]).

There is a special element, $\hat{v}_2 \in ER(2)^{48}$ that maps to $v_2^{-8} \in E(2)^{48}$. It is the periodicity element for $ER(2)$ and it makes our bookkeeping easier if we do away with it once and for all now by setting $\hat{v}_2 = 1$, and, in $E(2)^*$, the corresponding $v_2^{-8} = 1$. This makes our theories graded over $\mathbb{Z}/(48)$.

There are also elements $\hat{v}_1 \in ER(2)^{16}$ that maps to $v_1v_2^{-3} \in E(2)^{16}$ and $w \in ER(2)^{-8}$ mapping to $\hat{v}_1v_2^4 = v_1v_2 \in E(2)^{-8}$.

The theory $E(2)^*(-)$ is a complex orientable theory so $E(2)^*(\mathbb{CP}^\infty) = E(2)^*[[u]]$ where u is of degree 2. The only adjustment needed here is to define $\hat{u} = uv_2^3$, of degree -16. We write $E(2)^*(\mathbb{CP}^\infty) = E(2)^*[[\hat{u}]]$. Since v_2 is a unit, this is not a problem.

We also need the complex conjugate of $\hat{u}$, $c(\hat{u})$. There is a class, $\hat{p} \in ER(2)^{-32}(\mathbb{CP}^\infty)$, that maps to $\hat{u}\,c(\hat{u}) \in E(2)^{-32}(\mathbb{CP}^\infty)$, see [14]. This is a modified first Pontryagin class.

We can generalize this to $BU(n)$. Because $E(2)$ is a complex oriented theory, we have

$$E(2)^*(BU(n)) \cong E(2)^*[[c_1, \dots, c_n]].$$

Again, we need to modify the generalized Conner-Floyd Chern classes to $\hat{c}_k = v_2^{3k}c_k$, putting them in degree $-16k$.

We also have modified Pontryagin classes

$$\hat{P}_k \in ER(2)^{-32k}(BU(n)) \longrightarrow \sum_{\substack{i+j=2k \\ 0\le i,j\le n}} \hat{c}_i\, c(\hat{c}_j) \in E(2)^{-32k}(BU(n))$$

These elements are special to us because much of our answer is described in terms of them and they are familiar elements. In addition, they are necessary for us because their images are permanent cycles, making it possible to compute our d_3 and d_7 in our Bockstein spectral sequences. There are alternative elements that would work for our proofs just as well. There is a norm that creates an element that maps to $\hat{c}_k\, c(\hat{c}_k)$ in $E(2)^{-32k}(BU(n))$, so this image element is also a permanent cycle that would allow us to finish our proofs for d_3 and d_7. Both of these elements work for our proofs because their representation in our spectral sequence is the same. We do like the more traditional nature of the Pontryagin classes though. See Section 11, Definition 11.1 for the details.

We have "hatted" various otherwise familiar elements. See Remark 2.2 for some historical background.

Although we compute all of $ER(2)^*(-)$ for $\wedge^n\mathbb{CP}^\infty$ and $MU(n)$, the x^1-torsion generators are quite messy and have been left out of the introduction.

We let $\hat{u}_i$ be our $\hat{u}$ associated with the i-th term in the smash product of the $\mathbb{CP}^\infty$. Similarly, with $\hat{p}_i$. The clean results we can state nicely are presented in the next theorems. Keep in mind that because we use an auxiliary spectral sequence to compute d_1, our results are stated in terms of associated graded versions of E_i.

Theorem 1.1. *The associated graded versions of E_i for the Bockstein spectral sequence going from $E(2)^*(\wedge^n\mathbb{CP}^\infty)$ to $ER(2)^*(\wedge^n\mathbb{CP}^\infty)$ are as follows:* $E_1 =$

$$\begin{aligned} E(2)^*(\wedge^n\mathbb{CP}^\infty) &\cong E(2)^*[[\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n]]\{\hat{u}_1\hat{u}_2\cdots\hat{u}_n\} \\ &= \mathbb{Z}_{(2)}[\hat{v}_1][[\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n]]\{v_2^{0-7}\hat{u}_1\hat{u}_2\cdots\hat{u}_n\} \end{aligned}$$

$E_2 = E_3 =$

$$\mathbb{Z}/(2)[\hat{p}_n]\{v_2^{0,2,4,6}\hat{p}_1\hat{p}_2\dots\hat{p}_n\}$$

The x^3-torsion generators are represented by

$$\mathbb{Z}/(2)[\hat{p}_n]\{v_2^{0,4}\hat{p}_1\hat{p}_2\dots\hat{p}_n^2\}$$

$E_4 = E_5 = E_6 = E_7 =$

$$\mathbb{Z}/(2)\{v_2^{0,4}\hat{p}_1\hat{p}_2\dots\hat{p}_n\}$$

The x^7-torsion generator is represented by

$$\mathbb{Z}/(2)\{\hat{p}_1\hat{p}_2\dots\hat{p}_n\}$$

Theorem 1.2. *The associated graded versions of E_i for the Bockstein spectral sequence going from $E(2)^*(MU(2n))$ to $ER(2)^*(MU(2n))$ are as follows:* $E_1 =$

$$\begin{aligned} E(2)^*(MU(2n)) &\cong E(2)^*[[\hat{c}_1, \hat{c}_2, \dots, \hat{c}_{2n}]]\{\hat{c}_{2n}\} \\ &= \mathbb{Z}_{(2)}[\hat{v}_1][[\hat{c}_1, \hat{c}_2, \dots, \hat{c}_{2n}]]\{v_2^{0-7}\hat{c}_{2n}\} \end{aligned}$$

$E_2 = E_3 =$

$$\mathbb{Z}/(2)[\hat{v}_1][[\hat{P}_2, \hat{P}_4, \dots, \hat{P}_{2n}]]\{v_2^{0,2,4,6}\hat{P}_{2n}\}$$

The x^3-torsion generators are represented by

$$\begin{aligned} &\mathbb{Z}/(2)[\hat{v}_1][[\hat{P}_2, \hat{P}_4, \dots, \hat{P}_{2n}]]\{\hat{v}_1 v_2^{0,4}\hat{P}_{2n}\} = \\ &\mathbb{Z}/(2)[\hat{v}_1][[\hat{P}_2, \hat{P}_4, \dots, \hat{P}_{2n}]]\{\hat{v}_1\hat{P}_{2n}, w\hat{P}_{2n}\} \end{aligned}$$

$E_4 = E_5 = E_6 = E_7 =$

$$\mathbb{Z}/(2)[\hat{P}_2, \hat{P}_4, \dots, \hat{P}_{2n}]\{v_2^{0,4}\hat{P}_{2n}\}$$

The x^7-torsion generators are represented by

$$\mathbb{Z}/(2)[\hat{P}_2, \hat{P}_4, \dots, \hat{P}_{2n}]\{\hat{P}_{2n}\}$$

Theorem 1.3. *The associated graded versions of E_i for the Bockstein spectral sequence going from $E(2)^*(MU(2n+1))$ to $ER(2)^*(MU(2n+1))$ are as follows:* $E_1 =$

$$E(2)^*(MU(2n+1)) \cong E(2)^*[[\hat{c}_1, \hat{c}_2, \dots, \hat{c}_{2n+1}]]\{\hat{c}_{2n+1}\}$$
$$= \mathbb{Z}_{(2)}[\hat{v}_1][[\hat{c}_1, \hat{c}_2, \dots, \hat{c}_{2n+1}]]\{v_2^{0-7}\hat{c}_{2n+1}\}$$

$E_2 = E_3$, *for* $0 \leq b < n$

$$\mathbb{Z}/(2)[\hat{P}_2, \hat{P}_4, \dots, \hat{P}_{2b}, \hat{P}_{2b+1}, \hat{P}_{2b+3}, \dots, \hat{P}_{2n+1}]\{v_2^{0,2,4,6}\hat{P}_{2b+1}\hat{P}_{2n+1}\}$$

and

$$\mathbb{Z}/(2)[\hat{P}_2, \hat{P}_4, \dots, \hat{P}_{2n}, \hat{P}_{2n+1}]\{v_2^{0,2,4,6}\hat{P}_{2n+1}\}$$

The x^3-torsion generators are represented by

$$\mathbb{Z}/(2)[\hat{P}_2, \hat{P}_4, \dots, \hat{P}_{2b}, \hat{P}_{2b+1}, \hat{P}_{2b+3}, \dots, \hat{P}_{2n+1}]\{v_2^{0,4}\hat{P}_{2b+1}\hat{P}_{2n+1}\} \quad 0 \leq b \leq n$$

$E_4 = E_5 = E_6 = E_7 =$

$$\mathbb{Z}/(2)[\hat{P}_2, \hat{P}_4, \dots, \hat{P}_{2n}]\{v_2^{0,4}\hat{P}_{2n+1}\}$$

The x^7-torsion generators are represented by

$$\mathbb{Z}/(2)[\hat{P}_2, \hat{P}_4, \dots, \hat{P}_{2n}]\{\hat{P}_{2n+1}\}$$

The elements $\hat{v}_1$, w, $\hat{p}_i$, and $\hat{P}_i$ all exist in the appropriate $ER(2)^*(X)$. It is worth noting that all of the x^3-torsion generators are well-defined in $ER(2)^*(MU(2n))$ (likewise with the x^7-torsion generators in all three cases). Consequently, new elements don't have to be created and named. We often deal only with elements in degrees $16*$. To see these, just modify the statements in the theorems to eliminate the $v_2^{2,4,6}$. In fact, we can handle elements in degrees $8*$ quite easily. In the case of the above theorems, just keep the $v_2^{0,4}$ and eliminate the $v_2^{2,6}$. By definition, the x^i-torsion generators inject to $E(2)^*(X)$.

The following is useful for computations and relations.

Theorem 1.4. *For $X = \wedge^n \mathbb{CP}^\infty$ and $MU(n)$, $ER(2)^{8*}(X) \to E(2)^{8*}(X)$ injects.*

Remark 1.5. In the kernel of $ER(2)^{4*}(\wedge^n \mathbb{CP}^\infty) \to E(2)^{4*}(\wedge^n \mathbb{CP}^\infty)$, there is only one element, namely, $x^4\hat{p}_1\hat{p}_2 \dots \hat{p}_n$. Similarly, in degrees $(8*-6)$ we have only $x^6\hat{p}_1\hat{p}_2 \dots \hat{p}_n$.

We do our general preliminaries in Section 2. In Section 3 we sketch out our approach in both cases in rather general terms to give some idea of how we go about our computations. We define a crucial filtration in Section 4. Then we spend a few sections doing the computation for $\wedge^n \mathbb{CP}^\infty$. When that is done, we begin preliminaries for $BU(n)$ in Section 10. We do the main calculation for $MU(n)$ starting in Section 14 going to the end of the paper.

2. Preliminaries

There are many ways to describe $ER(2)^*$, but we will stick mainly with the description given in [13, Remark 3.4].

We have traditionally given the name α to the element $\hat{v}_1$, but this is gradually being phased out. We also have elements α_i, $0 < i < 4$, with degree $-12i$. We often extend this notation to $\alpha_0 = 2$. These elements map to $2v_2^{2i} \in E(2)^*$. For the last non-torsion algebra generator, we have w of degree -8, which maps to $\hat{v}_1 v_2^4 = v_1 v_2 \in E(2)^*$.

Torsion is generated by the element $x \in ER(2)^{-17}$. It has $2x = 0$ and $x^7 = 0$. Keep in mind that $ER(2)^*$ is 48 periodic. We use, for efficient notation, $x^{3-6} = \{x^3, x^4, x^5, x^6\}$ and $R\{a, b\}$ for the free R-module on a and b. Similarly, we use notation like $v_2^{0,2,4,6} = \{1, v_2^2, v_2^4, v_2^6\}$ and $v_2^{2,4} = \{v_2^2, v_2^4\}$.

Fact 2.1. [9, Proposition 2.1] $ER(2)^*$ is:

$$\begin{array}{cc} \mathbb{Z}_{(2)}[\hat{v}_1]\{1, w, \alpha_1, \alpha_2, \alpha_3\} \quad \text{with} & 2w = \alpha\alpha_2 = \hat{v}_1\alpha_2 \\ \mathbb{Z}/(2)[\hat{v}_1]\{x^{1-2}, x^{1-2}w\} & \mathbb{Z}/(2)\{x^{3-6}\}. \end{array}$$

Remark 2.2. So far we have defined several "hatted" elements just by multiplying the originals by a unit. If we look more generally and let $\hat{v}_i = v_k v_n^{-(2^n-1)(2^k-1)}$, we still have

$$E(n)^* = \mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{n-1}, v_n^{\pm 1}]$$

The $\hat{v}_k$ all lift to $ER(n)^*$ as in [13, Theorem 3.1], but the original v_k do not. This good fortune carries over to statements like

$$ER(n)^*(RP^\infty) \simeq ER(n)^*[[\hat{u}]]/([2](\hat{u}))$$

of [10, Theorem 1.2] and of [13, Theorem 1.1] computing $ER(n)^*(BO(q))$ in terms of the $\hat{c}_k$. The first Pontryagin class, $\hat{p}$ was studied in [14], and we need to add the general Pontryagin class, $\hat{P}_i$, to our collection of well-behaved elements with respect to the restriction $ER(2) \to E(2)$.

Before we do that, we should recall the mathematics behind our hatted elements. We need this in our construction of the $\hat{P}_i$ in Section 11.

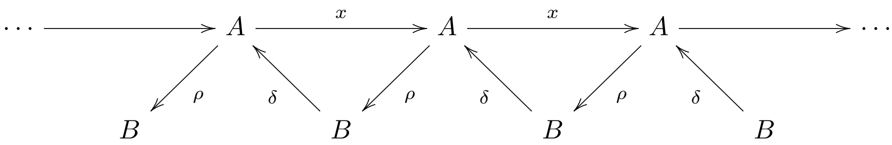
Let $\mathbb{E}(n)$ denote Real Johnson-Wilson, a $\mathbb{Z}/2$-equivariant spectrum, and let $ER(n)$ denote its fixed points. Recall that the $RO(\mathbb{Z}/2)$-graded coefficients of $\mathbb{E}(n)$ contain a class $y(n) \in \pi_{\lambda+\alpha}\mathbb{E}(n)$ (from [8], with $\lambda = 2(2^n-1)^2 - 1$) which is invertible. Its underlying nonequivariant class is $v_n^{2^n-1}$. For any $\mathbb{Z}/2$-space X, we may shift any class in $\mathbb{E}(n)^\star(X)$ into integer degrees by multiplying by the appropriate power of $y(n)$. When we do this to a class in degree a multiple of the regular representation $z \in \mathbb{E}(n)^{k(1+\alpha)}(X)$, we define $\hat{z} := y(n)^k z \in \mathbb{E}(n)^{k(1-\lambda)}(X)$. The image of $\hat{z}$ in $E(n)^*(X)$ is given by $z v_n^{k(2^n-1)}$ and we abuse notation by denoting the image in $E(n)^*(X)$ by the same name. The following diagram commutes:

$$\begin{array}{ccccl} \mathbb{E}(n)^{*(1+\alpha)}(Y) & \xrightarrow{\ \hat{}\ } & \mathbb{E}(n)^{*(1-\lambda)}(Y) & & \text{if } |z| = k(1+\alpha),\ z \mapsto \hat{z} := zy(n)^k \\ \downarrow \rho & & \downarrow \rho & & \\ E(n)^{2*}(Y) & \xrightarrow{\ \hat{}\ } & E(n)^{*(1-\lambda)}(Y) & & \text{if } |z| = 2k,\ z \mapsto \hat{z} := z v_n^{k(2^n-1)} \end{array}$$

Remark 2.3. A major theme in this paper will be to look at elements in degrees $16*$ (and sometimes even $8*$). We have $ER(2)^{16*} = \mathbb{Z}_{(2)}[\hat{v}_1]$. In addition, the x^1-torsion generators in degree $16*$ are given by $\mathbb{Z}_{(2)}[\hat{v}_1]\{2\}$, the x^3-torsion generators, $\mathbb{Z}/(2)[\hat{v}_1]\{\hat{v}_1\}$, and the only x^7-torsion generator is $\mathbb{Z}/(2)$.

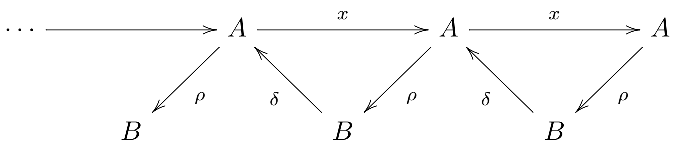
The fibration $\Sigma^{17}ER(2) \longrightarrow ER(2) \longrightarrow E(2)$ gives rise to an exact couple and a convergent Bockstein Spectral Sequence that begins with $E(2)^*(X)$ and where there can only be differentials d_1 through d_7.

Remark 2.4. We have used two versions of this Bockstein spectral sequence in the past so we should explain the difference. We'll do without indices to make the discussion clearer. The untruncated version allows the exact couple to go on forever to the left and right:



If A is all x-torsion (and in our case x^7 kills everything), then this spectral sequence starts with B and converges to zero.

If we truncate this on the right as



the spectral sequence converges to A.

Where the computations can be done globally and give elegant results, the preference is to use the truncated version, such as we have done in [5] and [13]. On the other hand, when computations get grubby, using the untruncated version has the advantage that it gets smaller after each differential, making the bookkeeping much easier to manage. We have used the untruncated version in [10], [11], and [6].

Both spectral sequences contain the same information. We find it to our advantage to use the untruncated version that converges to zero in this paper.

We give a simplified summary of the untruncated Bockstein Spectral Sequence (BSS) we use for computing $ER(2)^*(X)$ from $E(2)^*(X)$.

Theorem 2.5 *([10][Theorem 4.2]).*

(1) The exact couple gives a spectral sequence, E_r, of $ER(2)^*$ modules, starting with

$$E_1 \simeq E(2)^*(X) \quad \textit{and ending with} \quad E_8 = 0.$$

(2) $d_1(y) = v_2^{-3}(1-c)(y)$ where $c(v_i) = -v_i$ and c comes from complex conjugation.
(3) The degree of d_r is $17r + 1$.
(4) The targets of the d_r represent the x^r-torsion generators of $ER(2)^*(X)$.

Definition 2.6. Let K_i be the kernel of x^i on $ER(2)^*(X)$ and let M_i be the image of K_i in $ER(2)^*(X)/(xER(2)^*(X)) \subset E(2)^*(X)$. We call $M_r/M_{r-1} \simeq$ image d_r the x^r**-torsion generators.** We are aware that these are not technically generators, just cosets, but this is what the spectral sequence computes when it is computing the x^r-torsion, so the terminology from this definition simplifies our work. Of course any element not divisible by x can be thought of as an x^i-torsion generator, and all such elements map non-trivially to $E(2)^*(X)$.

Remark 2.7. All of our BSSs in this paper have only even degree elements, so we always have $d_2 = d_4 = d_6 = 0$. In fact, d_5 never shows up here although we have seen it with other even degree spaces, for example, in [6] for $\mathbb{CP}^{8i+1}$ and $\mathbb{CP}^{8i+5}$.

Remark 2.8 *(The BSS on the coefficients.).* For our purposes, it is important to know how this works for the cohomology of a point ([13, Theorem 3.1]). The differential d_1 is on $E(2)^* = \mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 1}]$, which can now be rewritten as $\mathbb{Z}_{(2)}[\hat{v}_1]\{v_2^{0-7}\}$. The differential, d_1, commutes with $\hat{v}_1$ and v_2^2 so all that matters here is $d_1(v_2) = 2v_2^{-2}$.

The E_2 term becomes $\mathbb{Z}/(2)[\hat{v}_1]\{v_2^{0,2,4,6}\}$. We have d_3 commutes with $\hat{v}_1$ and v_2^4, and $d_3(v_2^2) = \hat{v}_1 v_2^{-4}$.

This leaves us with only $\mathbb{Z}/(2)\{v_2^{0,4}\}$. We have d_7 commutes with $v_2^8 = \hat{v}_2^{-1} = 1$ and $d_7(v_2^4) = \hat{v}_2 v_2^{-8} = \hat{v}_2^2 = v_2^{-16} = 1$, so $E_8 = 0$.

Using this approach to $ER(2)^*$ we see that the x^1-torsion is generated by $\mathbb{Z}_{(2)}[\hat{v}_1]\{2v_2^{0,2,4,6}\}$, the x^3-torsion by $\mathbb{Z}/(2)[\hat{v}_1]\{\hat{v}_1 v_2^{0,4}\}$, and the x^7-torsion by $\mathbb{Z}/(2)$. The previous description of $ER(2)^*$ is easy to relate to this now. The x-torsion is given by $\mathbb{Z}_{(2)}[\hat{v}_1]$ on the α_i, $0 \le i < 4$. The x^3-torsion is generated over $\mathbb{Z}/(2)[\hat{v}_1]$ on $\hat{v}_1 = \alpha$ and w. Finally, the x^7-torsion is given by $\mathbb{Z}/(2)$.

The complex conjugate of the BSS comes from $E(2)$, but Lorman shows in [14, Lemma 4.1] that the complex conjugate of $\hat{u} \in E(2)^{-16}(\mathbb{CP}^\infty)$, $c(\hat{u})$, can be calculated using the formal group law for $E(2)$ from $\hat{F}(\hat{u}, c(\hat{u})) = 0$.

Remark 2.9. The standard formal group law for $E(2)$ is $F(x, y)$ with the degrees of x and y equal to two. The element $F(x, y)$ also has degree two. Let $\hat{x} = v_2^3 x$ and $\hat{y} = v_2^3 y$. Replace v_i in F with $\hat{v}_i$. This gives us $\hat{F}(\hat{x}, \hat{y}) = v_2^3 F(x, y)$ of degree -16.

We need some basic easily computed formulas, which we just quote here. We use Araki's generators. These are all modulo $x^i y^j$, $i + j > 4$ or $\hat{u}^5$.

$$\begin{aligned}\hat{F}(\hat{x}, \hat{y}) &= \hat{x} + \hat{y} + \hat{v}_1 \hat{x}\hat{y} + \hat{v}_1^2(\hat{x}^2\hat{y} + \hat{x}\hat{y}^2) \\ &\quad + (\frac{6}{7}\hat{v}_1^3 + \frac{2}{7}\hat{v}_2)(\hat{x}^3\hat{y} + \hat{x}\hat{y}^3) + (\frac{16}{7}\hat{v}_1^3 + \frac{3}{7}\hat{v}_2)\hat{x}^2\hat{y}^2 \\ c(\hat{u}) &= -\hat{u} + \hat{v}_1\hat{u}^2 - \hat{v}_1^2\hat{u}^3 + (\frac{10}{7}\hat{v}_1^3 + \frac{1}{7}\hat{v}_2)\hat{u}^4\end{aligned}$$

We collect the basics we need:

Lemma 2.10.

$$\begin{array}{rcll} c(\hat{u}) & = & -\hat{u} + \hat{v}_1\hat{u}^2 & \bmod (\hat{u}^3) \\ c(\hat{u}) & = & \hat{u} + \hat{v}_1\hat{u}^2 + \hat{v}_1^2\hat{u}^3 + \hat{v}_2\hat{u}^4 & \bmod (2, \hat{u}^5) \\ \hat{p} = \hat{u}c(\hat{u}) & = & -\hat{u}^2 & \bmod (\hat{u}^3) \end{array}$$

where $\hat{p} \in ER(2)^{-32}(\mathbb{CP}^\infty)$ maps to $\hat{p} = \hat{u}\, c(\hat{u}) \in E(2)^{-32}(\mathbb{CP}^\infty)$ and is a modified first Pontryagin class.

Proof. This all follows from the preceding formulas. □

Recall that

$$d_1(y) = v_2^{-3}(1 - c)(y).$$

We rewrite some of our basic facts from Lemma 2.10 in our present terminology keeping in mind that in $E(2)^*(-)$, $\hat{v}_2 = 1 = v_2^{-8}$ and $\hat{p} = \hat{u}\, c(\hat{u})$.

Lemma 2.11.

$$
\begin{array}{lll}
c(\hat{u}) & = -\hat{u} - \hat{v}_1\hat{p} & \text{mod } (\hat{p}\hat{u}) \\
c(\hat{v}_1) & = \hat{v}_1 & \\
c(v_2) & = -v_2 & \\
d_1(\hat{u}) & = 2v_2^{-3}\hat{u} & \text{mod } (\hat{p}) \\
d_1(v_2\hat{u}) & = 0 & \text{mod } (\hat{p}) \\
d_1(\hat{u}) & = v_2^{-3}\hat{v}_1\hat{p} & \text{mod } (2, \hat{p}\hat{u}) \\
d_1(\hat{u}) & = v_2^{-3}(\hat{v}_1\hat{p} + \hat{v}_1^3\hat{p}^2 + \hat{p}^2) & \text{mod } (2, \hat{p}^2\hat{u}) \\
d_1(v_2) & = 2v_2^{-2} & \\
d_1(v_2\hat{p}) & = 2v_2^{-2}\hat{p} & \text{mod } (\hat{p}\hat{u})
\end{array}
$$

Proof. There is one minor new thing here, the formula for $d_1(\hat{u})$ mod $(2, \hat{p}^2\hat{u})$. We do have $d_1(\hat{u}) = v_2^{-3}(\hat{v}_1\hat{u}^2 + \hat{v}_1^2\hat{u}^3 + \hat{u}^4)$ and $\hat{p} = \hat{u}\, c(\hat{u}) = \hat{u}(\hat{u} + \hat{v}_1\hat{u}^2 + \hat{v}_1^2\hat{u}^3) = \hat{u}^2 + \hat{v}_1\hat{u}^3 + \hat{v}_1^2\hat{u}^4$. Replace the $\hat{u}^2$ with $\hat{p} + \hat{v}_1\hat{u}^3 + \hat{v}_1^2\hat{u}^4$ to get $d_1(\hat{u}) = v_2^{-3}(\hat{v}_1\hat{p} + \hat{v}_1^2\hat{u}^3 + \hat{v}_1^3\hat{u}^4 + \hat{v}_1^2\hat{u}^3 + \hat{u}^4)$. Two the terms cancel out and, modulo higher terms, $\hat{u}^4 = \hat{p}^2$. □

3. A sketch of the approach

The Bockstein spectral sequence for a general space X, $E(2)^*(X)$ to $ER(2)^*(X)$, concludes with $E_8 = 0$. In the two cases of interest to us, namely, $\wedge^n\mathbb{C}\mathbb{P}^\infty$ and $MU(n)$, the spectral sequence is even degree. In fact, the only differentials are d_1, d_3, and d_7. The last two are quite easy to do once d_1 has been computed. Although d_1 is complicated, we have an explicit algebraic formula for it. We require a spectral sequence to compute d_1 though. The spectral sequence we use for computing d_1 is broken up into $n+1$ parts. We evaluate d_1 on various subsets and denote those maps by $d_{1,0}, d_{1,1}, \dots, d_{1,n}$. After computing $d_{1,j}$, we call the result $E_{1,j+1}$. The $E_{1,n+1}$ is an associated graded version for E_2 of the Bockstein spectral sequence.

The sketch approach in this section works for both $\wedge^n\mathbb{C}\mathbb{P}^\infty$ and $MU(n)$. We do this here without inserting the necessary technical details in hopes of clarifying our computations. When it comes time to actually do the computations, we can adjust what we present here to be rigorous, and, in the process, add the gruesome technical details.

Our general description begins with an E_1 similar to the following:

$$R\{v_2^{0-7}\hat{u}^\epsilon\} \text{ with } \hat{u}^\epsilon = \hat{u}_1^{\epsilon_1}\hat{u}_2^{\epsilon_2}\dots\hat{u}_n^{\epsilon_n} \quad \epsilon_k \le 1.$$

Our R has no torsion and d_1 commutes with R and v_2^2.

Definition 3.1. Define W_j to be the set of $\hat{u}^\epsilon$ with $\epsilon_k = 0$ for $k < j$ and $\epsilon_j = 1$. We also include W_{n+1} with all $\epsilon_k = 0$.

This breaks our problem up to the following form:

$$R\{v_2^{0-7}W_1, v_2^{0-7}W_2, \dots, v_2^{0-7}W_n, v_2^{0-7}W_{n+1}\}$$

The filtration we use cannot be based on degree because we are $\mathbb{Z}/(48)$-periodic. It is also not indexed over $\mathbb{Z}$, an additional complication. The first step in defining our filtration looks a lot like using the standard cohomology degrees of the pre-hatted elements, which we call "length", but this can wait.

Our plan is quite simple. First we have to compute $d_{1,0}$, which we do below. This makes everything mod 2. Next we inductively (on j) compute our d_1 in the spectral sequence on $R\{v_2^{0-7}W_j\}$ (really a quotient of this). We call the restriction of d_1 to this our $d_{1,j}$ and the resulting quotient, $E_{1,j+1}$. We say quotient instead of subquotient because we find that $d_{1,j}$ is injective.

Since $\epsilon_j = 1$ in W_j, we find that, due to the filtration, all we need to compute $d_{1,j}$ on is $\hat{u}_j$. This will get rid of $\hat{u}_j$ so the target will always end up in $R\{v_2^{0-7}W_k\}$ for $k > j$. As mentioned already, we find that $d_{1,j}$ computed like this is injective, thus eliminating W_j completely. When all is said and done, the final answer after computing $d_{1,n}$ must be a quotient of $R\{v_2^{0-7}W_{n+1}\}$ and is just our associated graded object for E_2 of the Bockstein spectral sequence.

Remember, each map, $d_{1,j}$ computes some x^1-torsion elements and then throws them away so they don't clutter things up. Of course, we have to keep track of them, but the elements remaining in this quotient of $R\{v_2^{0-7}W_{n+1}\}$ are just the elements left over that can be x^3 and x^7-torsion elements.

That is our brief summary of how $d_{1,j}$, $j > 0$, behaves. Before we can do those computations though, we need to compute $d_{1,0}$. We can do that here in a general way that actually gives the result for our two cases.

We start with our

$$R\{v_2^{0-7}\hat{u}^\epsilon\} \text{ with } \hat{u}^\epsilon = \hat{u}_1^{\epsilon_1}\hat{u}_2^{\epsilon_2}\dots\hat{u}_n^{\epsilon_n} \quad \epsilon_k \le 1.$$

We need a new definition:

$$s(\epsilon) = \sum \epsilon_k.$$

Our $d_{1,0}$ kills off lots of elements and 2. (Mod higher filtrations.) Recall that $c(\hat{u}) = -\hat{u}$ modulo pretty much anything. We compute $d_{1,0}$ using the formula for d_1:

$$\begin{aligned} d_{1,0}(\hat{u}^\epsilon) &= v_2^{-3}(\hat{u}^\epsilon - c(\hat{u}^\epsilon)) \\ &= v_2^{-3}(\hat{u}^\epsilon - \prod_{\epsilon_k=1} c(\hat{u}_k)) = v_2^{-3}(\hat{u}^\epsilon - (-1)^{s(\epsilon)}\hat{u}^\epsilon) \end{aligned}$$

So,

$$\begin{aligned} d_{1,0}(\hat{u}^\epsilon) &= 2v_2^{-3}\hat{u}^\epsilon \qquad s(\epsilon) \text{ odd} \\ d_{1,0}(\hat{u}^\epsilon) &= 0 \qquad s(\epsilon) \text{ even} \end{aligned}$$

With the v_2 in front, knowing $c(v_2) = -v_2$, we get

$$\begin{aligned} d_{1,0}(v_2\hat{u}^\epsilon) &= 2v_2^{-2}\hat{u}^\epsilon \qquad s(\epsilon) \text{ even} \\ d_{1,0}(v_2\hat{u}^\epsilon) &= 0 \qquad s(\epsilon) \text{ odd} \end{aligned}$$

The end result of the computation is

$$E_{1,1} = R/(2)\{v_2v_2^{0,2,4,6}\hat{u}^\epsilon \quad s(\epsilon) \quad \text{odd}\} \oplus R/(2)\{v_2^{0,2,4,6}\hat{u}^\epsilon \quad s(\epsilon) \quad \text{even}\}$$

We can make a dramatic simplification with better notation.

$$\begin{aligned} v_2^{o/e} &= v_2 \qquad s(\epsilon) \text{ odd} \\ v_2^{o/e} &= 1 \qquad s(\epsilon) \text{ even} \end{aligned}$$

Note that the $v_2^{o/e}$ is a function of $\hat{u}^\epsilon$. Using this notation, the result cleans up as:

$$E_{1,1} = R/(2)\{v_2^{o/e}v_2^{0,2,4,6}\}$$

Remark 3.2. It is important to note that after our $d_{1,0}$, which is just the first step in our spectral sequence for computing d_1, we are working mod (2) in a very strong sense. Normally, in a spectral sequence, after our computation of $d_{1,0}$, this would mean that 2 times an element in the associated graded object is really just an element represented by some higher filtration term. However, because $2x = 0$, we do not have such extension problems. Two times any element is definitely killed by x and so is actually zero in the spectral sequence. For reference, we state this as a lemma.

Lemma 3.3 ***(Two is zero)***. *Two times an element in $E_{1,1}$ for $\wedge^n \mathbb{CP}^\infty$ and $MU(n)$ is zero. It is not represented in the associated graded object by a non-zero element in a higher filtration.*

4. The filtration

By complex orientability, we have

$$E(2)^*(\wedge^n \mathbb{CP}^\infty) \cong E(2)^*[[\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n]]\{\hat{u}_1 \hat{u}_2 \cdots \hat{u}_n\}$$

The class $\hat{u}\, c(\hat{u})$ coming from $\hat{p}$ is a permanent cycle. Let $\hat{p}_i$ be the class associated with the i-th copy of $\mathbb{CP}^\infty$ in our smash product.

Because $\hat{p} = -\hat{u}^2$ mod higher powers, we can replace our description of $E(2)^*(\wedge^n \mathbb{CP}^\infty)$. We need some notation first.

$$\begin{array}{lll} I = (i_1, i_2, \ldots, i_n) & s(I) = \sum i_k & i_k \geq 0 \\ \epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) & s(\epsilon) = \sum \epsilon_k & \epsilon_k = 0 \text{ or } 1 \end{array}$$

Define

$$\hat{p}^I \hat{u}^\epsilon = \hat{p}_1^{i_1} \hat{u}_1^{\epsilon_1} \hat{p}_2^{i_2} \hat{u}_2^{\epsilon_2} \cdots \hat{p}_n^{i_n} \hat{u}_n^{\epsilon_n}$$

and define the length of (I, ϵ), $\ell(I, \epsilon)$ to be $2s(I) + s(\epsilon)$.

Note that this length is just degree of the corresponding (unhatted) elements in mod 2 cohomology.

We want to rewrite $E(2)^*(\wedge^n \mathbb{CP}^\infty)$ in terms of our Pontryagin classes, but the smash product requires some awkward notation. We require $i_k + \epsilon_k = 1$ to get

$$E(2)^*(\wedge^n \mathbb{CP}^\infty) \cong E(2)^*[[\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n]]\{\hat{p}^I \hat{u}^\epsilon\} \cong \mathbb{Z}_{(2)}[\hat{v}_1][[\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n]]\{v_2^{0-7} \hat{p}^I \hat{u}^\epsilon\}$$

Once we have our filtration and look at the associated graded object, it will become, now with $i_k + \epsilon_k > 0$:

$$E(2)^*\{\hat{p}^I \hat{u}^\epsilon\} \cong \mathbb{Z}_{(2)}[\hat{v}_1]\{v_2^{0-7} \hat{p}^I \hat{u}^\epsilon\}$$

We now need to put our filtration on this.

Definition 4.1. We put an order on the pairs (I, ϵ) as follows. If $\ell(I', \epsilon') > \ell(I, \epsilon)$, then $(I', \epsilon') > (I, \epsilon)$. If $\ell(I', \epsilon') = \ell(I, \epsilon)$ and $2i'_j + \epsilon'_j = 2i_j + \epsilon_j$ for $k < j \leq n$, and $2i'_k + \epsilon'_k < 2i_k + \epsilon_k$ then $(I', \epsilon') > (I, \epsilon)$.

The (I, ϵ) now form an ordered set and we can use them to give a filtration on $E(2)^*(\wedge^n \mathbb{CP}^\infty)$ as follows:

$$F(I, \epsilon) = \mathbb{Z}_{(2)}[\hat{v}_1]\{v_2^{0-7} \hat{p}^{I'} \hat{u}^{\epsilon'}\} \qquad (I', \epsilon') > (I, \epsilon)$$

The associated graded object still looks the same:

$$E_{1,0}(I, \epsilon) = E(2)^*(\wedge^n \mathbb{CP}^\infty) \cong \mathbb{Z}_{(2)}[\hat{v}_1]\{v_2^{0-7} \hat{p}^I \hat{u}^\epsilon\} \qquad i_k + \epsilon_k > 0$$

Note that in degrees $16*$, this is just $\mathbb{Z}_{(2)}[\hat{v}_1]\{\hat{p}^I\hat{u}^\epsilon\}$, $i_k+\epsilon_k>0$.

In general, we will suppress the (I,ϵ) notation associated with this filtration. We will use it, but the associated graded object will be implicit, not explicit. A certain amount of clutter is avoided without loss, we hope, of clarity.

5. Computing $d_{1,0}$ for $\wedge^n\mathbb{CP}^\infty$

The setup of our computation in Section 3 now applies. The zeroth differential is computed there giving us:

Proposition 5.1. *After computing $d_{1,0}$ for $\wedge^n\mathbb{CP}^\infty$, we get*

$$E_{1,1}\cong\mathbb{Z}/(2)[\hat{v}_1]\{v_2^{o/e}v_2^{0,2,4,6}\hat{p}^I\hat{u}^\epsilon\}$$

with $i_k+\epsilon_k>0$. The x^1-torsion generators detected by $d_{1,0}$ are represented by:

$$\mathbb{Z}_{(2)}[\hat{v}_1]\{2v_2^{o/e}v_2^{0,2,4,6}\hat{p}^I\hat{u}^\epsilon\}=\mathbb{Z}_{(2)}[\hat{v}_1]\{v_2^{o/e}\alpha_i\hat{p}^I\hat{u}^\epsilon\}$$

6. Computing $d_{1,1}$ for $\wedge^n\mathbb{CP}^\infty$

After $d_{1,0}$, we are working mod (2). Following Section 3, we start our computation of d_1 on elements with $\epsilon_1=1$. The main formula we need now is: $c(\hat{u})=\hat{u}+\hat{v}_1\hat{p}$ modulo $(2,\hat{p}\hat{u})$, where we are now invoking the filtration and looking only at the representative in the associated graded object. We call the map restricted to the $\hat{u}^\epsilon$ with $\epsilon_1=1$, $d_{1,1}$.

Our d_1, and so our $d_{1,1}$, commutes with v_2^2, $\hat{p}_i$ and $v_2^{o/e}$.

$$\begin{aligned}d_{1,1}(v_2^{o/e}\hat{u}^\epsilon)&=v_2^{-3}v_2^{o/e}(\hat{u}^\epsilon+c(\hat{u}^\epsilon))\\&=v_2^{-3}v_2^{o/e}(\hat{u}^\epsilon+\prod_{\epsilon_k=1}(\hat{u}_k+\hat{v}_1\hat{p}_k)\hat{u}^{\epsilon-\Delta_k})\end{aligned}$$

The $\hat{u}^\epsilon$ cancels out. If we keep 2 or more of the $\hat{p}_k$, the length is greater than if we just keep one. Modulo those terms of higher length, i.e. retaining only those with one $\hat{p}_k$, we have:

$$d_{1,1}(v_2^{o/e}\hat{u}^\epsilon)=v_2^{-3}v_2^{o/e}(\sum_{\epsilon_k=1}\hat{v}_1\hat{p}_k\hat{u}^{\epsilon-\Delta_k})$$

These terms all have the same length, but when $k=1$, we have $\hat{v}_1\hat{p}_1$ when d_1 acts on $\hat{u}_1$ and the others have $\hat{u}_1$. Our filtration gives us the $v_2^{-3}v_2^{o/e}(\hat{v}_1\hat{p}_1\hat{u}^{\epsilon-\Delta_1})$ is the term with lowest filtration. We have just computed:

Proposition 6.1. *After computing $d_{1,1}$ for $\wedge^n\mathbb{CP}^\infty$, we get*

$$E_{1,2}\cong\mathbb{Z}/(2)\{v_2^{o/e}v_2^{0,2,4,6}\hat{p}^I\hat{u}^\epsilon\}\quad\epsilon_1=0$$

with $i_k+\epsilon_k>0$. The x^1-torsion generators detected by $d_{1,1}$ are represented by:

$$\mathbb{Z}/(2)[\hat{v}_1]\{v_2^{o/e}v_2^{0,2,4,6}\hat{v}_1\hat{p}^I\hat{u}^\epsilon\}\quad\epsilon_1=0$$

Note that because we are in the smash product, $\epsilon_1 = 0$ implies that $i_1 > 0$.

As mentioned in Section 3, $d_{1,1}$ is injective on terms with $\epsilon_1 = 1$, so all remaining terms have $\epsilon_1 = 0$.

It might be premature to discuss such things, but the above is consistent with the results for $ER(2)^*(\mathbb{CP}^\infty)$ from [6, Theorems 3.1 and 4.1], i.e. the $n = 1$ case, even if, at first glance, they don't look the same.

7. Computing $d_{1,j}$ for $\wedge^n \mathbb{CP}^\infty$

By induction, when we start our work with $d_{1,j}$, we find that all we have left are $\hat{u}^\epsilon$ with $i_1 = i_2 = \cdots = i_{j-1} = 0$.

From Lemma 2.11, we have $d_1(\hat{u}) = v_2^{-3}(\hat{v}_1\hat{p} + \hat{v}_1^3\hat{p}^2 + \hat{p}^2) \mod (2, \hat{p}^2\hat{u})$. When we computed $d_{1,1}$, we were working in $E_{1,1}$, and this was already strongly mod 2. All we needed of this formula was the lower length term $\hat{v}_1\hat{p}$, which made this term zero in $E_{1,2}$. However, unlike with 2, where there were no extension problems, $\hat{v}_1$ times $\hat{p}$ is not zero, but can be represented by terms of higher filtration, namely $\hat{v}_1\hat{p} + \hat{v}_1^3\hat{p}^2 + \hat{p}^2 = 0$. If we apply this formula to the middle term, we get even higher filtrations, so it goes away, and, because we are working mod 2, we get the main new formula used in this section, previously used as Equation (4.2) from [6]:

$$0 = \hat{v}_1\hat{u}^2 + \hat{v}_2\hat{u}^4 = \hat{v}_1\hat{p} + \hat{p}^2 \qquad \mod (\hat{p}^2\hat{u}) \qquad \text{or} \qquad \hat{v}_1\hat{p} = \hat{p}^2 \tag{7.1}$$

Proposition 7.2. *After computing $d_{1,j}$ for $\wedge^n\mathbb{CP}^\infty$, we get*

$$E_{1,j+1} \cong \mathbb{Z}/(2)\{v_2^{o/e} v_2^{0,2,4,6}\hat{p}^I\hat{u}^\epsilon\}$$

for $1 < j \le n$. We have $i_k + \epsilon_k > 0$.

$$\epsilon_k = 0 \quad \textit{for } k \le j, \qquad i_k = 1 \quad \textit{for } k < j$$

The x^1-torsion generators detected by $d_{1,j}$ are represented by:

$$\mathbb{Z}/(2)\{v_2^{o/e} v_2^{0,2,4,6}\hat{p}^I\hat{u}^\epsilon\} \quad \epsilon_k = 0 \quad \textit{for } k \le j$$
$$i_k = 1 \quad \textit{for } k < j-1, \qquad i_{j-1} > 1$$

We have $E_{1,n+1} = E_{1,\infty}$, which is our associated graded object for the BSS E_2 for computing $ER(2)^*(\wedge^n\mathbb{CP}^\infty)$ from $E(2)^*(\wedge^n\mathbb{CP}^\infty)$, is

$$\mathbb{Z}/(2)\{v_2^{0,2,4,6}\hat{p}^I\} \qquad i_k = 1 \quad \textit{for } k < n$$

or

$$\mathbb{Z}/(2)\{v_2^{0,2,4,6}\hat{p}_1\hat{p}_2\ldots\hat{p}_{n-1}\hat{p}_n^{i_n}\}.$$

Note that if all $\epsilon_k = 0$, $s(\epsilon)$ is even.

Proof. We have already computed $E_{1,2}$, so our induction is started. Assume we have computed $E_{1,j'+1}$ and $d_{1,j'}$ for $j' < j$. We need to compute $d_{1,j}$ on $E_{1,j}$ to get $E_{1,j+1}$. We compute $d_{1,j}$ only on those $\hat{u}^\epsilon$ with $i_j = 1$, i.e. on the W_j of Section 3.

We use our filtration to get $d_1(\hat{u}) = v_2^{-3}\hat{v}_1\hat{p}$ from Lemma 2.11. As in the case of $j = 1$, when $\epsilon_j = 1$, d_1 applied to a $\hat{u}_k$, with $k > j$, increases the filtration more than d_1 applied to $\hat{u}_j$ does. This gives:

$$d_{1,j}(v_2^{o/e}v_2^{0,2,4,6}\hat{p}^I\hat{u}^\epsilon) = v_2^{-3}\hat{v}_1 v_2^{o/e}v_2^{0,2,4,6}\hat{p}^{I+\Delta_j}\hat{u}^{\epsilon-\Delta_j}$$

Unfortunately, $\hat{v}_1$ doesn't show up in the associated graded object for $E_{1,j}$ so we need to find an equivalent element that represents this. Note that $i_k = 1$ for $k < j-1$, and $i_{j-1} > 0$. We use formula (7.1), $\hat{v}_1\hat{p} = \hat{p}^2$. The lowest i with $\hat{p}_i^2 \neq 0$ in $E_{1,j}$ is $i = j-1$, so, this term is, mod higher filtrations, represented by:

$$v_2^{-3}v_2^{o/e}v_2^{0,2,4,6}\hat{p}^{I+\Delta_j+\Delta_{j-1}}\hat{u}^{\epsilon-\Delta_j}.$$

The result follows. □

Remark 7.3. At this stage, we are done with d_1 for degree reasons, but also we see that all the remaining terms have $d_1 = 0$ on them as they are all cycles. We have computed E_2 for Theorem 1.1.

8. Summary of the x^1-torsion generators for $ER(2)^*(\wedge^n\mathbb{CP}^\infty)$

We just collect from the previous sections:

Theorem 8.1. *Representatives for the x^1-torsion generators in our associated graded object for $ER(2)^*(\wedge^n\mathbb{CP}^\infty)$ are given by:*

$$\mathbb{Z}_{(2)}[\hat{v}_1]\{v_2^{o/e}\alpha_i\hat{p}^I\hat{u}^\epsilon\} \quad 0 \leq i < 4$$
$$\mathbb{Z}/(2)[\hat{v}_1]\{v_2^{o/e}v_2^{0,2,4,6}\hat{v}_1\hat{p}^I\hat{u}^\epsilon\} \quad \epsilon_1 = 0$$

For $1 < j \leq n$,

$$\mathbb{Z}/(2)\{v_2^{o/e}v_2^{0,2,4,6}\hat{p}^I\hat{u}^\epsilon\} \quad \epsilon_k = 0 \ \text{ for } k \leq j$$
$$i_k = 1 \ \text{ for } k < j-1, \quad i_{j-1} > 1$$

9. Computing d_3 and d_7 for $\wedge^n\mathbb{CP}^\infty$

We have finished our computation of d_1 and we get $E_2 = E_3$.

Proposition 9.1. *Our associated graded version of the BSS E_4 for computing $ER(2)^*(\wedge^n\mathbb{CP}^\infty)$ from $E(2)^*(\wedge^n\mathbb{CP}^\infty)$ is*

$$E_4 = E_5 = E_6 = E_7 = \mathbb{Z}/(2)\{v_2^{0,4}\hat{p}^I\} \qquad i_k = 1$$

The x^3-torsion generators are represented by

$$\mathbb{Z}/(2)\{v_2^{0,4}\hat{p}^I\} \qquad i_k = 1 \ \text{ for } k < n \qquad i_n > 1$$

Remark 9.2. By definition, all x^i-torsion generators inject into $E(2)^*(-)$. In particular, the x^1-torsion generators (all are of even degree) inject. The x^3-torsion generators are all in degrees $8*$. The degree of x is -1 mod (8) so for x^3-torsion, we only have x and x^2 times elements in degree $8*$. Consequently, all of the elements in degrees $4*$ that we have studied so far inject. Lots of elements have x^2 times them non-zero, so there are many elements in degrees -2 mod (8) that don't inject.

Proof. All of the $\hat{p}_k$ are permanent cycles ([14, Proposition 5.1]) and d_3 commutes with v_2^4, so the computation of d_3 is based on:

$$d_3(v_2^2) = \hat{v}_1 v_2^{-4}$$

from the action on the coefficients. We have the relation:

$$\hat{v}_1 \hat{p}_n = \hat{p}_n^2$$

Because $i_k = 1$ for $k < n$, we have

$$d_3(v_2^2 \hat{p}^I) = v_2^{-4} \hat{v}_1 \hat{p}^I = v_2^{-4} \hat{p}^{I+\Delta_n}$$

From this we get our $E_4 = E_5 = E_6 = E_7$ (for degree reasons). □

Starting with our E_7 and recalling from the coefficients that

$$d_7(v_2^4) = 1,$$

we get:

Proposition 9.3. *For the BSS for computing $ER(2)^*(\wedge^n \mathbb{CP}^\infty)$ from $E(2)^*(\wedge^n \mathbb{CP}^\infty)$, we have $E_8 = 0$. The x^7-torsion generator is*

$$\mathbb{Z}/(2)\{\hat{p}_1 \hat{p}_2 \ldots \hat{p}_n\}$$

This generator is in a degree that is a multiple of 16. More precisely, it is in degree $-32n = 16n$.

Remark 9.4. The only element divisible by x in degree 4 mod (8) is $x^4 \hat{p}_1 \hat{p}_2 \ldots \hat{p}_n$ (to be more precise, this is in degree $16(n-1) - 4$ mod (48). Consequently, it is the only element in the kernel of the map in degrees $4*$. Similarly, $x^6 \hat{p}_1 \hat{p}_2 \ldots \hat{p}_n$ in degree $16n - 6$ is the only element in degree $8 * -6$ divisible by x. This concludes the proof of Theorem 1.1 and part of Theorem 1.4, and the remark that follows. If you want particularly clean statements, stick with elements in degree $16*$. In all our statements, just require $s(\epsilon)$ to be even and ignore the $v_2^{2,4,6}$. Historically, those are the only elements that have mattered to us, but it takes so little effort to get the injection for $8*$, it seems obligatory. Here we still require $s(\epsilon)$ to be even, but we only ignore $v_2^{2,6}$, leaving $v_2^{0,4}$.

10. Preliminaries for $BU(n)$

Because of the stable splitting, $BU(n) = MU(n) \vee BU(n-1)$, [15], we can compute $ER(2)^*(MU(n))$ instead of $ER(2)^*(BU(n))$.

So, rather than study the map

$$\prod^n \mathbb{CP}^\infty \longrightarrow BU(n)$$

we will mainly look at:

$$\wedge^n \mathbb{CP}^\infty \longrightarrow MU(n)$$

Because $E(2)$ is a complex orientable theory, we have the usual

$$E(2)^*(BU(n)) \cong E(2)^*[[c_1, c_2, \ldots, c_n]]$$

where the c_k are the generalized Conner-Floyd Chern classes. To see $E(2)^*(MU(n))$, we just look at the ideal generated by c_n. So, we have:

$$E(2)^*(MU(n)) \cong E(2)^*[[c_1, c_2, \ldots, c_n]]\{c_n\}$$

We need to 'hat' these Chern classes just as we did with $\hat{u}$ for $\mathbb{CP}^\infty$. Define (keeping in mind that v_2 is a unit):

$$\hat{c}_k = v_2^{3k} c_k.$$

This puts $\hat{c}_k$ in degree $2k - 18k = -16k$ and we have

$$E(2)^*(MU(n)) \cong E(2)^*[[\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n]]\{\hat{c}_n\}$$

We need to use the well-known fact that for complex oriented theories, $G^*(BU(n))$ injects into $G^*(\prod^n \mathbb{CP}^\infty)$. Each c_k, or, respectively, $\hat{c}_k$, goes to the k-th symmetric function on the u_i, respectively, $\hat{u}_i$. Similarly for the map of the smash product to $MU(n)$. Here, we have $\hat{c}_n$ goes to $\hat{u}_1\hat{u}_2 \cdots \hat{u}_n$.

For $J = (j_1, j_2, \ldots, j_n)$, let

$$\hat{c}^J = \hat{c}_1^{j_1} \hat{c}_2^{j_2} \cdots \hat{c}_n^{j_n}.$$

After we go to our associated graded object, we can write $E(2)^*(MU(n))$ as

$$E(2)^*\{\hat{c}^J\} \qquad j_n > 0.$$

We can view

$$E(2)^*(MU(n)) \subset E(2)^*(\wedge^n \mathbb{CP}^\infty)$$

and we know how to write elements of $E(2)^*(\wedge^n \mathbb{CP}^\infty)$ in terms of

$$\hat{p}^I \hat{u}^\epsilon = \hat{p}_1^{i_1} \hat{u}_1^{\epsilon_1} \hat{p}_2^{i_2} \hat{u}_2^{\epsilon_2} \cdots \hat{p}_n^{i_n} \hat{u}_n^{\epsilon_n}$$

Every element $z \in E(2)^*(\wedge^n \mathbb{CP}^\infty)$ can be written as a sum of such elements (with coefficients). These elements are ordered using the order on (I, ϵ) from 4.1.

Definition 10.1. The **leading term** of $z \in E(2)^*(\wedge^n \mathbb{CP}^\infty)$ is the term of lowest order.

The leading term of any symmetric function must be of the form $\hat{p}^I \hat{u}^\epsilon$ with

$$2i_1 + \epsilon_1 \geq \cdots \geq 2i_k + \epsilon_k \geq 2i_{k+1} + \epsilon_{k+1} \geq \ldots \geq 2i_n + \epsilon_n > 0$$

Definition 10.2. We call this **property A** and use it constantly from here on, but without having to repeat the above often.

Although many symmetric functions could have the same leading term, given a $\hat{p}^I\hat{u}^\epsilon$ with **property A**, we can construct a unique symmetric function, $w_{I,\epsilon}$, with this as its leading term. Our $w_{I,\epsilon}$ is just the sum of all distinct permutations of our $\hat{p}^I\hat{u}^\epsilon$, keeping in mind that the $\hat{p}_i$ and the $\hat{u}_i$ move together. These symmetric functions $w_{I,\epsilon}$ generate $E(2)^*(MU(n)) \subset E(2)^*(\wedge^n \mathbb{CP}^\infty)$. Our computations will take place entirely in this image.

We can consider $E(2)^*(MU(n))$ in $E(2)^*(\wedge^n \mathbb{CP}^\infty)$ and write the associated graded object as $\mathbb{Z}_{(2)}[\hat{v}_1] \times \{v_2^{0-7} w_{I,\epsilon}\}$. For our filtration, we just use the order, 4.1, on the (I,ϵ), which all have **property A**. This is the same as using it on the leading term.

Recall that d_1 commutes with $\hat{p}_i$, and v_2^2.

Similar to the computation in Section 5, we can compute $d_{1,0}$ (modulo higher filtration) on every term of $w_{I,\epsilon}$ to get

$$d_{1,0}(w_{I,\epsilon}) = 2v_2^{-3} w_{I,\epsilon} \quad s(\epsilon) \text{ odd}$$
$$d_{1,0}(v_2 w_{I,\epsilon}) = 2v_2^{-2} w_{I,\epsilon} \quad s(\epsilon) \text{ even}$$

Proposition 10.3. *With* ***property A****, after computing $d_{1,0}$ for $MU(n)$, we have:*

$$E_{1,1} \cong \mathbb{Z}/(2)[\hat{v}_1]\{v_2^{o/e} v_2^{0,2,4,6} w_{I,\epsilon}\}$$

The x^1-torsion generators detected by $d_{1,0}$ are represented by:

$$\mathbb{Z}_{(2)}[\hat{v}_1]\{2v_2^{o/e} v_2^{0,2,4,6} w_{I,\epsilon}\} = \mathbb{Z}_{(2)}[\hat{v}_1]\{v_2^{o/e} \alpha_i w_{I,\epsilon}\}$$

11. Different descriptions of $E(2)^*(MU(n))$

We find it easiest to make our computations with the $w_{I,\epsilon} \in E(2)^*(\wedge^n \mathbb{CP}^\infty)$, but it would be more traditional to think in terms of Chern classes in $E(2)^*(MU(n))$. So, we now show how to relate the $\hat{c}^J$ to the $w_{I,\epsilon}$.

In the product, the image of $\hat{c}_k$ is the k-th symmetric function on the $\hat{u}_i$. The leading term in the sum that makes up the symmetric function is:

$$\hat{u}(k) = \hat{u}_1\hat{u}_2 \cdots \hat{u}_k.$$

Modulo higher terms in the filtration, we have $\hat{u}^2 = -\hat{u}\,(-\hat{u}) = -\hat{u}\,c(\hat{u}) = -\hat{p}$, so, in the smash product, the leading term of the image of $\hat{c}_k\hat{c}_n$ is (modulo higher terms):

$$\hat{u}(k)\hat{u}(n) = \hat{u}_1^2\hat{u}_2^2 \cdots \hat{u}_k^2\hat{u}_{k+1} \cdots \hat{u}_n = (-1)^k \hat{p}_1\hat{p}_2 \cdots \hat{p}_k\hat{u}_{k+1}\hat{u}_{k+2} \cdots \hat{u}_n$$

We have the leading term of the image of c^J

$$\hat{u}(1)^{j_1}\hat{u}(2)^{j_2} \cdots \hat{u}(n)^{j_n} \longrightarrow \hat{u}_1^{\sum_{i=1}^n j_i}\hat{u}_2^{\sum_{i=2}^n j_i} \cdots \hat{u}_k^{\sum_{i=k}^n j_i} \cdots \hat{u}_n^{j_n}$$

We prefer to replace all the $\hat{c}_k^2$ with the Pontryagin classes $\hat{P}_k$, but to do that, we need to take a break to define them.

Pontryagin classes

We will use $\mathbb{BU}(k)$ to denote the space $BU(k)$ with $\mathbb{Z}/2$ acting by complex conjugation and $BU(k)$ to denote the space with trivial $\mathbb{Z}/2$-action. By an equivariant analog of the Atiyah-Hirzebruch spectral sequence, we have

$$\mathbb{E}(n)^\star(\mathbb{BU}(k)) = \mathbb{E}(n)^\star[[c_1,\dots,c_k]], \qquad |c_k| = k(1+\alpha)$$

Consider the (equivariant) map $f : BU(k) \to BSO(2k) \to \mathbb{BU}(2k)$ classifying the complexification of the underlying real bundle of the tautological complex k-plane bundle.

Definition 11.1. For $1 \le m \le k$, define the mth hatted Pontryagin class $\hat{P}_m \in ER(n)^*(BU(k))$ to be

$$\hat{P}_m := \widehat{f^*(c_{2m})} \in \mathbb{E}(n)^{2m(1-\lambda)}(BU(k)) = ER(n)^{2m(1-\lambda)}(BU(k))$$

Lemma 11.2.

$$\hat{P}_m \in ER(n)^{m2^{n+3}(1-2^{n-1})}(BU(k)) \quad \textit{maps to} \quad \sum_{\substack{i+j=2m,\\ 0\le i,j\le k}} \hat{c}_i\, c(\hat{c}_j) \in E(n)^{m2^{n+3}(1-2^{n-1})}(BU(k))$$

Note that in our $n = 2$ case, $\hat{P}_m$ is in degree $-32m$ and the image is a permanent cycle in our Bockstein spectral sequence.

Proof. The fact that for a complex vector bundle V, the complexification of its underlying real bundle splits as a direct sum of V and its complex conjugate, $(V_{\mathbb{R}}) \otimes \mathbb{C} = V \oplus \overline{V}$, means that the following diagram commutes up to homotopy:

$$\begin{array}{ccccc}
BU(k) & \xrightarrow{\Delta} & BU(k)\times BU(k) & \xrightarrow{1\times c} & BU(k)\times BU(k) \\
\downarrow & & & & \downarrow \\
BSO(2k) & & \longrightarrow & & BU(2k)
\end{array}$$

where the right vertical map classifies the direct sum. Applying $E(n)$-cohomology to this diagram, we see that the image of the total hatted Chern class $(\sum \hat{c}_i)$ in $E(n)^*(BU(2k))$ is given by the total hatted Chern class in $E(n)^*BU(k)$ times its conjugate, $(\sum \hat{c}_i)(\sum c(\hat{c}_j))$. It follows that the image of c_{2m} under f^* is as claimed. □

Remark 11.3. Working mod 2 in our associated graded object, we have $c(\hat{c}_i)$ is represented by the same term as $\hat{c}_i$. The terms $\hat{c}_i\, c(\hat{c}_j)$ and $\hat{c}_j\, c(\hat{c}_i)$ cancel out and we are left with $\hat{c}_m\, c(\hat{c}_m) = \hat{c}_m^2$ represents $\hat{P}_m$.

We will not use it in this paper, we could in fact produce a class in $ER(n)^*(BU(k))$ whose image in $E(n)^*(BU(k))$ is given by $\hat{c}_m\, c(\hat{c}_m)$ on the nose as follows. Let $\mathbb{MU}(n)$ denote the 2-local Real bordism spectrum with v_n inverted–it is a commutative $\mathbb{Z}/2$-ring spectrum (see e.g. [7, Lemma 4.2]). Applying the norm $N_{\{e\}}^{\mathbb{Z}/2}$ to $c_m \in MU(n)^{2m}(BU(k))$ yields a class in $\mathbb{MU}(n)^{2m(1+\alpha)}(BU(k))$ whose underlying nonequivariant class is $c_m\, c(c_m)$ by the double coset formula. Mapping from $\mathbb{MU}(n)$-cohomology to $\mathbb{E}(n)$-cohomology and applying the hat construction gives $\hat{c}_m\, c(\hat{c}_m)$ as desired.

Either this element or the Pontryagin class could be used later to show that our d_3 and d_7 only operate on the coefficient ring. We have chosen the more traditional Pontryagin classes. Neither one is necessary to compute d_1 because $\hat{c}_m\, c(\hat{c}_m)$ has d_1 equal to zero on it. It is only for the higher differentials that we need the permanent cycles that the Pontryagin classes give us.

Recall that we can compute c on any element of $E(2)^*(BU(n))$ by naturality because it injects into $E(2)^*(\wedge^n \mathbb{CP}^\infty)$ and we know c on $E(2)^*(\mathbb{CP}^\infty)$.

The leading term of the image of $\hat{P}_k$ is $(-1)^k \hat{u}(k)^2 = \hat{p}_1\hat{p}_2\dots\hat{p}_k$.

To avoid the power series, we rewrite the associated graded object for $E(2)^*(MU(n))$ as

$$E(2)^*\{\hat{P}_1^{k_1}\hat{c}_1^{r_1}\hat{P}_2^{k_2}\hat{c}_2^{r_2}\dots\hat{P}_n^{k_n}\hat{c}_n^{r_n}\} \quad 0 < k_n + r_n \quad r_k \leq 1$$

or, simply as:

$$E(2)^*\{\hat{P}^K\hat{c}^r\} \quad 0 < k_n + r_n$$

Define s_i, e_i, g_i and ϵ_i as follows:

$$s_i = k_i + k_{i+1} + \cdots + k_n \text{ and } e_i = r_i + r_{i+1} + \cdots + r_n = 2g_i + \epsilon_i \quad \epsilon_i \leq 1.$$

Let $i_j = s_j + g_j$, then, using our injection, we have, modulo higher filtration:

$$\hat{P}^K\hat{c}^r \text{ maps to } \pm w_{I,\epsilon}$$

where $w_{I,\epsilon}$ is the symmetric function, with leading term

$$\hat{p}_1^{s_1+g_1}\hat{u}_1^{\epsilon_1}\hat{p}_2^{s_2+g_2}\hat{u}_2^{\epsilon_2}\cdots\hat{p}_n^{s_n+g_n}\hat{u}_n^{\epsilon_n} = \hat{p}^I\hat{u}^\epsilon.$$

Note that, by construction, this satisfies **property A**.

Reversing the process to go from $w_{I,\epsilon}$ to $\hat{P}^K\hat{c}^r$ is unpleasant. It is trivial to go back to $\hat{p}^I\hat{u}^\epsilon$, but that is still inside $E(2)^*(\wedge^n\mathbb{CP}^\infty)$. It is best to see $w_{I,\epsilon}$ in terms of Chern classes. Modulo higher filtration, we have,

$$w_{I,\epsilon} = \hat{c}_1^{2i_1+\epsilon_1-2i_2-\epsilon_2}\hat{c}_2^{2i_2+\epsilon_2-2i_3-\epsilon_3}\cdots\hat{c}_n^{2i_n+\epsilon_n}. \tag{11.4}$$

Note that when $\epsilon_1 = \epsilon_2 = 0$, we get $\hat{c}_1^{2(i_1-i_2)} = \pm\hat{P}_1^{(i_1-i_2)}$ mod higher filtration. This comes in handy later.

Although we don't need to be able to completely reverse the process to go from $w_{I,\epsilon}$ to $\hat{P}^K\hat{c}^r$, we do need to keep track of the parity of $s(\epsilon)$.

Lemma 11.5. *If $w_{I,\epsilon}$ is the image of $\hat{c}^J = \hat{c}_1^{j_1}\hat{c}_2^{j_2}\dots\hat{c}_n^{j_n}$, then the parity of $s(\epsilon)$ is the same as the parity of $j_1 + j_3 + j_5 + \cdots$, or, equivalently, $r_1 + r_3 + r_5 + \cdots$ from above.*

Proof. The proof is easy. From Equation (11.4), we have $j_1+j_3+j_5+\cdots$ is, mod (2), just $\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4+\cdots$ and this has the same parity as $s(\epsilon)$. Using the r, we have, mod 2, $s(\epsilon) = r_1+2r_2+3r_3+\cdots+nr_n$. Deleting all the even terms gives the same result. □

12. Lemmas for our $MU(n)$ d_1 computations

We are going to compute d_1 in the Bockstein spectral sequence using a spectral sequence. Our computations will be done on the image of $E(2)^*(MU(n))$ in $E(2)^*(\wedge^n\mathbb{CP}^\infty)$. This is generated by the symmetric functions $w_{I,\epsilon}$ where the leading term is $\hat{p}^I\hat{u}^\epsilon$ with **property A**. The spectral sequence we use to compute d_1 is based on the filtration we have given using the ordering on the (I,ϵ). Since $d_1(w_{I,\epsilon})$ is also a symmetric function, to compute the spectral sequence we need to know its leading term in the associated graded object, i.e. the lowest filtration term of $d_1(w_{I,\epsilon})$. In principle, to do this, we have to compute d_1 on every one of the distinct permutations that make up $w_{I,\epsilon}$.

We reduce that onerous task significantly in this section by a series of simplifications. First, we recall that we are working mod (2) in a very strong sense now that we have computed $d_{1,0}$. In our actual computations, it turns out that we never need to consider raising our filtration so much that the length of (I, ϵ), $\ell(I, \epsilon)$, is raised by more than 3. We don't prove that here, that just comes out of the computations. What we do here is show how to compute when you keep the increase in length to less than or equal to 3.

Our differentials only act on the $\hat{u}$ part of $w_{I,\epsilon}$. We show that if d_1 acts on more than one $\hat{u}$ at a time, it increases the length by more than 3. The consequence of this is that we only have to take d_1 of one $\hat{u}_k$ at a time in each term of $w_{I,\epsilon}$. That's still a lot to do, but is already a significant simplification.

A leading term of $d_1(w_{I,\epsilon})$ must come from d_1 acting on some distinct permutation, $\hat{p}^J\hat{u}^r$, of the leading term $\hat{p}^I\hat{u}^\epsilon$, and we need only consider d_1 on one of the $\hat{u}$ in $\hat{u}^r$ at a time. To be a distinct permutation other than the leading term, it cannot have **property A**. For d_1 of it to be a leading term, d_1 of it must have a term with **property A**. If it doesn't have such a term with **property A**, then we don't have to concern ourselves with it as it cannot be the leading term of $d_1(w_{I,\epsilon})$.

There are many possible distinct permutations. Taking d_1 of all of them, even using only one $\hat{u}$ at a time, results in a large number of terms. Using the considerations just discussed, we will be able to eliminate from consideration almost all of them. We reduce the relevant permutations and computations to a very few special cases.

That is the goal of this section.

We recall from Lemma 2.11 (our long version of d_1):

$$d_1(\hat{u}) = v_2^{-3}(\hat{v}_1\hat{p} + \hat{v}_1^3\hat{p}^2 + \hat{p}^2) \qquad \text{mod } (2, \hat{p}^2\hat{u})$$

This is our main source of information for computing d_1 because these are all the terms of d_1 we need.

Remark 12.1 ***(Powers of*** v_2*)*. We have already introduced the notation $v_2^{o/e}$. If we apply our above d_1 to a $\hat{u}_k$, we decrease the number of $\hat{u}$ in $\hat{u}^\epsilon$ by one, thus changing the parity of $s(\epsilon)$. On the other hand, the v_2^{-3} changes the parity for $v_2^{o/e}$, so the parity of $v_2^{o/e}w_{I,\epsilon}$ stays aligned as we do differentials. In fact, we can generally ignore the powers of v_2 when working with d_1 because they take care of themselves.

Conventions 12.2. Now that we have established that the $v_2^{o/e}$ that depends on $s(\epsilon)$ takes care of itself, for the part of this section before our important lemmas, we will ignore the powers of v_2. They will be re-introduced when we get to our lemmas.

Definition 12.3 ***(Short version of*** d_1*)*. Following Convention 12.2, the *short version* of d_1 is:

$$d_1(\hat{u}) = \hat{v}_1\hat{p} \qquad \text{mod } (2, \hat{p}\hat{u})$$

This is much of what we need, but it does run into problems that require the long version of the formula. When we apply this to just one $\hat{u}_k$ and one term of the symmetric function, we get

$$d_1(\hat{p}^J\hat{u}^r) = \hat{v}_1\hat{p}^{J+\Delta_k}\hat{u}^{r-\Delta_k}$$

If this element exists and is of lowest filtration for our choice of k, we usually don't have to go further. If there is no $\hat{v}_1$ on such an element, it doesn't mean it is zero as is the case with 2. Instead, it means that the element can be represented in a higher filtration. Since all of our elements start off with a $\hat{v}_1$, if it isn't there, it means that the short version of d_1 has already come along to hit it. That doesn't make it zero, but since the image of d_1 is zero, it means we have:

$$\hat{v}_1\hat{p} = \hat{v}_1^3\hat{p}^2 + \hat{p}^2 \qquad \text{mod } (2, \hat{p}^2\hat{u})$$

Always in such cases, the $\hat{v}_1^3\hat{p}^2$ isn't there as well and so belongs in a higher filtration giving us the relation.

Relation 12.4.

$$\hat{v}_1\hat{p} = \hat{p}^2 \qquad \text{mod } (2, \hat{p}^2\hat{u})$$

This is Equation (7.1), and it was proven there.

This increase in filtration is significant as it involves an increase in the length of (I, ϵ), $\ell(I, \epsilon)$. Note that the short version of d_1 increases length by one and the relation above by another 2. We are fortunate that we never have to go beyond an increase of length 3. Note that in the long version of d_1 above, we only raise length by 3 if we need to use the $\hat{p}^2$ term as well. When this happens, it is always because the terms with $\hat{v}_1$ have proven useless. In these cases we can move on to:

Definition 12.5 ***(Long version of*** d_1***).*** Following Convention 12.2 when the $\hat{v}_1$ term proves useless, the *long version* of d_1 is:

$$d_1(\hat{u}) = \hat{p}^2 \qquad \text{mod } (2, \hat{p}^2\hat{u})$$

Our d_1 acts only on the $\hat{u}_k$ because d_1 commutes with the $\hat{p}_i$ and v_2^2, but we show now that if we act on more than one $\hat{u}_k$ at a time, the result is in a high enough length we don't need to worry about it.

If we apply d_1 to two of our $\hat{u}$ at the same time, we get

$$d_1(\hat{p}^J\hat{u}^r) = \hat{v}_1^2\hat{p}^{J+\Delta_i+\Delta_j}\hat{u}^{r-\Delta_i-\Delta_j}$$

This raises length by 2. In our situations, if $\hat{v}_1$ is around, it would be unnecessary to use 2 different $\hat{u}$. We could just use one of $\hat{u}_i$ or $\hat{u}_j$, choosing the lower of i and j to get the lowest filtration element.

We need to consider the case where there is no $\hat{v}_1$ in the associated graded object on

$$\hat{p}^{J+\Delta_i+\Delta_j}\hat{u}^{r-\Delta_i-\Delta_j}.$$

To get rid of a $\hat{v}_1$ using the formula 12.4, we have to add two more to the length, and, again, we are out of bounds for our work, having increased the length by 4.

Remark 12.6 ***(Only one*** $\hat{u}_k$ ***at a time).*** We will never need to apply d_1 to more than one $\hat{u}_k$ at a time in each of the distinct permutations. This simplifies things dramatically.

We need to identify the leading term of $d_1(w_{I,\epsilon})$ in our spectral sequence for d_1. We will do this inductively by computing the map $d_{1,j}$, which is just our d_1 in our spectral sequence, restricted to $w_{I,\epsilon}$ with $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_{j-1} = 0$ and $\epsilon_j = 1$, that is, our W_j of Section 3.

Since $d_1(w_{I,\epsilon})$ is a symmetric function, the leading term must be a term of $d_1(\hat{p}^J\hat{u}^r)$, where $\hat{p}^J\hat{u}^r$ is a distinct permutation of the leading term for $w_{I,\epsilon}$, i.e. $\hat{p}^I\hat{u}^\epsilon$. If $\hat{p}^J\hat{u}^r$ is anything other than the leading term, it cannot have **property A** in order to be a distinct permutation. However, if it is going to create a leading term for $d_1(w_{I,\epsilon})$, a term of $d_1(\hat{p}^J\hat{u}^r)$ must have **property A**.

There can be many distinct permutations on our leading term to make up a $w_{I,\epsilon}$. The two properties listed above restrict the permutations we need to be concerned with.

Only a few things can happen with our d_1. The first thing that always happens is to take a $\hat{p}_k^{i_k}\hat{u}_k$ to $\hat{v}_1\hat{p}_k^{i_k+1}$. Sometimes this is enough because our associated graded object is free over $\mathbb{Z}/(2)[\hat{v}_1]$ and our choice of k gives the lowest filtration. Often it is not enough because the term with $\hat{v}_1$ is not there in the associated graded object and we need to apply the relation $\hat{v}_1\hat{p}_h^{i_h} = \hat{p}_h^{i_h+1}$ for some h and get

$$\hat{p}^{J+\Delta_h+\Delta_k}\hat{u}^{r-\Delta_k} \tag{12.7}$$

In special cases we have to go straight to the long form of d_1 and take $\hat{p}_k^{i_k}\hat{u}_k$ directly to $\hat{p}_k^{i_k+2}$.

Unfortunately, we cannot write down a general formula that works in all of our cases. Our computations depend too much on the state of the associated graded object at the time of the computation. There are, however, some recurring standard computations that we can discuss. Before we look at these general cases, it is illuminating to look at some small special cases.

We begin with $w_{(1,0),(0,1)} = \hat{p}_1\hat{u}_2 + \hat{u}_1\hat{p}_2$, which has leading term $\hat{p}^{(1,0)}\hat{u}^{(0,1)} = \hat{p}_1\hat{u}_2$. If we take d_1 of this using the short version of d_1, we get

$$d_1(\hat{p}_1\hat{u}_2 + \hat{u}_1\hat{p}_2) = \hat{v}_1(\hat{p}_1\hat{p}_2 + \hat{p}_1\hat{p}_2) = 2\hat{v}_1\hat{p}_1\hat{p}_2 = 0.$$

In cases (and there are many) like this, we call on the long form of d_1 where we have established that we can ignore the $\hat{v}_1$'s. Now we get

$$d_1(\hat{p}_1\hat{u}_2 + \hat{u}_1\hat{p}_2) = \hat{p}_1\hat{p}_2^2 + \hat{p}_1^2\hat{p}_2 = w_{(2,1),(0,0)}.$$

Our leading term for this is $\hat{p}_1^2\hat{p}_2$, and this is a case where the leading term of $d_1(w_{I,\epsilon})$ does not come from d_1 on the leading term of $w_{I,\epsilon}$, something that would make our lives much easier.

Stepping up to the similar situation for $n = 3$, consider

$$w_{(1,1,0),(0,0,1)} = \hat{p}_1\hat{p}_2\hat{u}_3 + \hat{p}_1\hat{u}_2\hat{p}_3 + \hat{u}_1\hat{p}_2\hat{p}_3$$

This time, applying the short version of d_1 gives us

$$\hat{v}_1\hat{p}_1\hat{p}_2\hat{p}_3 + \hat{v}_1\hat{p}_1\hat{p}_2\hat{p}_3 + \hat{v}_1\hat{p}_1\hat{p}_2\hat{p}_3 = 3\hat{v}_1\hat{p}_1\hat{p}_2\hat{p}_3$$

We have two possibilities at this point. If the associated graded object is free over $\mathbb{Z}/(2)[\hat{v}_1]$, we are done. If $\hat{v}_1$ is zero on the associated graded object, we could, in principle, get $w_{I,\epsilon}$ with leading term $\hat{p}_1^2\hat{p}_2\hat{p}_3$. In fact, in the $n = 3$ case this doesn't happen (as we shall see) but it still illustrates a point because related things like this do happen when $n > 3$. The same is true about the next example as well.

Consider

$$w_{(2,2,0),(0,0,1)} = \hat{p}_1^2\hat{p}_2^2\hat{u}_3 + \hat{p}_1^2\hat{u}_2\hat{p}_3^2 + \hat{u}_1\hat{p}_2^2\hat{p}_3^2$$

Start by using the short version of d_1 to get

$$\hat{v}_1\hat{p}_1^2\hat{p}_2^2\hat{p}_3 + \hat{v}_1\hat{p}_1^2\hat{p}_2\hat{p}_3^2 + \hat{v}_1\hat{p}_1\hat{p}_2^2\hat{p}_3^2 = \hat{v}_1 w_{(2,2,1),(0,0,0)}$$

If this is an element, we are done. If $\hat{v}_1 = 0$ here, we have to apply Relation 12.4. The obvious choice gives us $w_{(3,2,1),(0,0,0)}$, but if this is not an element in our associated graded object, we would have to apply Relation 12.4 to the $i_3 = 1$ term giving us $3w_{(2,2,2),(0,0,0)}$.

It is worth keeping these simple examples in mind as we try to look at some general cases.

We are now going to prove some highly technical lemmas that will help us get through our rough computations later. Each of our $E_{1,j}$ comes in two parts, a $\mathbb{Z}/(2)[\hat{v}_1]$ free part and a part where $\hat{v}_1$ is zero on the associated graded object. Dealing with the $\mathbb{Z}/(2)[\hat{v}_1]$ free part is fairly easy, so we start with it. We don't have to know much right now about $E_{1,j}$, except that the elements $w_{I,\epsilon}$ all have $\epsilon_k = 0$ for $k < j$ and we are only interested in computing $d_{1,j}$ on the elements with $\epsilon_j = 1$.

As we will use the following lemmas in our main computation, we abandon the use of the Convention 12.2.

Lemma 12.8 ***(The*** $\hat{v}_1$ ***free part).*** *Given* $w_{I,\epsilon} \in E_{1,j}$ *with* $\epsilon_j = 1$ *in the* $\mathbb{Z}/(2)[\hat{v}_1]$ *free part of* $E_{1,j}$ *for* $MU(n)$ *such that either*

$$i_{j-s} = i_{j-s+1} = \cdots = i_{j-2} = i_{j-1} = i_j + 1$$

with s *maximal and even or* $i_{j-1} > i_j + 1$ *(the equivalent of* $s = 0$*). Then*

$$d_{1,j}(v_2^{o/e} w_{I,\epsilon}) = v_2^{-3} v_2^{o/e} \hat{v}_1 w_{I+\Delta_j,\epsilon-\Delta_j}$$

Proof. First note that $(I + \Delta_j, \epsilon - \Delta_j)$ has **property A** because all we changed was i_j and it was raised by 1 to be less than or equal to i_{j-1}.

Second, we want to show how to get such a term, and then we will show that no other term with **property A** has a lower filtration.

We start with the $i_{j-1} = i_j + 1$ option. We can consider all of the permutations where all we have done is moved $\hat{p}_j^{i_j} \hat{u}_j$ to the left in the place of $\hat{p}_{j-k}^{i_{j-k}}$ for k from 1 to s (there is no $\hat{u}_{j-k}$ by **property A** and the description of $E_{1,j}$). When we apply our short d_1 to each of these terms, with our $\hat{u}_j$ in the $j-k$ place, we have $(s + 1)$ terms all the same as our desired result. Since s is even, we have our required term.

If $i_{j-1} > i_j + 1$, we can just apply the short d_1 to $\hat{u}_j$ to get the required term. Note here that if we try to shift the $\hat{u}_j$ term to the left, we get a term without **property A**, such that when we apply the short d_1 to it, it still does not have **property A**. This is really just the $s = 0$ version of the lemma.

Now we have to show that we cannot achieve a lower filtration element using any other $\hat{u}_k$ and/or permutation.

We pick a $\hat{p}_k^{j_k} \hat{u}_k$ in some permutation, $\hat{p}^J \hat{u}^r$ of $\hat{p}^I \hat{u}^\epsilon$ to apply our short d_1 to. If we remove $\hat{p}_k^{j_k} \hat{u}_k$ from $\hat{p}^J \hat{u}^r$, we must have **property A**. If not, we cannot get **property A** when we apply d_1 to $\hat{u}_k$. And so, what remains, must be a subsequence of $\hat{p}^I \hat{u}^\epsilon$ with just one term missing, $\hat{p}_h^{i_h} \hat{u}_h$. The permutation is to just move $\hat{p}_h^{i_h} \hat{u}_h$ to $\hat{p}_k^{j_k} \hat{u}_k$ leaving all other terms fixed. By this we mean that $i_h = j_k$. If $h < k$, we have moved $\hat{p}_h^{i_h} \hat{u}_h$ to the right. For this to be a distinct permutation, we must have $2i_h + 1 > 2i_k + \epsilon_k$. It is because of this term that this distinct permutation has a higher filtration than the leading term. Since we are going to then replace $\hat{u}_k$ with $\hat{v}_1 \hat{p}_k$, we are going to increase the filtration even further. Since this situation can only happen when $j \le h < k$, $(\epsilon_h = 0,\ h < j)$, this is of a higher filtration than the element we have already discussed.

We have shown that, in this case, the only relevant permutation consist of sliding some $\hat{p}_k^{j_k} \hat{u}_k$ to the left, because we have shown that going to the right results in higher filtration elements.

Our first computation involves $\hat{u}_j$ and permutations that involve sliding it to the left, so all we have to do now is eliminate sliding $\hat{u}_k$ to the left when $j < k$. To get a distinct permutation, we must have $2i_{k-1} + \epsilon_{k-1} > 2i_k + 1 (= \epsilon_k)$. We must slide the term in the k-th place passed the one in the $(k-1)$-st place and then apply the d_1 to the moved $\hat{u}_k$. That gives us the same length, but the increase in the k-th place by this permutation gives it a higher filtration than the term we have already obtained. □

Remark 12.9 ***(Limits on permutations).*** The above lemma took care of all of the $\mathbb{Z}/(2)[\hat{v}_1]$ issues we will come up against. The differential $d_{1,j}$ on the part of $E_{1,j}$ with $\epsilon_j = 1$ and $\hat{v}_1$ equal to zero on it always raises the length of (I, ϵ) by 3 either because we use the long version of d_1 or the short version followed by the Relation 12.4. To compare filtrations, we have to use the criteria for the order on the (I, ϵ) other than the length.

We want to limit the types of permutations we need to consider. We only look at the two step process where we use the short d_1 and then the relation. The proof of the case using the long d_1 is similar to the previous lemma.

The first assumption we make is that we can find a non-zero $w_{I+\Delta_h+\Delta_j,\epsilon-\Delta_j}$ term in $d_1(w_{I,\epsilon})$ with $h < j$ with **property A**. We will have to do this with computations in our lemmas, but we just assume it here.

Now we want to eliminate all but a few permutations from our consideration.

Consider some permutation, $\hat{p}^J\hat{u}^r$, of our leading term, $\hat{p}^I\hat{u}^\epsilon$. We plan on applying the short d_1 to some $\hat{u}_k$ and then using Relation 12.4 on some $\hat{p}_h^{j_h}$. If we remove these two terms from $\hat{p}^J\hat{u}^r$, what remains must have **property A**, and so is a subsequence of $\hat{p}^I\hat{u}^\epsilon$. Consequently, we can describe our permutation of $\hat{p}^I\hat{u}^\epsilon$ to $\hat{p}^J\hat{u}^r$ as just moving two terms around, namely some $\hat{p}_{k'}^{i_{k'}}\hat{u}_{k'}$ moving to $\hat{p}_k^{j_k}\hat{u}_k$ with $i_{k'} = j_k$ and some $\hat{p}_{h'}^{i_{h'}}\hat{u}_{h'}^{\epsilon_{h'}}$ moving to $\hat{p}_h^{j_h}\hat{u}_h^{r_h}$ with $i_{h'} = j_h$ and $\epsilon_{h'} = r_h$. All our permutation does is slide these two terms around, either to the left or right in $\hat{p}^I\hat{u}^\epsilon$.

Because we have assumed the existence of a certain type of element in $d_1(w_{I,\epsilon})$, we can see immediately that any change to the right of the $\hat{u}_j$ place, either due to d_1 or the permutation, will result in a higher filtration term, much as in the previous lemma.

Since we can't mess with things to the right of $\hat{u}_j$, we must have $k' = j$. The only permutation that $\hat{p}_j^{i_j}\hat{u}_j$ can be involved with is a shift to the left. Likewise, the $\hat{p}_{h'}^{i_{h'}}\hat{u}_{h'}^{\epsilon_{h'}}$ term above cannot be to the right of the $\hat{u}_j$ term, but must be to the left. That means that $\epsilon_{h'} = r_h = 0$. If we try to shift our $\hat{p}_{h'}^{i_{h'}}$ to the right, we automatically end up with something of higher filtration again, so this term too must shift only to the left it at all.

There are limitations when shifting to the left as well. If we try to shift $\hat{p}_j^{i_j}\hat{u}_j$ to the left, we can only go passed terms with $i_k = i_j + 1$. Otherwise, when we change $\hat{p}_j^{i_j}\hat{u}_j$ to $\hat{p}_j^{i_j+1}$ we would not have **property A**. Similarly, if we try to shift $\hat{p}_h^{i_h}$ to the left, it can only go passed terms with $i_{h'} = i_h + 1$ or we will not have **property A** when we apply Relation 12.4.

Lemma 12.10. *Given $w_{I,\epsilon} \in E_{1,j}$ with $\epsilon_j = 1$ in the part of $E_{1,j}$ of $MU(n)$ that has $\hat{v}_1 = 0$ on it such that*

$$i_{j-s} = i_{j-s+1} = \cdots = i_{j-2} = i_{j-1} = i_j + 1$$

with s maximal and odd and

$$i_{j-s-t} = i_{j-s-t+1} = \cdots = i_{j-s-2} = i_{j-s-1} = i_{j-s} + 1$$

with t maximal and even. Then

$$d_{1,j}(v_2^{o/e} w_{I,\epsilon}) = v_2^{-3} v_2^{o/e} w_{I+\Delta_{j-s}+\Delta_j,\epsilon-\Delta_j}$$

Proof. First note that $(I + \Delta_{j-s} + \Delta_j, \epsilon - \Delta_j)$ has **property A**. The Δ_j part is for the same reason as in the previous lemma. We also know that $i_{j-s-1} > i_{j-s}$ by definition, so adding the Δ_{j-s} preserves **property A**.

Second, we want to show how to get such a term, and then we will show that no other term of $d_{1,j}(w_{I,\epsilon})$ with **property A** has a lower filtration.

We can consider all of the permutations where all we have done is moved $\hat{p}_j^{i_j}\hat{u}_j$ to the left in the place of $\hat{p}_{j-k}^{i_{j-k}}$ for k from 1 to s (there is no $\hat{u}_{j-k}$). When we apply our short d_1 to each of these terms, with our $\hat{u}_j$ in the $j-k$ place, we have $(s+1)$ terms all the same, but this time, we have an even number of them and so this is zero. So, the $\hat{v}_1$ part of d_1 has proven useless on these terms. Moving on to the long form of d_1, we replace the $\hat{u}_j$ with $\hat{p}_{j-k}$ in each $(j-k)$ place of the various permutations. These terms are all now in different filtrations. The lowest filtration version gives the answer we are looking for.

The above covers the $t = 0$ case, i.e. where $i_{j-s-1} > i_{j-s} + 1$ and deals with the first few possible permutations of the $t > 0$ case, i.e. where $i_{j-s-1} = i_{j-s} + 1$. In this case though, there are other possible permutations. We cannot do anything with i_b where $b < j - s - t$ because we have already used the long d_1 and there is nothing else to do. However, we can shift i_{j-s} to the left from 1 to t times. Then our permutation on the (I, ϵ) of $\hat{p}^I\hat{u}^\epsilon$ looks like

$$(I - \Delta_{j-s-c} + \Delta_{j-s}, \epsilon)$$

for each c from 1 to t. For each such c, we can consider the permutations that just slides $\hat{p}_j^{i_j}\hat{u}_j$ to the left, but we can now only do this $(s-1)$ times, giving us a total of s equal terms. Since s is odd, this gives us

$$v_2^{-3}v_2^{o/e}\hat{v}_1\hat{p}^{I-\Delta_{j-s-c}+\Delta_{j-s}+\Delta_j}\hat{u}^{\epsilon-\Delta_j}$$

To make this has **property A**, we have to apply Relation 12.4 to $\hat{v}_1\hat{p}_{j-s-c}$. Together with the first case that left $\hat{p}_{j-s}^{i_{j-s}}$ where it was, we have $(t+1)$ of these, but since t is even, our final result is the desired

$$v_2^{-3}v_2^{o/e}\hat{p}^{I+\Delta_{j-s}+\Delta_j}\hat{u}^{\epsilon-\Delta_j}.$$

Now we have to show that we cannot achieve a lower filtration element in this situation using any other $\hat{u}_k$ and/or permutation.

Remark 12.9 restricted the permutations we needed to deal with. It forced us to start with $\hat{u}_j$ for d_1 and then deal with $\hat{p}_h$ with $h < j$ with the Relation 12.4 if need be. This is indeed, exactly what we did, so we see that this is the only possibility. □

Lemma 12.11. *We start with $w_{I,\epsilon} \in E_{1,j}$ with $\epsilon_j = 1$ in the part of $E_{1,j}$ for $MU(n)$ that has $\hat{v}_1 = 0$ on it. We assume that*

$$i_{j-s} = i_{j-s+1} = \cdots = i_{j-2} = i_{j-1} = i_j + 1$$

with s maximal and even. We also assume that, for some $k < j-s$, we have

$$i_{k-t} = i_{k-t+1} = \cdots = i_{k-2} = i_{k-1} = i_k + 1$$

with t maximal and even. We further assume that k is the smallest number such that $w_{I+\Delta_k+\Delta_j,\epsilon-\Delta_j}$ is in $E_{1,j}$. Then

$$d_{1,j}(v_2^{o/e}w_{I,\epsilon}) = v_2^{-3}v_2^{o/e}w_{I+\Delta_k+\Delta_j,\epsilon-\Delta_j}$$

Remark 12.12. This seems highly technical, but it covers a lot of territory for us. It even covers more than is obvious. If $s = 0$, that is the same a $i_{j-1} > i_j + 1$ and if $t = 0$, that is the same as $i_{k-1} > i_k + 1$.

Proof. It is easy to see that our term has **property A**. We just need to see that we can obtain it, but by now, that is straightforward. With s even, we know the permutations from Lemma 12.8 that give us the short d_1 on our leading term along with these permutations. Note that as in Remark 12.12, this is even easier if $s = 0$ as there are no relevant permutations. We get

$$v_2^{-3}v_2^{o/e}\hat{v}_1 w_{I+\Delta_j,\epsilon-\Delta_j}$$

Now, using similar permutations and t even, we can apply Relation 12.4 to the $(t+1)$ permutations to get the same term, namely the desired

$$v_2^{-3}v_2^{o/e}w_{I+\Delta_k+\Delta_j,\epsilon-\Delta_j}.$$

We have to rule out one possible glitch. If $i_{j-s}+1 = i_{j-s-1}$, we could try to shift the term in the $(j-s)$ place to the $(j-s-1)$ place or lower, we could have something like what happened in the previous lemma, but we don't. If we do this, the possible shifts on the term in the j-th coordinate are to move it to the left from 1 to $(s-1)$ times. This would give s identical terms when we applied the short d_1, but s is even, so we would have to go to the long d_1. Using the same argument as the previous lemma, that would raise i_{j-s+1} by one, and this would make it automatically have a higher filtration than the term we have already found. □

13. Computing $d_{1,j}$, low j, for $MU(n)$

We recall the definition of **property A**.

$$2i_1+\epsilon_1 \geq \cdots \geq 2i_k+\epsilon_k \geq 2i_{k+1}+\epsilon_{k+1} \geq \ldots \geq 2i_n+\epsilon_n > 0$$

We start the computation of d_1 on $E_{1,1}$ only using the $w_{I,\epsilon}$ with $\epsilon_1 = 1$. We call this map $d_{1,1}$ and the result of this computation, $E_{1,2}$. This is all very similar to the work in Section 6 but we have to contend with the symmetric function now in our computation.

Proposition 13.1. *With $\epsilon_1 = 0$ and* **property A**, *$E_{1,2}$ for $MU(n)$ is:*

$$\mathbb{Z}/(2)[\hat{v}_1]\{v_2^{o/e}v_2^{0,2,4,6}w_{I,\epsilon}\} \quad i_1 = i_2$$

and

$$\mathbb{Z}/(2)\{v_2^{o/e}v_2^{0,2,4,6}w_{I,\epsilon}\} \quad i_1 > i_2$$

The x^1-torsion generators detected by $d_{1,1}$ are represented by:

$$\mathbb{Z}_{(2)}[\hat{v}_1]\{\hat{v}_1 v_2^{o/e}v_2^{0,2,4,6}w_{I,\epsilon}\} \quad i_1 > i_2$$

Proof. Recall that we are now working mod (2) and that d_1 commutes with $\hat{p}_i$ and v_2^2, so we can concentrate on $v_2^{o/e}w_{I,\epsilon}$ from $E_{1,1}$ with $\epsilon_1 = 1$.

All we have to do is apply Lemma 12.8 with $s = 0$, giving us:

$$d_{1,1}(v_2^{o/e}w_{I,\epsilon}) = v_2^{-3}v_2^{o/e}\hat{v}_1 w_{I+\Delta_1,\epsilon-\Delta_1}.$$

Note that the first part of $E_{1,2}$ is there because $i_1 = i_2$ with $\epsilon_1 = 0$ (and therefore $\epsilon_2 = 0$), cannot be the target of our differential. The result follows. □

Remark 13.2. If $n = 1$, the above is consistent with the results for $ER(2)^*(\mathbb{CP}^\infty)$ from [6, Theorems 3.1 and 4.1], i.e. the $n = 1$ case, even if, at first glance, they don't look the same. Here, the only $w_{I,\epsilon}$ we have left for E_2 are the $\hat{p}_1^i$, which is the same as $\hat{P}_1^i$.

Our proofs can generously be called tedious. More detail would not make them more user friendly. The die-hard reader who really cares about the details will have to put in serious effort. To begin the induction, it isn't necessary to compute all of the $E_{1,3-5}$, but, speaking from experience, they are invaluable guides to the general inductive case and so we have left them in.

Proposition 13.3. *With $\epsilon_1 = \epsilon_2 = 0$ and* **property A**, *$E_{1,3}$ for $MU(n)$ is:*

$$\mathbb{Z}/(2)[\hat{v}_1]\{v_2^{o/e}v_2^{0,2,4,6}w_{I,\epsilon}\} \quad i_1 = i_2$$

and

$$\mathbb{Z}/(2)\{v_2^{o/e} v_2^{0,2,4,6} w_{I,\epsilon}\} \quad i_1 > i_2 = i_3$$

The x^1-torsion generators detected by $d_{1,2}$ are represented by:

$$\mathbb{Z}_{(2)}\{v_2^{o/e} v_2^{0,2,4,6} w_{I,\epsilon}\} \quad i_1 > i_2 > i_3.$$

Proof. Because $\epsilon_2 = 0$ already on the first part of $E_{1,2}$, we have no $d_{1,2}$ on this part.

For the second part, with $i_1 > i_2$, our proof comes in two stages. First we assume that $i_1 > i_2 + 1$. In this case we just apply Lemma 12.11 with $s = t = 0$ and $k = 1$ to get

$$v_2^{-3} v_2^{o/e} w_{I+\Delta_1+\Delta_2,\epsilon-\Delta_2}.$$

If $i_1 = i_2 + 1$, we use Lemma 12.10 with $s = 1$ to get the same result. This eliminates the $i_1 > i_2$ terms with $\epsilon_2 = 1$ as sources and the $i_1 > i_2$ terms with $\epsilon_2 = 0$ as targets, missing only the $i_2 = i_3$ terms. This concludes the proof. □

Remark 13.4. If $n = 2$, we would be done computing an associated graded version of E_2 for the Bockstein spectral sequence. There appear to be two parts to the answer, but there are no $w_{I,\epsilon}$ with $i_2 = i_3$ because there is no i_3. Consequently, the answer is entirely in the first part, namely

$$\mathbb{Z}/(2)[\hat{v}_1]\{v_2^{0,2,4,6} w_{I,\epsilon}\} \quad i_1 = i_2 \quad \epsilon_1 = \epsilon_2 = 0.$$

These $w_{I,\epsilon}$ are no more than just $\hat{p}_1^i \hat{p}_2^i \in E(2)^*(\wedge^2 \mathbb{CP}^\infty)$, which is the image of $\hat{P}_2^i \in E(2)^*(MU(2))$.

Proposition 13.5. *With $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ and* **property A**, *$E_{1,4}$ for $MU(n)$ is:*

$$\mathbb{Z}/(2)[\hat{v}_1]\{v_2^{o/e} v_2^{0,2,4,6} w_{I,\epsilon}\} \quad i_1 = i_2 \quad i_3 = i_4$$

and

$$\mathbb{Z}/(2)\{v_2^{o/e} v_2^{0,2,4,6} w_{I,\epsilon}\} \quad i_1 > i_2 = i_3$$
$$\mathbb{Z}/(2)\{v_2^{o/e} v_2^{0,2,4,6} w_{I,\epsilon}\} \quad i_1 = i_2 \quad i_3 > i_4$$

The x^1-torsion generators detected by $d_{1,3}$ are represented by:

$$\mathbb{Z}_{(2)}[\hat{v}_1]\{\hat{v}_1 v_2^{o/e} v_2^{0,2,4,6} w_{I,\epsilon}\} \quad i_1 = i_2 \quad i_3 > i_4$$

Proof. This one is fairly easy. For the second part of $E_{1,3}$ we have $i_2 = i_3$, but we also have $\epsilon_2 = 0$, so we must also have $\epsilon_3 = 0$. Therefore there is no $d_{1,3}$ on this second part.

As for the first part, because we want to consider $\epsilon_3 = 1$ with $\epsilon_1 = \epsilon_2 = 0$, we must have $2i_2 \geq 2i_3 + 1 (= \epsilon_3)$, so $i_2 > i_3$. Applying $d_{1,3}$ using Lemma 12.8, we get

$$v_2^{-3} v_2^{o/e} \hat{v}_1 w_{I+\Delta_3,\epsilon-\Delta_3}$$

This leaves our conditions $i_1 = i_2$ and $i_3 = i_4$ on the first part (because they are missed), and the quotient of $d_{1,3}$ on the first part gives us the $i_1 = i_2$, $i_3 > i_4$ of the second part. □

Remark 13.6. If $n = 3$, we are done. Because in the first part, $i_3 = i_4$ and there is no i_4, there is no contribution to the answer from this first part.

For the second part, we can always write our answer in terms of:

$$\hat{c}_1^{2(i_1-i_2)}\hat{c}_2^{2(i_2-i_3)}\hat{c}_3^{2i_3} \quad i_3 > 0.$$

We have conditions on i_j. In the first case with $i_1 > i_2 = i_3$, we get

$$\hat{P}_1^i\hat{P}_3^j \quad i, j > 0$$

In the second case we have $i_1 = i_2 \geq i_3$. This gives us

$$\hat{P}_2^i\hat{P}_3^j \quad i \geq 0 \quad j > 0$$

This last example can be used to ground our induction.

Proposition 13.7. *With $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0$ and* **property A**, *$E_{1,5}$ for $MU(n)$ is:*

$$\mathbb{Z}/(2)[\hat{v}_1]\{v_2^{o/e}v_2^{0,2,4,6}w_{I,\epsilon}\} \quad i_1 = i_2 \quad i_3 = i_4$$

and

$$\mathbb{Z}/(2)\{v_2^{o/e}v_2^{0,2,4,6}w_{I,\epsilon}\} \quad i_1 > i_2 = i_3 \quad i_4 = i_5$$
$$\mathbb{Z}/(2)\{v_2^{o/e}v_2^{0,2,4,6}w_{I,\epsilon}\} \quad i_1 = i_2 \quad i_3 > i_4 = i_5$$

The x^1-torsion generators detected by $d_{1,4}$ are represented by:

$$\mathbb{Z}/(2)\{v_2^{o/e}v_2^{0,2,4,6}w_{I,\epsilon}\} \quad i_1 > i_2 = i_3 \quad i_4 > i_5$$
$$\mathbb{Z}/(2)\{v_2^{o/e}v_2^{0,2,4,6}w_{I,\epsilon}\} \quad i_1 = i_2 \quad i_3 > i_4 > i_5$$

Proof. The easy part is the first part, we must have $\epsilon_4 = 0$, so there is no differential. On the rest, there are many cases to consider. Note that after we apply d_1 to $\hat{u}_4$, we can never hit $i_4 = i_5$ (because of **property A**), so we will have that condition in the end.

We first look at the $i_1 > i_2 = i_3$ part of $E_{1,4}$. By **property A**, we also have $i_3 > i_4$. If $i_4 + 1 = i_3$ we use Lemma 12.11 with $s = 2$, $t = 0$, and $k = 1$, to get

$$v_2^{-3}v_2^{o/e}w_{I+\Delta_1+\Delta_4,\epsilon-\Delta_4}.$$

If $i_4 + 1 < i_3$, we use Lemma 12.11 with $s = t = 0$ and $k = 1$ to get the same result. It wasn't really necessary to break this into two pieces since Lemma 12.11 handled both.

This gives us everything in the first part of our non-$\hat{v}_1$ part of $E_{1,5}$ except when $i_1 = i_2 + 1$. We already had $i_1 > i_2$ and we added 1 to i_1. We can fix this by looking at the second part when we have $i_1 = i_2 = i_3$. We know $i_4 < i_3$. If $i_4 + 1 = i_3$, we use Lemma 12.10 with $s = 3$ and $t = 0$. If $i_4 + 1 < i_3$, we use Lemma 12.11 with $s = t = 0$ and $k = 1$. This now gives us our $i_1 = i_2 + 1$ case.

It is time to take stock of where we are. We have acquired all of the first part of our answer and used up the $i_1 = i_2 = i_3 > i_4$ part of the second part of $E_{1,4}$ as sources.

We still need to hit, as targets, all of the $w_{I,\epsilon}$ with $i_1 = i_2 \geq i_3 > i_4 > i_5$ when $\epsilon_4 = 0$. The $i_4 > i_5$ always takes care of itself.

For sources, we need to use the $i_1 = i_2 > i_3 > i_4$ with $\epsilon_4 = 1$. It will complete the proof if we can show that for these source (I, ϵ), we have:

$$d_{1,4}(v_2^{o/e}w_{I,\epsilon}) = v_2^{-3}v_2^{o/e}w_{I+\Delta_3+\Delta_4,\epsilon-\Delta_4}.$$

We cannot replace the Δ_3 with Δ_1 because our element would not be in $E_{1,4}$. If we try to replace it with Δ_2, the term does not have **property A**. If $i_3 > i_4 + 1$, we just apply Lemma 12.11 with $k = 3$, $s = 0$ and $t = 0$ unless $i_2 = i_3 + 1$, in which case we use $t = 2$. If $i_3 = i_4 + 1$, we use Lemma 12.10 with $s = 1$ and $t = 0$ unless $i_2 = i_3 + 1$, in which case we use $t = 2$. □

Remark 13.8. If $n = 4$, we are done. Because in the second part, $i_4 = i_5$ and there is no i_5, there is no contribution to the answer from this second part.

For the first part, our leading term for $w_{I,\epsilon}$ is just $\hat{p}_1^i\hat{p}_2^i\hat{p}_3^j\hat{p}_4^j$ with $i \geq j > 0$. This is the image of $\hat{P}_2^{(i-j)}\hat{P}_4^j$.

14. Computing $E_{1,j+1}$ for $MU(n)$

We recall the definition of **property A**.

$$2i_1 + \epsilon_1 \geq \cdots \geq 2i_k + \epsilon_k \geq 2i_{k+1} + \epsilon_{k+1} \geq \ldots \geq 2i_n + \epsilon_n > 0$$

We are using an auxiliary spectral sequence that comes from the filtration defined by the ordering on the (I,ϵ) to compute the d_1 for the Bockstein spectral sequence. Following our description of the process in Section 3, we compute our spectral sequence for d_1 by induction on j using the $w_{I,\epsilon}$ with $\epsilon_k = 0$ for $k < j$ and $\epsilon_j = 1$, i.e., the W_j of Section 3. We call this map $d_{1,j}$ and it is defined on $E_{1,j}$ and the result gives us $E_{1,j+1}$. As in Section 3, the map $d_{1,j}$ is injective on W_j so we are left with $\epsilon_j = 0$ in $E_{1,j+1}$. When we have done $d_{1,n}$ and computed $E_{1,n+1}$ (as a quotient of W_{n+1}), we will be done, giving an associated graded version of the E_2 of the Bockstein spectral sequence. Since at this stage all $\epsilon_k = 0$, $s(\epsilon) = 0$ and is even, making $v_2^{o/e} = 1$.

Theorem 14.1. *For the spectral sequence for the calculation of E_2 for the Bockstein spectral sequence from $E(2)^*(MU(n))$ to $ER(2)^*(MU(n))$, we always have* **property A**. *For $E_{1,j+1}$, $1 \leq j \leq n$, we have $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_j = 0$. There are two parts to $E_{1,j+1}$. First:*

$$\mathbb{Z}/(2)[\hat{v}_1]\{v_2^{o/e}v_2^{0,2,4,6}w_{I,\epsilon}\} \quad \textit{with} \quad i_{2b-1} = i_{2b} \quad 0 < 2b \leq j+1$$

Second, for b with $0 < 2b + 2 \leq j + 1$, let :

$$i_{2c-1} = i_{2c} \quad 0 < 2c \leq 2b, \quad i_{2b+1} > i_{2b+2}, \quad i_{2a} = i_{2a+1} \quad 2b < 2a < j+1$$

Then we have:

$$\mathbb{Z}/(2)\{v_2^{o/e}v_2^{0,2,4,6}w_{I,\epsilon}\}$$

When $j = 2q + 1$, the x^1-torsion detected by $d_{1,j}$ is represented by:

$$\mathbb{Z}/(2)[\hat{v}_1]\{\hat{v}_1 v_2^{o/e}v_2^{0,2,4,6}w_{I,\epsilon}\} \quad i_{2b-1} = i_{2b} \quad 0 < b \leq q \quad i_j > i_{j+1}$$

When $j = 2q$, the x^1-torsion detected by $d_{1,j}$ is the same as the second part of $E_{1,j+1}$ but with $i_j > i_{j+1}$.

Remark 14.2. It is easy enough to read off the terms in the theorem that are in degrees $8*$. It requires $s(\epsilon)$ to be even, forcing $v_2^{o/e} = 1$. Then just eliminate the $v_2^{2,6}$ as well. To get just terms in degrees $16*$, also eliminate v_2^4. All x^i-torsion generators inject to $E(2)^*(-)$, so we see that the x^1-torsion generators of degree $8*$ inject, giving part of Theorem 1.4.

Remark 14.3. When $j = n = 2q + 1$, the condition on the $\mathbb{Z}/(2)[\hat{v}_1]$ free part has $i_n = i_{n+1}$, but since there is no i_{n+1}, this condition is never met and there is no $\mathbb{Z}/(2)[\hat{v}_1]$ free part. When $j = n = 2q$, the condition on the part with $\hat{v}_1 = 0$ has $i_n = i_{n+1}$, but since there is no i_{n+1}, this condition is never met and there is no part with $\hat{v}_1 = 0$.

Proof. Our proof is by induction. We assume we have computed $d_{1,j'}$ for $j' < j$. We need to compute $d_{1,j}$ on $E_{1,j}$ and show our result gives $E_{1,j+1}$. We have computed $E_{1,2}$ through $E_{1,5}$ to begin our induction. In fact, we need the $d_{1,4}$ to ground our induction.

There are some, but not enough, easy parts to this. First, if $j = 2q$, $d_{1,j} = 0$ on the first part because we have $i_{j-1} = i_j$ and so $\epsilon_j = 0$. Likewise, if $j = 2q + 1$, $d_{1,j} = 0$ on the second part because we have $i_{j-1} = i_j$ and so $\epsilon_j = 0$.

When $j = 2q + 1$, computing $d_{1,j}$ on the first part is just Lemma 12.8. This misses the usual $i_j = i_{j+1}$, but, because the $\hat{v}_1$ is there, this creates the $b = q$ part of $E_{1,j+1}$ in the second part, the only piece of the second part that wasn't there already in $E_{1,j}$. The rest of $E_{1,j}$ remains unchanged and carries over to $E_{1,j+1}$.

What remains now is to deal with $j = 2q$. The $\mathbb{Z}/(2)[\hat{v}_1]$ free part of $E_{1,j}$ is uninvolved and carries over to be exactly the same for the first part of $E_{1,j+1}$.

In the second part of $E_{1,j}$, the range of b does not change between $E_{1,j}$ and $E_{1,j+1}$. However, the change does allow for a to be q, giving $i_j = i_{j+1}$. We expect this and can now forget about it. To compute d_1 on $\hat{u}_j$, we can never end up with $i_j = i_{j+1}$, which explains how this condition comes about. Otherwise, the descriptions of $E_{1,j}$ and $E_{1,j+1}$ are the same except, of course, we end up with $\epsilon_j = 0$.

Let's take a look at what we have to accomplish yet. We have to compute $d_{1,j}$ in such a way that all the $\hat{u}_j$ go away. Our map $d_{1,j}$ has to take the second part of $E_{1,j}$ with $\epsilon_j = 1$ and $i_{2q} \geq i_{2q+1}$ (sources) and put it in 1-1 correspondence with the second part of $E_{1,j}$, with $i_{2q} > i_{2q+1}$ (targets) and $\epsilon_j = 0$. Recall that our $2q = j$.

First let us work with the $b = 0$ case. We want all $b = 0$ terms with $\epsilon_j = 0$ and $i_j > i_{j+1}$ to be hit as targets. We need to find the sources to do this with. Our sources must have $\epsilon_j = \epsilon_{2q} = 1$, so we have $i_{2q-1} > i_{2q}$ by **property A** and the fact that $\epsilon_{2q-1} = 0$. We first restrict our attention to source terms with $b = 0$.

We use Lemma 12.11 to get

$$d_{1,j}(w_{I,\epsilon}) = w_{I+\Delta_1+\Delta_j,\epsilon-\Delta_j}.$$

In this application, the t of Lemma 12.11 is zero and $k = 1$, but the s could range from 0 to $j-2 = 2q-2$ (by twos) depending on I. This hits all elements in $E_{1,j}$ we need to have as targets with $b = 0$ and $i_1 > i_2 + 1$.

As targets, we have not yet hit the $b = 0$ terms with $i_1 = i_2 + 1$, i.e. (I, ϵ) with $\epsilon_j = 0$, $i_j > i_{j+1}$ and $i_1 = i_2 + 1$. The source that works here is $(J, r) = (I - \Delta_1 - \Delta_j, \epsilon + \Delta_j)$. To see this, recall that for $b = 0$, we have $i_{2a} = i_{2a+1}$ for $0 < 2a < 2q$. Find the $q > b' > 0$ such that

$$i_1 - 1 = i_2 = \cdots = i_{2b'+1} > i_{2b'+2}$$

In almost all cases, we can apply Lemma 12.11 to (J, r) to get the desired result using $k = 1$, $t = 0$, and s can go from 0 to $2q - 2b' - 2$ by twos, depending on I.

There is one place where Lemma 12.11 does not apply and we must use Lemma 12.10. That is when $b' = q - 1$ and $i_{2q-1} = i_{2q} + 1$. Here $s = 2q - 1$ and $t = 0$.

Note that this turns a term associated with $b' > 0$ into one with $b = 0$.

For targets, we have hit all of our $b = 0$, $\epsilon_j = 0$, $i_j > i_{j+1}$. For sources, we have used all of b with $i_1 = \cdots = i_{2b+1} > i_{2b+2}$ and $\epsilon_j = 1$, $i_j \geq i_{j+1}$ for $b = 0$ to $q - 1$. Note that this includes all of the $b = 0$, $\epsilon_j = 1$, $i_j \geq i_{j+1}$ terms.

Summary 14.4. The unused terms we need as sources are all of the $q > b \geq 1$, $\epsilon_j = 1$, with $i_j \geq i_{j+1}$, excluding terms with

$$i_1 = i_2 = \cdots = i_{2b} = i_{2b+1} > i_{2b+2}$$

The unused terms we need as targets are $b \geq 1$, $\epsilon_j = 0$, with $i_j > i_{j+1}$.

We must now do $b > 0$.

Moving on, we want to find all of the $b = 1$ terms as targets. We do much that is similar to the $b = 0$ case. We begin with source terms that also have $b = 1$. When $b = 1$, we have $i_3 > i_4$, and since we have excluded $i_1 = i_2 = i_3 > i_4$, we always have $i_1 = i_2 > i_3 > i_4$. Clarity is often thwarted by the necessity to handle special cases. We want to apply our lemmas to get

$$d_{1,j}(w_{I,\epsilon}) = w_{I+\Delta_3+\Delta_j,\epsilon-\Delta_j}.$$

We see that this has **property A** because $i_2 > i_3$ and $i_{j-1} > i_j$. We cannot replace Δ_3 with Δ_1 because that term does not exist in $E_{1,j}$. We cannot replace it with Δ_2 because that term does not have **property A**.

Generally, we can do this using Lemma 12.11 when we are not dealing with the special cases. In our use we have $t = 0$ or $t = 2$ (if $i_2 = i_3 + 1$), $k = 3$, and s can be anywhere from 0 to $2q - 4$ (by twos).

In the special case of source with $j = 4$ and $i_1 = i_2 > i_3 = i_4 + 1$ and $\epsilon_4 = 1$, we have to use Lemma 12.10 with $s = 1$, $k = 3$, and $t = 0$ unless $i_2 = i_3 + 1$, in which case $t = 2$.

We had $i_3 > i_4$ and we added Δ_3 so we missed the cases where $i_3 = i_4 + 1$. We are left with the need to hit these cases. Again, this is just like the $b = 0$ case. As targets, we have not yet hit the $b = 1$ terms (I, ϵ) with $\epsilon_j = 0$, $i_j > i_{j+1}$ and $i_3 = i_4 + 1$. The source that works here is $(J, r) = (I - \Delta_3 - \Delta_j, \epsilon + \Delta_j)$. To see this, recall that for $b = 1$, we have $i_{2a} = i_{2a+1}$ for $2 < 2a < 2q$. Find the $q > b' > 0$ such that

$$i_3 - 1 = i_4 = \cdots = i_{2b'+1} > i_{2b'+2}$$

In almost all cases, we can apply Lemma 12.11 to get the desired result. using $k = 3$, $t = 0$ or $t = 2$ (if $i_2 = i_3 + 1$), and s can go from 0 to $2q - 2b' - 2$ by twos, depending on I.

Of course, if $2b' + 1 = 2q - 1$ AND $i_{2q-1} = i_{2q} + 1$, then we have to use Lemma 12.10. Here we have $s = 2q - 3$, $t = 0$ or $t = 2$ (if $i_2 = i_3 + 1$).

We need to identify all of the targets hit so far and all of the sources used so far.

We have hit all elements as targets with $b = 0$ or $b = 1$, $\epsilon_j = 0$ and $i_j > i_{j+1}$.

We have used all terms as sources with $b = 0$ and $b = 1$ with $\epsilon_j = 1$ and $i_j \geq i_{j+1}$. In addition, we have used all terms with $i_1 = \cdots = i_{2b'+1} > i_{2b'+2}$ for $b' > 0$ and all terms with $i_1 = i_2 > i_3 = \cdots = i_{2b'+1} > i_{2b'+2}$ for $b' > 1$. Combined, that is $i_1 = i_2 \geq i_3 = \cdots = i_{2b'+1} > i_{2b'+1}$.

Summary 14.5. The unused terms we need as sources are all of the $q > b \geq 2$, $\epsilon_j = 1$, $i_j \geq i_{j+1}$, excluding terms with

$$i_1 = i_2 \geq i_3 = i_4 = \cdots = i_{2b} = i_{2b+1} > i_{2b+2}$$

The unused terms we need as targets are $b \geq 2$, $\epsilon_j = 0$, with $i_j > i_{j+1}$.

We are getting close to our induction statement where we will set things up to do $d_{1,j}$ for $b \geq 2$ using the induction.

Our $d_{1,j}$ on what is left cannot involve i_1 or i_2 because $(I + \Delta_1 + \Delta_j, \epsilon - \Delta_j)$ does not give a term in $E_{1,j}$ and $(I + \Delta_2 + \Delta_j, \epsilon - \Delta_j)$ does not have **property A**.

Thus, we can ignore i_1 and i_2. What is left of (I, ϵ) if we remove them is an I' of length $n-2$. More importantly, $i_j = i_{2q}$ moves down to the new i'_{2q-2} and the $b \geq 2$ condition moves to a $b' \geq 1$ condition.

This translates our $b \geq 2$, n, $j = 2q$ problem, Summary 14.5, to our $b' \geq 1$, $n-2$, $j-2 = 2q-2$ problem, Summary 14.4. They are identical, so, by induction, having already solved the later problem, we solve the present problem.

Because of the idiosyncrasies of the $b = 0$ case, we couldn't just go from $b \geq 1$ to $b' \geq 0$, but had to do the induction from $b \geq 2$ to $b' \geq 1$.

Because we must use $b = 2$ and we have $2b+2 \leq j+1$ and we must have $j = 2q$, our lowest computation here is for $E_{1,7}$, so, to use induction, we needed to have computed our $E_{1,5}$, which we did in the previous section. □

Remark 14.6. Rather than the downward induction we have done, we could equally well have done an induction on b. All that would be necessary would be to replace the 2 in 14.5 with a k and do the induction on k. The statement of the excluded terms would be a bit more complicated and showing that the lower i_t aren't involved would also be a bit more complicated. But, on the whole, the argument would be roughly equivalent.

15. All the $MU(n)$ theorems

Proofs of Theorems 1.2 and 1.3. We begin with $n = 2q$. In Theorem 14.1, for the part with $\hat{v}_1 = 0$, we have $i_n = i_{n+1}$, but since there is no i_{n+1}, this cannot happen and there is no contribution to the answer from this second part. We apply Equation (11.4) to the $\mathbb{Z}/(2)[\hat{v}_1]$ free part of Theorem 14.1. Since $s(\epsilon) = 0$, we have $v_2^{o/e} = 1$. We get, modulo higher filtrations,

$$w_{I,\epsilon} = \hat{c}_1^{2i_1-2i_2}\hat{c}_2^{2i_2-2i_3}\cdots\hat{c}_n^{2i_n} = \hat{P}_1^{i_1-i_2}\hat{P}_2^{i_2-i_3}\cdots\hat{P}_n^{i_n}$$

We have $i_{2b-1} = i_{2b}$ for all $0 < b \leq q$, so we end up with

$$\hat{P}_2^{i_2}\hat{P}_4^{i_4}\cdots\hat{P}_{2q}^{i_{2q}}.$$

Of course, **property A** requires that $i_{2q} > 0$. This gives us the E_2 of Theorem 1.2.

Moving on to d_3, because there is no $\hat{u}^\epsilon$ anymore and all the $\hat{P}_k$ are permanent cycles, all of our $w_{I,\epsilon}$ for E_2 are permanent cycles. Our entire d_3 is given by what happens on the coefficient ring. Using Remark 2.8, $d_3(v_2^2) = \hat{v}_1 v_2^{-4}$, we get the E_4 term and the x^3-torsion generators. The differential d_7 is again all on the coefficients so we have $d_7(v_2^4) = \hat{v}_2 v_2^{-8} = 1$, and we our x^7-torsion generators.

The proof for the $n = 2q+1$ case is a bit different. We can eliminate the $\mathbb{Z}/(2)[\hat{v}_1]$ free part from consideration because it requires $i_n = i_{n+1}$ and there is no i_{n+1}. We also have $v_2^{o/e} = 1$. The reduction to Pontryagin classes is the same idea, but our differential on the coefficients $d_3(v_2^2) = \hat{v}_1 v_2^{-4}$ gives us a $\hat{v}_1$ that we don't have. In our $w_{I,\epsilon}$ we want to apply our usual Relation 12.4, but if we do that, we must be sure that the resulting $w_{I+\Delta_k,0}$ exists. If $i_{2b} > i_{2b+1}$ we can just use $\hat{v}_1\hat{p}_{2b+1}^{i_{2b+1}} = \hat{p}_{2b+1}^{i_{2b+1}+1}$. Anything lower than that does not exist. If, however, $i_{2b} = i_{2b+1}$, we cannot do that but we can use $\hat{v}_1\hat{p}_{2b'+1}^{i_{2b'+1}} = \hat{p}_{2b'+1}^{i_{2b'+1}+1}$ where we have b' is the smallest number with $i_{2b'+1} = \cdots = i_{2b+1}$. This has **property A** and takes an element of type b to one of type b'. This allows us to hit all elements except when $b = q$ and $i_{2q+1} = 1$. This gives us both our x^3-torsion description and our E_4 term of Theorem 1.3. There is no mystery now to d_7 or the x^7-torsion. This is just computed on the coefficients as with $n = 2q$. □

Remark 15.1. All the terms in the theorems that are in degrees $8*$ can be found just by eliminating the $v_2^{2,6}$. To see degrees $16*$, eliminate the v_2^4 as well. All x^3-torsion generators are in degrees $8*$ and the

x^7-torsion generators are in degrees $16*$. Since x^i-generators inject to $E(2)^*(-)$, this concludes the proof of Theorem 1.4.

All that remains is to give a more $MU(n)$ associated description of the x^1-torsion generators. They are all recoverable from Theorem 14.1 where they are written in terms of symmetric functions. Here, we rewrite this in terms of Pontryagin and Chern classes to give it more the look of $MU(n)$. Again, we rely on Equation (11.4). We can just read this off from 14.1.

Recall from Lemma 11.5 that when we write our elements in terms of Chern classes, our $v_2^{o/e}$ is determined by the parity of $j_1 + j_3 + j_5 + \cdots$, for $\hat{c}^J$.

Theorem 15.2. *Representatives for the x^1-torsion generators in the associated graded object for $ER(2)^*(MU(n))$ start with:*

$$\mathbb{Z}_{(2)}[\hat{v}_1][[\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n]]\{2v_2^{o/e}v_2^{0,2,4,6}\hat{c}_n\} \cong \mathbb{Z}_{(2)}[\hat{v}_1][[\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n]]\{v_2^{o/e}\alpha_i\hat{c}_n\} \quad 0 \le i < 4$$

For $1 \le j = 2b+1 \le n$, we have

$$\mathbb{Z}/(2)[\hat{v}_1][[\hat{P}_2^{i_2}, \hat{P}_4^{i_4}, \cdots, \hat{P}_{2b}^{i_{2b}}, \hat{P}_j^{i_j}, \hat{c}_{j+1}^{i_{j+1}}, \hat{c}_{j+2}^{i_{j+2}}, \cdots, \hat{c}_n^{i_n}]]\{v_2^{o/e}v_2^{0,2,4,6}\hat{v}_1\hat{P}_j\hat{c}_n\}$$

except when $j = n$, then we do not need the $\hat{c}_n$ at the end. The parity that determines $v_2^{o/e}$ is the parity of $j_{2b+3} + j_{2b+5} + j_{2b+7} + \cdots$.

For $0 \le 2b < j = 2q \le n$ we get

$$\mathbb{Z}/(2)[\hat{P}_2^{i_2}, \hat{P}_4^{i_4}, \cdots, \hat{P}_{2b}^{i_{2b}}, \hat{P}_{2b+1}^{i_{2b+1}}, \hat{P}_{2b+3}^{i_{2b+3}}, \cdots, \hat{P}_{j-1}^{i_{j-1}}, \hat{P}_j^{i_j}, \hat{c}_{j+1}^{i_{j+1}}, \hat{c}_{j+2}^{i_{j+2}}, \cdots, \hat{c}_n^{i_n}]\{v_2^{o/e}v_2^{0,2,4,6}\hat{P}_{2b+1}\hat{P}_j\hat{c}_n\}$$

except when $j = n$, then we do not need the $\hat{c}_n$ at the end. The parity that determines $v_2^{o/e}$ is the parity of $j_{2q+3} + j_{2q+3} + j_{2q+5} + \cdots$.

Remark 15.3. To get the x^1-torsion generators in degrees $8*$, we have to have $v_2^{o/e} = 1$ and we only use $v_2^{0,4}$. For degrees $16*$, we must have $v_2^{o/e} = 1$ and no powers of v_2.

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