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W. Stephen Wilson

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The Omega spectrum for mod 2 *KO*-theory

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The 8-periodic theory that comes from the *KO*-theory of the mod 2 Moore space is the same as the real first Morava *K*-theory obtained from the homotopy fixed points of the $\mathbb{Z}/(2)$ action on the first Morava *K*-theory. The first Morava *K*-theory, $K(1)$, is just mod 2 *KU*-theory. We compute the homology Hopf algebras for the spaces in this Omega spectrum.

1. Introduction

We have stable maps $2 : S^0 \to S^0$ and $\eta : S^1 \to S^0$ and we get a stable diagram

$$\begin{array}{ccccc}
S^1 & \xrightarrow{2} & S^1 & \longrightarrow & \Sigma^1 M \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\
S^0 & \xrightarrow{2} & S^0 & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow \\
N & \xrightarrow{2} & N & \longrightarrow & NM
\end{array}$$

with M the mod 2 Moore space and N and NM the appropriate cofibers.

If we smash this diagram with connective K-theory, bo, and then only look at the low dimensional spaces in the Omega spectrum where we get periodicity, we get the diagram of fibrations

$$\begin{array}{ccccccc}
\underline{KO}_{i+1} & \xrightarrow{2} & \underline{KO}_{i+1} & \xrightarrow{\rho} & \underline{KR(1)}_{i+1} & \xrightarrow{\delta} & \underline{KO}_{i+2} \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\
\underline{KO}_{i} & \xrightarrow{2} & \underline{KO}_{i} & \xrightarrow{\rho} & \underline{KR(1)}_{i} & \xrightarrow{\delta} & \underline{KO}_{i+1} \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\
\underline{KU}_{i} & \xrightarrow{2} & \underline{KU}_{i} & \xrightarrow{\rho} & \underline{K(1)}_{i} & \xrightarrow{\delta} & \underline{KU}_{i+1} \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\
\underline{KO}_{i+2} & \xrightarrow{2} & \underline{KO}_{i+2} & \xrightarrow{\rho} & \underline{KR(1)}_{i+2} & \xrightarrow{\delta} & \underline{KO}_{i+3}
\end{array} \tag{1.1}$$

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The $\underline{KU}_i$ are the usual 2-periodic spaces for complex K-theory and the $\underline{KO}_i$ the 8-periodic spaces for real K-theory. The $\underline{K(1)}_i$ are 2-periodic and they are just the mod 2 KU-theory, or the first Morava K-theory. The spaces of interest are the $\underline{KR(1)}_i$, which are simultaneously the real version of the first Morava K-theory (see [Hu and Kriz 2001, Theorem 3.32]) and the mod 2 KO-theory.

Our interest is in computing the Hopf algebra $H_*(\underline{KR(1)}_i)$. We work with $\mathbb{Z}/(2)$ coefficients in homology. Our notation is that P is a polynomial algebra, E is an exterior algebra, $TP_4(x)$ is $P(x)/(x^4)$. The Frobenius F is just the map that takes x to x^2. The Verschiebung V is the dual of the Frobenius and gives us the coproduct structure on our Hopf algebras. Our notation is such that the subscript of an element denotes the degree it resides in. Keep in mind that the k used as a subscript for the tensor product in the main theorem is not a field, but an index. This is always the case throughout the paper when the tensor symbol has a subscript.

Our main theorem is easy to state.

Theorem 1.2. *The homology of the connected component of $\underline{KR(1)}_i$ is as follows. If the Verschiebung isn't described, it is zero. The index k runs over all $k > 0$.*

$i=0$	$E(x_k)\otimes_k P(y_{4k+2})$	$V(x_{2k})=x_k$
$i=1$	$P(x_{2k+1})\otimes_k P(y_{4k+2})$	$V(y_{4k+2})=x_{2k+1}$
$i=2$	$P(x_{8k+2})\otimes_k P(y_{4k+3})$	
$i=3$	$E(x_{8k+3})\otimes_k P(y_{8k+4})$	
$i=4$	$E(x_{4k})\otimes_k E(y_{8k+5})$	$V(x_{8k})=x_{4k}$
$i=5$	$E(x_{4k+1})\otimes_k E(y_{2k})$	$V(y_{4k})=y_{2k},\ V(y_{8k+2})=x_{4k+1}$
$i=6$	$\otimes_k TP_4(x_k)$	$V(x_{2k})=x_k$
$i=7$	$E(x_{2k})\otimes_k P(y_{2k+1})$	$V(x_{4k})=x_{2k}$

Remark 1.3. We began this research trying to give meaningful names to all of the algebra generators. Eventually, it became clear that it was easier to compute just using the degrees of the generators. We do know good names for all of the generators of $H_*(\underline{KO}_*)$, $H_*(\underline{KU}_*)$, and $H_*(\underline{K(1)}_*)$, and we are able to relate our poorly named generators to generators we are more familiar with, thus solving the naming problem after the fact. In order to be explicit about these results, we have to write down the known homologies first. We put that off until the next section. We can give the nonexplicit answer here.

Theorem 1.4. *The maps of the connected components $\underline{KO}_i \xrightarrow{\rho} \underline{KR(1)}_i \xrightarrow{\delta} \underline{KO}_{i+1}$ give rise to maps on homology*

$$H_*(\underline{KO}_i) \xrightarrow{\rho_*} H_*(\underline{KR(1)}_i) \xrightarrow{\delta_*} H_*(\underline{KO}_{i+1})$$

that are exact in the category of Hopf algebras at the middle term. For $i = 1, 2, 5$ and 6, this is a short exact sequence of Hopf algebras. In the case $i = 0$ we have a *long exact sequence*

$$H_*(\underline{KO}_1) \xrightarrow{\eta_*} H_*(\underline{KO}_0) \xrightarrow{2_*} H_*(\underline{KO}_0) \xrightarrow{\rho_*} H_*(\underline{KR(1)}_0) \xrightarrow{\delta_*} H_*(\underline{KO}_1) \xrightarrow{\rho_*} H_*(\underline{KU}_1).$$

In the above diagram there are 20 distinct spaces as i varies, KU and $K(1)$ are 2-periodic, and KO and $KR(1)$ are 8-periodic. We know the homology of 12 of them. It is the other 8 associated with $KR(1)$ that we are interested in. Counting the suspension maps, there are 98 maps to evaluate, 48 of them involving the $KR(1)$ spaces. For each map, there is a spectral sequence, and 56 of them involve the $KR(1)$ spaces. It is not necessary to know all of them to get our main results, but it is often helpful. Because I want to have access to this information, it has been written up as a supplement to this paper. Once you know the homology of all the spaces and also know the maps, it is fairly easy to figure out how all the spectral sequences behave. Also, for my personal benefit to have a reference, the long exact sequences of homotopy groups have been put in the appendix as well. In this paper we state, compute, and use, only what we need, but we assume results not involving the $\underline{KR(1)}_i$.

The spaces $\underline{KR(1)}_i$ have been around for a long time. When I tried to find a reference for the homotopy groups, the experts informed me that they were known in the 1960s to Mahowald and that there wasn't a reference because everyone already knew them. What might be new is that the $\underline{KR(1)}_i$ are also the real first Morava $K(1)$-theory. This comes from the work of Hu and Kriz [2001], where they compute the homotopy of all of the real Morava $K(n)$, $KR(n)$. This project got started because I thought $KR(2)$ would be interesting but that I should quickly take a look at $KR(1)$ first. From the point of view of personal satisfaction, the homology, $H_*(\underline{KR(1)}_6)$, was both the most difficult to compute and the most interesting. In the beginning, motivation was easy. I was hoping to find something interesting. After the fact, it isn't clear how to motivate. However, a quick look at the appendix might make this paper look elegant.

In Section 2 we give the homology of the spaces that are known already as well as state the details of Theorem 1.4. In Section 3 we state the spectral sequences we use and discuss how Hopf algebras help us with our computations. After that, each section is just the computation of some $H_*(\underline{KR(1)}_i)$. They are somewhat in order except that to do $H_*(\underline{KR(1)}_6)$, we need to have $H_*(\underline{KR(1)}_7)$ first, which is computed from $H_*(\underline{KR(1)}_0)$.

2. Connecting to known results

Our preferred generators for $H_*(\underline{KU}_*)$ and $H_*(\underline{KO}_*)$ come from Hopf rings. They are given elegant descriptions in [Cowen Morton and Strickland 2002]. In [Kitchloo and Wilson 2007, Section 25], there is an alternative Hopf ring description for

$H_*(\underline{KO}_*)$ and one can read off that for $H_*(\underline{KU}_*)$ from [Ravenel and Wilson 1977]. We do not write down these descriptions in this paper. It is enough to know they have nice Hopf ring names. In the case of $H_*(\underline{KR(1)}_*)$, we do not get Hopf ring names because $KR(1)$ is not a ring spectrum.

We give the descriptions of the homologies we need in this paper.

Theorem 2.1. *The homology of the connected component of $\underline{KU}_i$ is as follows. If the Verschiebung isn't described, it is zero. The index k runs over all $k > 0$.*

$$\begin{array}{lll} i=0 & \bigotimes_k P(x_{2k}) & V(x_{4k}) = x_{2k} \\ i=1 & \bigotimes_k E(x_{2k+1}) & \end{array}$$

Theorem 2.2. *The homology of the connected component of $\underline{KO}_i$ is as follows. If the Verschiebung isn't described, it is zero. The index k runs over all $k > 0$.*

$$\begin{array}{lll} i=0 & \bigotimes_k P(x_k) & V(x_{2k}) = x_k \\ i=1 & \bigotimes_k P(x_{2k+1}) & \\ i=2 & \bigotimes_k P(x_{4k+2}) & \\ i=3 & \bigotimes_k E(x_{4k+3}) & \\ i=4 & \bigotimes_k P(x_{4k}) & V(x_{8k}) = x_{4k} \\ i=5 & \bigotimes_k E(x_{4k+1}) & \\ i=6 & \bigotimes_k E(x_{2k}) & V(x_{4k}) = x_{2k} \\ i=7 & \bigotimes_k E(x_k) & V(x_{2k}) = x_k \end{array}$$

Theorem 2.3. *The homology of the connected component of $\underline{K(1)}_i$ is as follows. If the Verschiebung isn't described, it is zero. The index k runs over all $k > 0$.*

$$\begin{array}{lll} i=0 & TP_4(x_{4k+3}) \otimes_k E(y_{4k}) \otimes_k E(z_{8k+2}) & \begin{array}{c} V(y_{8k}) = y_{4k}, \ V(y_{16k+4}) = z_{8k+2}, \\ V(y_{16k+12}) = (x_{4k+3})^2 \end{array} \\ i=1 & E(x_{4k+1}) \otimes_k P(y_{4k+2}) & V(y_{8k+2}) = x_{4k+1} \end{array}$$

In the paper this is from, [Wilson 1984], we computed $H_*(\underline{K(n)}_*)$ for all n and all primes. Slight adjustments had to be made all along the way for $p = 2$, and it seems that they weren't all made.

In the paper, we write

$$H_*(\underline{K(1)}_0) \simeq E(x_{4k+3}) \otimes_k E(x_{2k}),$$

but we missed the extension $x_{4k+3}^2 = x_{8k+6}$. So, what is in the paper is an associated graded version. When the spectral sequence there is used to compute $H_*(\underline{K(1)}_1)$, deep down in the gruesome depths of the paper there is a d_1, so the resulting answer is correct. Explicitly, what it shows in that paper is that we need (in the notation of the paper) $(e_1 a_{(0)})^2 = b_{(0)} b_{(1)}$. The rest follows from Hopf ring considerations as our generators there all have nice Hopf ring names. Something similar happens for $K(n)$ in that paper, but again, only for $p = 2$.

We can now use these results to connect to our new results.

Theorem 2.4. *The exactness at the middle term of*

$$H_*(\underline{KO}_i) \to H_*(\underline{KR(1)}_i) \to H_*(\underline{KO}_{i+1})$$

of Theorem 1.4 is given explicitly as follows, where, if not described, the element maps to zero. The index j runs over all $j > 0$.

$$i=0 \qquad \bigotimes_j P(y_j) \xrightarrow{y_j \mapsto z_j} E(z_j) \otimes_j P(zz_{4j+2}) \xrightarrow{zz_{4j+2} \mapsto (w_{2j+1})^2} \bigotimes_j P(w_{2j+1})$$

$$i=1 \qquad \bigotimes_j P(y_{2j+1}) \xrightarrow{y_{2j+1} \mapsto z_{2j+1}} P(z_{2j+1}) \otimes_j P(zz_{4j+2}) \xrightarrow{zz_{4j+2} \mapsto w_{4j+2}} \bigotimes_j P(w_{4j+2})$$

$$i=2 \qquad \bigotimes_j P(y_{4j+2}) \xrightarrow[y_{8j+6} \mapsto (zz_{4j+3})^2]{y_{8j+2} \mapsto z_{8j+2}} P(z_{8j+2}) \otimes_j P(zz_{4j+3}) \xrightarrow{zz_{4j+3} \mapsto w_{4j+3}} \bigotimes_j E(w_{4j+3})$$

$$i=3 \qquad \bigotimes_j E(y_{4j+3}) \xrightarrow{y_{8j+3} \mapsto z_{8j+3}} E(z_{8j+3}) \otimes_j P(zz_{8j+4}) \xrightarrow{zz_{8j+4} \mapsto w_{8j+4}} \bigotimes_j P(w_{4j})$$

$$i=4 \qquad \bigotimes_j P(y_{4j}) \xrightarrow{y_{4j} \mapsto z_{4j}} E(z_{4j}) \otimes_j E(zz_{8j+5}) \xrightarrow{zz_{8j+5} \mapsto w_{8j+5}} \bigotimes_j E(w_{4j+1})$$

$$i=5 \qquad \bigotimes_j E(y_{4j+1}) \xrightarrow{y_{4j+1} \mapsto z_{4j+1}} E(z_{4j+1}) \otimes_j E(zz_{2j}) \xrightarrow{zz_{2j} \mapsto w_{2j}} \bigotimes_j E(w_{2j})$$

$$i=6 \qquad \bigotimes_j E(y_{2j}) \xrightarrow{y_{2j} \mapsto (z_j)^2} \bigotimes_j TP_4(z_j) \xrightarrow{z_j \mapsto w_j} \bigotimes_j E(w_j)$$

$$i=7 \qquad \bigotimes_j E(y_j) \xrightarrow{y_{2j} \mapsto z_{2j}} E(z_{2j}) \otimes_j P(zz_{2j+1}) \xrightarrow{zz_{2j+1} \mapsto w_{2j+1}} \bigotimes_j P(w_j)$$

Remark 2.5. The long exact sequence for $i = 0$ of Theorem 1.4 consists of the above maps spliced together with well-understood maps that we will see throughout the paper.

3. Hopf algebras, fibrations, and spectral sequences

We need two spectral sequences. The homology version we use computes the homology of a base space from the homologies of the fiber and the total space. It is in [Moore 1961, Theorems 2.2 and 3.1]. I think of it as the bar spectral sequence, but it should perhaps be called the Moore spectral sequence. Unfortunately, Moore doesn't indulge appropriately with Hopf algebras as he clearly could have. Rothenberg and Steenrod [1965] really bring in the Hopf algebras, but neglected to do the more general case where the total space isn't contractible. Everyone seems to think they can do it by just slightly extending Rothenberg and Steenrod's proof, except those who think it is already in their paper. The cohomology version computes the cohomology of the fiber from the cohomologies of the base space and the total space. This seems to originate with Eilenberg and Moore [1966]. However, my favorite reference here is [Smith 1970] because this is where I learned to compute with Hopf algebras in these spectral sequences.

We state the two spectral sequences for the record and then discuss the use of Hopf algebras in their computations.

Proposition 3.1. *Let $F \to E \to B$ be a fibration of infinite loop spaces and maps.*

(1) *There is a first quadrant homology spectral sequence of Hopf algebras*

$$E^2_{*,*} = \operatorname{Tor}^{H_*(F)}_{*,*}(H_*(E), \mathbb{Z}/(2)) \Rightarrow H_*(B)$$

with $d_r : E_{u,v} \to E_{u-r,v+r-1}$.

(2) *There is a second quadrant cohomology spectral sequence of Hopf algebras*

$$E_2^{*,*} = \operatorname{Tor}^{*,*}_{H^*(B)}(H^*(E), \mathbb{Z}/(2)) \Rightarrow H^*(F)$$

with $d_r : E^{u,v} \to E^{u+r,v-r+1}$

Discussion of Hopf algebras,* Tor*, and differentials. Combining the above spectral sequences with Hopf algebras makes for a powerful tool. We only discuss the homology version but everything carries over to the cohomology version. The general reference for Hopf algebras is [Milnor and Moore 1965], but my computational reference is [Smith 1970].

We work with mod 2 homology throughout. The Borel structure theorem (see [Milnor and Moore 1965]) for our graded Hopf algebras over $\mathbb{Z}/(2)$ is that they are the tensor products of algebras of the form $P(x_i)$ (polynomial), $E(x_i)$ (exterior), and $TP_{2^j}(x_i) = P(x_i)/(x_i^{2^j})$ (truncated polynomial). (Recall our notation is that x_i is of degree i.) Sub-Hopf algebras of polynomial algebras must also be polynomial. In our Hopf algebras, we have $2_* = FV = VF$, where F is the Frobenius (i.e., $x \mapsto x^2$) and V is the Verschiebung (i.e., the dual of the Frobenius on cohomology). The Hopf algebra $\Gamma[x_i]$ is dual to $P(y_i)$ with y_i primitive. As such, it is $\mathbb{Z}/(2)$-free on elements $\gamma_k(x_i)$ in degree ki. As an algebra, it is an exterior algebra on the generators $\gamma_{2^j}(x_i)$ of degree $2^j i$. We have $V(\gamma_{2^{j+1}}(x_i)) = \gamma_{2^j}(x_i)$.

There are a number of situations that arise frequently in our computations. For example, we might find that we have an associated graded object that is $\bigotimes_i E(x_i)$, but we know that when the extensions are solved it must be polynomial. This becomes $\bigotimes_i P(x_{2i+1})$ for degree reasons. Similarly $\bigotimes_i E(x_{2i})$ and $\bigotimes_i E(x_{4i})$, if they are really polynomial algebras, become $\bigotimes_i P(x_{4i+2})$ and $\bigotimes_i P(x_{8i+4})$. If we have $\bigotimes_i \Gamma[x_i]$ as an associated graded object for what we know is polynomial, we get $\bigotimes_i P(x_i)$.

On the other hand, if there are no extension problems, as algebras, we have that $\bigotimes_i \Gamma[x_{2i+1}]$ is just $\bigotimes_i E(y_i)$, and $\bigotimes_i \Gamma[x_{4i+2}]$ is just $\bigotimes_i E(y_{2i})$.

If we have the differential Hopf algebra $E(x_i) \otimes P(y_{i+1})$ with $d^1(x_i) = y_{i+1}$, we know that the homology in positive degrees is zero. We are often confronted with the dual of this situation, where we have $E(x_i) \otimes \Gamma[y_{i+1}]$ with $d_1(y_{i+1}) = x_i$. Again, our homology here is zero in positive degrees. It is not always that simple though. It often happens that we have $E(x_{2i+1}) \otimes \Gamma[y_{i+1}]$ and have $d_2(\gamma_2(y_{i+1})) = x_{2i+1}$. This leaves $E(y_{i+1})$ as its homology. When this happens, we abuse notation and write

$$E(x_{2i+1}) \otimes \Gamma[y_{i+1}] \simeq E(x_{2i+1}) \otimes E(y_{i+1}) \otimes \Gamma[y_{2i+2}]$$

so we can see the differential and results more clearly. This is just the associated graded object we get from the short exact sequence of Hopf algebras

$$E(y_{i+1}) \to \Gamma[y_{i+1}] \to \Gamma[y_{2i+2}],$$

where we have written $\gamma_2(y_{i+1}) = y_{2i+2}$. Similarly, worse happens and we need

$$E(x_{4i+3}) \otimes \Gamma[y_{i+1}] \simeq E(x_{4i+3}) \otimes E(y_{i+1}) \otimes E(y_{2i+2}) \otimes \Gamma[y_{4i+4}],$$

where we have a differential taking y_{4i+4} to x_{4i+3} leaving only $E(y_{i+1}) \otimes E(y_{2i+2})$ but with $V(y_{2i+2}) = y_{i+1}$.

To deal with our spectral sequences, we must be able to evaluate Tor. The simple case of $\mathrm{Tor}_{0,*}$ is the Hopf algebra cokernel of the map $H_*(F) \to H_*(E)$. There are no differentials on this zero filtration and what remains after differentials hit it is a sub-Hopf algebra of $H_*(B)$, i.e., the image of $H_*(E) \to H_*(B)$. In general, for $\mathrm{Tor}_{i,j}$, this is our i-th filtration and an element has total degree $i + j$.

We have a few facts to accumulate.

(1) If A is the Hopf algebra kernel of the map $H_*(F) \to H_*(E)$, then the higher filtrations are given by $\mathrm{Tor}^A(\mathbb{Z}/(2), \mathbb{Z}/(2))$.

(2) Tor commutes with tensor products.

(3) $\mathrm{Tor}^{E(x_i)}(\mathbb{Z}/(2), \mathbb{Z}/(2)) = \Gamma[y_{i+1}]$ with y_{i+1} in bidegree $(1, i)$ and $\gamma_{2^j}(y_{i+1})$ in bidegree 2^j times this.

(4) $\mathrm{Tor}^{P(x_i)}(\mathbb{Z}/(2), \mathbb{Z}/(2)) = E(y_{i+1})$ with y_{i+1} in bidegree $(1, i)$.

(5) $\mathrm{Tor}^{TP_{2^k}(x_i)}(\mathbb{Z}/(2), \mathbb{Z}/(2)) = E(y_{i+1}) \otimes \Gamma[z_{2^k i+2}]$ with y_{i+1} in bidegree $(1, i)$ and $z_{2^k i+2}$ in bidegree $(2, 2^k i)$.

(6) Elements in filtrations zero and one are permanent cycles.

If the kernel A is trivial, the spectral sequence collapses and the cokernel is $H_*(B)$, giving us a short exact sequence $H_*(F) \to H_*(E) \to H_*(B)$.

Since the kernel A is a Hopf algebra, Borel's theorem applies and the above allows us to compute Tor completely. Differentials must start on the second or higher filtration and they must take generators to primitives. The primitives all live in filtrations 0, 1, or 2 and the primitives in filtrations 1 and 2 are all generators. All generators in filtrations 2 or higher are of even degree. Thus the targets of differentials must be odd degree elements in filtrations 0 or 1. A fact that we often use is that *any even degree element in filtrations* 0 *or* 1 *must survive*.

There is one more special case we need to discuss. If we have a short exact sequence $E(y_{2i}) \to TP_4(x_i) \to E(x_i)$ that takes y_{2i} to $(x_i)^2$, we can compute Tor of $TP_4(x_i)$ as above and get $E(z_{i+1}) \otimes \Gamma[w_{4i+2}]$. If we didn't know there was the square $x_i^2 = y_{2i}$ in the middle term, but thought the middle term might be $E(y_{2i}) \otimes E(x_i)$, then Tor would be $\Gamma[z_{i+1}] \otimes \Gamma[u_{2i+1}]$. If we had a reason to know that this was not correct, then $d_1(\gamma_2(z_{i+1})) = u_{2i+1}$ would leave us with the correct answer.

4. $H_*(\underline{KR(1)}_0)$

We begin with the spectral sequence for

$$\underline{KO}_0 \xrightarrow{2} \underline{KO}_0 \to \underline{KR(1)}_0.$$

Computing 2_* is easy: we have

$$2_*(x_{2i}) = FV(x_{2i}) = F(x_i) = (x_i)^2 \quad \text{and} \quad 2_*(x_{2i+1}) = 0.$$

We can read off the cokernel as $\bigotimes_i E(x_i)$ and the kernel as $\bigotimes_i P(x_{2i+1})$. Computing Tor on the kernel, we get $\bigotimes_j E(y_{2j})$. Since all of these generators are in filtrations zero and one, the spectral sequence collapses. What we know at this stage is that we have

$$\bigotimes_i E(x_i) \subset H_*(\underline{KR(1)}_0), \qquad V(x_{2i}) = x_i$$

with quotient having an associated graded object $\bigotimes_j E(y_{2j})$.

We move now to a different spectral sequence, the one for

$$\underline{KO}_0 \to \underline{KR(1)}_0 \to \underline{KO}_1.$$

We have computed the image of $H_*(\underline{KO}_0) \to H_*(\underline{KR(1)}_0)$. It is just $\bigotimes_i E(x_i)$. The cokernel is the object with associated graded object $\bigotimes_i E(y_{2i})$ above. That is our zero filtration for this spectral sequence. The generators are all in even degrees and so must survive. This is all of the zeroth filtration and the zero filtration must be a sub-Hopf algebra of $H_*(\underline{KO}_1)$ which is polynomial, so the cokernel must be polynomial, and for degree reasons, this must be $\bigotimes_j P(y_{4j+2})$. This splits as algebras and coalgebras and so completes our computation.

5. $H_*(\underline{KR(1)}_1)$

We start with the spectral sequence for the fibration

$$\underline{KO}_1 \xrightarrow{2} \underline{KO}_1 \to \underline{KR(1)}_1.$$

The map 2_* is zero because all of the generators x_{2i+1} for $H_*(\underline{KO}_1)$ are primitive, so $V(x_{2i+1}) = 0$, giving $2_*(x_{2i+1}) = FV(x_{2i+1}) = 0$. The cokernel is $H_*(\underline{KO}_1) = \bigotimes_i P(x_{2i+1})$ and so is the kernel. We now know the zeroth filtration and taking Tor of the kernel, we get exterior generators y_{2i} in filtration 1. The spectral sequence collapses because all the generators are in filtrations 0 and 1. We still have extension problems though. Again, we move to the next spectral sequence for

$$\underline{KO}_1 \to \underline{KR(1)}_1 \to \underline{KO}_2.$$

We have computed the image of $H_*(\underline{KO}_1) \to H_*(\underline{KR(1)}_1)$. It is just $\bigotimes_i P(x_{2i+1})$. There is no kernel, so the spectral sequence collapses and is just the cokernel in the zeroth filtration. This becomes a short exact sequence of Hopf algebras

$$H_*(\underline{KO}_1) \to H_*(\underline{KR(1)}_1) \to H_*(\underline{KO}_2).$$

But this is just

$$\bigotimes_i P(x_{2i+1}) \to H_*(\underline{KR(1)}_1) \to \bigotimes_i P(y_{4i+2})$$

and so splits as algebras, giving us most of our answer. There is an extension problem to solve to get $V(y_{4i+2}) = x_{2i+1}$.

For that we use the spectral sequence for

$$\underline{KR(1)}_0 \to * \to \underline{KR(1)}_1.$$

Computing Tor of $H_*(\underline{KR(1)}_0)$ we get

$$\Gamma[w_k] \otimes_k E(ww_{4k+3}).$$

We should note that we have to use the $\mathbb{Z}/(2)$ in degree zero for $H_0(\underline{KR(1)}_0)$ to get the w_1 above.

This is way too big. Remember, we know the answer as algebras here. To get this down to size, we must take the first possible differential, i.e., we must have $d_3(\gamma_4(w_k)) \neq 0$. This element is degree $4k$ (in the fourth filtration) so the differential hits an element in the first filtration in degree $4k-1$. There are two possibilities, but it must hit one of them, and we don't need to know which just yet. All that is left after these differentials is an exterior algebra $\bigotimes_i E(z_i)$ with generators in filtration 1 and an exterior algebra $\bigotimes_i E(\gamma_2(w_i))$ with generators in filtration 2. This is precisely the size of the known answer so these differentials must indeed happen.

We know that the answer is polynomial, so the Frobenius must be injective. The Frobenius cannot raise filtration so the injective Frobenius on the first filtration gives us $\bigotimes_i P(z_{2i+1})$, forcing (to get the correct answer) the Frobenius to inject on the second filtration to get $\bigotimes_i P(w_{4i+2})$. The only ambiguity in the first filtration is about which elements in degrees $4i+3$ have survived. We know that the element in degree $2i+1$ in the first filtration must square to the element in degree $4i+2$, and this is unambiguously $x_{4i+2} = V(\gamma_2(x_{4i+2}))$. But we know that we must have $\gamma_2(x_{4i+2}) = (\gamma_2(x_{2i+1}))^2$ because of the injectivity of F. But now we have just computed VF on $\gamma_2(x_{2i+1})$ and found it nonzero. Consequently, $VF = FV$ must also be nonzero so that V is nonzero. We get our result that V of every generator of $P(y_{4i+2})$ is a generator of $P(x_{2i+1})$ as desired.

6. $H_*(\underline{KR(1)}_2)$

We start with the spectral sequence for

$$\underline{KO}_2 \xrightarrow{2} \underline{KO}_2 \to \underline{KR(1)}_2.$$

The map 2_* is zero because all of the generators x_{4i+2} for $H_*(\underline{KO}_2)$ are primitive, so $V(x_{4i+2}) = 0$, giving $2_*(x_{4i+2}) = FV(x_{4i+2}) = 0$. The cokernel is $H_*(\underline{KO}_2) = \bigotimes_i P(x_{4i+2})$ and so is the kernel. We now know the zeroth filtration and taking Tor of the kernel, we get $\bigotimes_i E(y_{4i+3})$. The spectral sequence collapses because all the generators are in filtrations 0 and 1. We still have extension problems though.

What we have from the spectral sequence is the short exact sequence

$$\bigotimes_i P(x_{4i+2}) \to H_*(\underline{KR(1)}_2) \to \bigotimes_i E(y_{4i+3}).$$

There is an extension problem we need to solve, namely $(y_{4i+3})^2$ from filtration 1 is x_{8i+6} in filtration 0. Once this is done, we would have the algebra structure.

To solve this problem we look at the spectral sequence for

$$\underline{KR(1)}_2 \xrightarrow{\eta} \underline{KR(1)}_1 \to \underline{K(1)}_1.$$

We have maps

$$H_*(\underline{KR(1)}_2) \to P(z_{2i+1}) \otimes_i P(zz_{4i+2}) \to E(w_{4i+1}) \otimes_i P(ww_{4i+2}).$$

Our calculation so far shows that $H_*(\underline{KR(1)}_2)$ is generated by primitives. We know from our computation of $H_*(\underline{KR(1)}_1)$ that $V(zz_{4i+2}) = z_{2i+1}$, so all the primitives in $H_*(\underline{KR(1)}_1)$ are in $\bigotimes_i P(z_{2i+1})$. Since primitives map to primitives, we see that $\bigotimes_i P(zz_{4i+2})$ is in the cokernel. It is even degree and in filtration zero so is a subalgebra of $H_*(\underline{K(1)}_1)$, so it must be our $\bigotimes_i P(ww_{4i+2})$ in our known answer. This accounts for all of the even degree generators and squares in $H_*(\underline{K(1)}_1)$.

If the element y_{4i+3} from $H_*(\underline{KR(1)_2})$ is in the kernel, then Tor gives rise to an element in filtration 1 of degree $4i+4$. This element would have to survive, but we have all of the even degree generators and squares we need, so y_{4i+3} maps to z_{4i+3} because it is the only primitive in that degree. However, z_{4i+3} is a polynomial generator, so y_{4i+3} must also be a polynomial generator, solving our extension problem.

We can go one step further. If x_{8i+2} doesn't map to $(z_{4i+1})^2$, this last element would be even degree in the cokernel where we don't need any more even degree elements, so it does map accordingly. We get a rare short exact sequence:

$$H_*(\underline{KR(1)_2}) \to H_*(\underline{KR(1)_1}) \to H_*(\underline{K(1)_1}).$$

7. $H_*(\underline{KR(1)_3})$

We start with the spectral sequence for

$$\underline{KR(1)_2} \to * \to \underline{KR(1)_3}.$$

Since $H_*(\underline{KR(1)_2}) \simeq P(x_{8i+2}) \otimes_i P(x_{4i+3})$, computing Tor is easy: it is just

$$E(x_{8i+3}) \otimes_i E(x_{4i})$$

and since the generators are all in filtration 1, it collapses. All we have left are extension problems.

Next we use the spectral sequence for

$$\underline{KO_3} \xrightarrow{2} \underline{KO_3} \to \underline{KR(1)_3}.$$

The homology $H_*(\underline{KO_3})$ is generated by primitives, so 2_* is zero. We get that the cokernel is $\bigotimes_i E(x_{4i+3})$ and so is the kernel. The E^2 term of the spectral sequence is

$$E(x_{4i+3}) \otimes_i \Gamma[y_{4i}].$$

This is much too big compared with our first spectral sequence. The only way to cut it down to the right size is with

$$d_2(\gamma_2(y_{4i})) = x_{8i-1}.$$

This leaves $E(x_{8i+3}) \otimes_i E(y_{4i})$ as with the first one, but now we know that the $E(x_{8i+3})$ is the image of $H_*(\underline{KO_3})$ in $H_*(\underline{KR(1)_3})$ and the cokernel has an associated graded object of $\bigotimes_i E(y_{4i})$.

We can move on to the spectral sequence for

$$\underline{KO_3} \to \underline{KR(1)_3} \to \underline{KO_4}.$$

We just computed the cokernel. It is even degree in filtration zero and all of the elements must survive. Since this cokernel is a subalgebra of the polynomial algebra $H_*(\underline{KO}_4)$, this solves all of our extension problems, giving $(y_{4i})^2 = y_{8i}$. So we have the expected polynomial algebra $\bigotimes_i P(y_{8i+4})$, completing our computation.

8. $H_*(\underline{KR(1)}_4)$

We use the spectral sequence coming from

$$\underline{KO}_4 \xrightarrow{2} \underline{KO}_4 \to \underline{KR(1)}_4.$$

As in the $\underline{KR(1)}_0$ case, $H_*(\underline{KO}_4)$ is bipolynomial. The cokernel of 2_* is just $\bigotimes_i E(x_{4i})$ and the kernel is $P(x_{8i+4})$. We take Tor of this to get exterior generators y_{8i+5} in filtration 1. Since all our generators are in filtrations zero and one, the spectral sequence collapses. For purely degree reasons, there can be no extension problems given that we know $\bigotimes_i E(x_{4i})$ is a subalgebra.

9. $H_*(\underline{KR(1)}_5)$

We use the spectral sequence for the fibration

$$\underline{KR(1)}_4 \to * \to \underline{KR(1)}_5.$$

Since $H_*(\underline{KR(1)}_4) \simeq E(x_{4k}) \otimes_k E(y_{8k+5})$, Tor is

$$\Gamma[z_{4k+1}] \otimes_k \Gamma[w_{8k+6}].$$

The only possible sources for differentials are in (total) degrees divisible by 4, but the only odd degree primitives are in bidegree $(1, 4k)$, total degree $4k+1$, so there can be no differentials (lowering total degree by 1). Furthermore, there are no algebra extension problems. In filtration one there are only elements z_{4k+1} and w_{8k+6}, so there is nothing for them to square to. In filtration two, the elements are in degrees $8k+2$ and $16k+12$ and again there are no elements in filtrations one or two to square to. Continue inductively on filtration. The degrees never work out to have extensions. This spectral sequence gives a complete description of V as well.

10. $H_*(\underline{KR(1)}_7)$

Note that we have skipped $H_*(\underline{KR(1)}_6)$. It is the hardest to compute and all our previous techniques failed us. We need $H_*(\underline{KR(1)}_7)$ to solve the problems with $H_*(\underline{KR(1)}_6)$.

We use the cohomology spectral sequence for the fibration

$$\underline{KR(1)}_7 \to \underline{KO}_0 \xrightarrow{2} \underline{KO}_0.$$

The homology of $\underline{KO}_0$ is bipolynomial with $H_*(\underline{KO}_0) \simeq \bigotimes_i P(x_i)$ and $V(x_{2i}) = x_i$. So we get that $H^*(\underline{KO}_0)$ is the same. Evaluating 2^* gives $2^*(x_{2i}) = FV(x_{2i}) = F(x_i) = (x_i)^2$. The cokernel is $\bigotimes_i E(x_i)$ with V as before. Since $V(x_{2i+1}) = 0$, the kernel is $\bigotimes_i P(x_{2i+1})$. Tor of the kernel is $\bigotimes_i E(w_{2i})$ with generators in the first filtration. Since all of the generators are in the first 2 filtrations, the spectral sequence collapses. Since we know the V on filtration zero ($\bigotimes_i E(x_i)$), we can dualize and we get that the homology has $\bigotimes_i P(y_{2i+1})$ in it. There is the $\bigotimes_i E(ww_{2i})$ (dual to $\bigotimes_i E(w_{2i})$) as well, but it could have extension problems we need to solve.

To show that the $\bigotimes_i E(ww_{2i})$ really is an exterior algebra, we take a quick look at the homology spectral sequence for

$$\underline{KO}_7 \xrightarrow{2} \underline{KO}_7 \to \underline{KR(1)}_7.$$

The first map is zero because $H_*(\underline{KO}_7) \simeq \bigotimes_i E(z_i)$ and F is zero, so the cokernel contains $\bigotimes_i E(z_{2i})$ in the zero filtration and this subalgebra must survive. We now have the desired exterior subalgebra.

11. $H_*(\underline{KR(1)}_6)$

This is both the hardest to compute and the most interesting. We start with our usual fibration

$$\underline{KO}_6 \xrightarrow{2} \underline{KO}_6 \to \underline{KR(1)}_6.$$

$H_*(\underline{KO}_6) \simeq \bigotimes_k E(x_{2k})$, so $2_* = VF = 0$ because F is zero. That means all of $H_*(\underline{KO}_6)$ is the cokernel and it all survives because it is even degree. Using our second spectral sequence for

$$\underline{KO}_6 \to \underline{KR(1)}_6 \to \underline{KO}_7,$$

we know that the first map injects so there is no kernel. The spectral sequence collapses with $H_*(\underline{KO}_7)$, the cokernel of the map. This gives the short exact sequence

$$\bigotimes_i E(x_{2i}) \to H_*(\underline{KR(1)}_6) \to \bigotimes_i E(y_i).$$

The goal here is to solve the extension problem $(y_i)^2 = x_{2i}$. We do already know that $V(x_{4i}) = x_{2i}$ on the first part and $V(y_{2i}) = y_i$ on the second part.

We use the spectral sequence

$$\underline{KR(1)}_6 \to * \to \underline{KR(1)}_7$$

to prove our result. Observe that Tor of $E(x_{2i}) \otimes_i E(y_i)$ is

$$\Gamma[w_{2i+1}] \otimes_i \Gamma[ww_i] = E(w_{2i+1}) \otimes_i \Gamma[w_{4i+2}] \otimes_i E(ww_i) \otimes_i \Gamma[ww_{2i}]$$

and that Tor of $\bigotimes_i TP_4(y_i)$ is

$$\Gamma[w_{4i+2}] \otimes_i E(ww_i).$$

If the extension exists, there is a $d_1(ww_{2i}) = w_{2i-1}$. We don't need d_1 for our result though. Note that no matter what the extension is, in Tor we have

$$\Gamma[w_{4i+2}] \otimes_i E(ww_i).$$

We rewrite this just a bit as

$$\Gamma[w_{4i+2}] \otimes_i E(ww_{2i}) \otimes_i E(ww_{2i+1}).$$

Note that this is precisely the correct size for our known result of $H_*(\underline{KR(1)}_7)$. That doesn't prove our result yet though. We do know that any even degree element in filtration 1 or 2 must survive, and so we know we have $E(w_{4i+2}) \otimes_i E(ww_{2i})$ already no matter what extensions there are. One of w_{4i+2} or ww_{4i+2} must be exterior and the other must square. The only thing to square to is ww_{8i+4}. Because we know the answer and all these elements must survive, this must be part of the polynomial part of the answer, so we must have $(ww_{4i})^2 = ww_{8i}$. It doesn't really matter which of the elements is exterior. What we know from this is that we have all the elements we need in degrees $4i+2$ that are generators, primitives, or squares.

If we had a case where $(y_{2i})^2 = 0$ in $H_*(\underline{KR(1)}_6)$, then from the above discussion, we would have Tor giving us a $\Gamma[z_{2i+1}] = E(z_{2i+1}) \otimes \Gamma[z_{4i+2}]$. We would not have the $\Gamma[z_{4i+2}]$ unless this happens. The z_{4i+2} is in filtration 2 so must survive, but we already have enough elements in this degree, so this cannot happen.

We now know that $(y_{2i})^2 = x_{4i}$ always. We have

$$x_{2i} = V(x_{4i}) = VF(y_{2i}) = FV(y_{2i}) = F(y_i) = (y_i)^2.$$

This solves the extension problem for all y_i with i odd.

References

[Cowen Morton and Strickland 2002] D. S. Cowen Morton and N. Strickland, "The Hopf rings for KO and KU", *J. Pure Appl. Algebra* **166**:3 (2002), 247–265. MR Zbl

[Eilenberg and Moore 1966] S. Eilenberg and J. C. Moore, "Homology and fibrations, I: Coalgebras, cotensor product and its derived functors", *Comment. Math. Helv.* **40** (1966), 199–236. MR Zbl

[Hu and Kriz 2001] P. Hu and I. Kriz, "Real-oriented homotopy theory and an analogue of the Adams–Novikov spectral sequence", *Topology* **40**:2 (2001), 317–399. MR Zbl

[Kitchloo and Wilson 2007] N. Kitchloo and W. S. Wilson, "On the Hopf ring for $ER(n)$", *Topology Appl.* **154**:8 (2007), 1608–1640. MR Zbl

[Milnor and Moore 1965] J. W. Milnor and J. C. Moore, "On the structure of Hopf algebras", *Ann. of Math.* (2) **81** (1965), 211–264. MR Zbl

[Moore 1961] J. C. Moore, "Algèbre homologique et homologie des espaces classifiants", exp. 7 in *Périodicité des groupes d'homotopie stables des groupes classiques, d'après Bott*, vol. 1, Séminaire Henri Cartan, 12ième année (1959/60), Secrétariat mathématique, Paris, 1961. MR Zbl

[Ravenel and Wilson 1977] D. C. Ravenel and W. S. Wilson, "The Hopf ring for complex cobordism", *J. Pure Appl. Algebra* **9**:3 (1977), 241–280. MR Zbl

[Rothenberg and Steenrod 1965] M. Rothenberg and N. E. Steenrod, "The cohomology of classifying spaces of H-spaces", *Bull. Amer. Math. Soc.* **71** (1965), 872–875. MR Zbl

[Smith 1970] L. Smith, *Lectures on the Eilenberg–Moore spectral sequence*, Lecture Notes in Mathematics **134**, Springer, 1970. MR Zbl

[Wilson 1984] W. S. Wilson, "The Hopf ring for Morava K-theory", *Publ. Res. Inst. Math. Sci.* **20**:5 (1984), 1025–1036. MR Zbl

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W. STEPHEN WILSON: wsw@math.jhu.edu
Department of Mathematics, Johns Hopkins University, Baltimore, MD, United States

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