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The cohomology of the connective spectra for K-theory revisited

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ABSTRACT. The stable mod 2 cohomologies of the spectra for connective real and complex K-theories are well known and easy to work with. However, the known bases are in terms of the anti-automorphism of Milnor basis elements. We offer simple bases in terms of admissible sequences of Steenrod operations that come from the Adem relations. In particular, a basis for $H^*(bu)$ is given by those Sq^I with I admissible and no Sq^1 or Sq^{2^n+1} appearing for $n > 0$.

CONTENTS

1. Introduction

Our goal is to give simple bases for the mod 2 cohomologies, $H^*(bo)$ and $H^*(bu)$, for the connective real and complex K-theory spectra respectively.

Let $I = (i_1, i_2, \dots, i_k)$. We let $Sq^I = Sq^{i_1}Sq^{i_2}\cdots Sq^{i_k}$ be a composition of Steenrod squares. We have the length of I, given by $\ell(I) = k$, and the degree of I, given by $|I| = |Sq^I| = \sum i_s$. We say I is *admissible* if $i_s \ge 2i_{s+1}$ for all s. For I admissible, we have the excess, $e(I) = i_1 - i_2 - \cdots - i_k$. The admissible Sq^I form the Serre-Cartan basis for the mod 2 Steenrod algebra, $\mathcal{A}$ ([Ser53, Car55]). Let $\mathcal{A}_1$ be the sub-algebra generated by Sq^1 and Sq^2. Let E_1 be the sub-algebra generated by $Q_0 = Sq^1$ and $Q_1 = Sq^1Sq^2 + Sq^2Sq^1$.

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Without Maple software, the Induction step 4.2 would never have been found.

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Let $\mathbf{Z}_2$ be the integers mod 2. It has been known for a long time ([Sto63]) that $H^*(bo) = \mathcal{A} \otimes_{\mathcal{A}_1} \mathbf{Z}_2 = \mathcal{A}/\!/\mathcal{A}_1$ and $H^*(bu) = \mathcal{A} \otimes_{E_1} \mathbf{Z}_2 = \mathcal{A}/\!/E_1$. The usual basis for $H^*(bo)$ involves applying the anti-automorphism to Milnor basis ([Mil58]) elements $Sq(R)$ with $R = (4r_1, 2r_2, r_3, \dots)$. One can also extract an exotic basis for $H^*(bo)$ from [Mor07] that is probably related to the spaces in the Omega spectrum that the elements are created on.

We can now state our main theorem:

Theorem 1.1. *A basis for $H^*(bo)$ is given by all Sq^I with I admissible, no $i_s = 2^n + 1$ for $n \geq 1$ and $i_k \geq 4$. The case $H^*(bu)$ is the same except that $i_k \geq 2$.*

Along the way we needed some things about the Steenrod algebra that may be of independent interest.

Definition 1.2. Let $T_b \subset \mathcal{A}$ be the span of all admissible Sq^I with $i_1 \leq b$.

Proposition 1.3. *$Sq^a\, T_b \subset T_n$ where*

$$n = \begin{cases} a & \text{if } a \geq 2b \\ 2b - 1 & \text{if } 2b > a \geq b \\ a + b & \text{if } b > a > 0. \end{cases}$$

Remark 1.4. Note that it is always true that $Sq^a T_b \subset T_{a+b}$. The way we eliminate the Sq^{2^n+1} is as follows. We consider $J = (2^{n+1} + 1, i_0, \dots, i_k)$ admissible. If Sq^J is non-zero in $\mathcal{A}/\!/\mathcal{A}_1$, we can write it as a sum of Sq^K with K admissible and $k_1 \leq 2^{n+1}$. Although we don't need it in this paper, we also show that $e(J) > e(K)$ for every such K.

The subalgebra $\mathcal{A}_n$ of $\mathcal{A}$ is generated by $Sq^1, Sq^2, Sq^4, \cdots, Sq^{2^n}$. As usual, let $\alpha(n)$ be the number of ones in the binary expansion of n. We had a brief hope that a basis for $\mathcal{A}/\!/\mathcal{A}_n$ would be given by Sq^I with I admissible, $i_k \geq 2^{n+1}$ and with no i_s with $\alpha(i_s - 1) \leq n$. Unfortunately it was false already in degree 49 for $\mathcal{A}_2$. The anti-automorphism of the Milnor element $Sq(8, 4, 2, 1)$ is non-zero in degree 49. However, a short calculation shows that the suggested conjecture has no elements in degree 49.

One observation survived:

Proposition 1.5. *In $\mathcal{A}/\!/\mathcal{A}_n$, if $\alpha(m) \leq n$, then $Sq^{m+1} \in T_m$.*

Our initial interest was in tmf with $H^*(tmf) = \mathcal{A}/\!/\mathcal{A}_2$. However, it was clear that not only was nothing known here, but the same held true for $H^*(bo)$. Calculations led to the conjecture and eventually the theorem. A conjecture for $\mathcal{A}/\!/\mathcal{A}_2$ still eludes us.

We first prove the results about the Steenrod algebra. Then we apply these results to prove Theorem 1.1.

2. Results on the Steenrod algebra

We will make constant use of the Thom spectrum, MO, for the unoriented cobordism case. From [Thom54] we know that $H^*(MO)$ is free over

$\mathcal{A}$ and that one copy of $\mathcal{A}$ sits on the Thom class $U \in H^0(MO)$. We need the Stiefel-Whitney (S-W) classes, $w_i \in H^i(BO)$, and the Thom isomorphism $H^*(BO) \cong H^*(MO)$ that takes w_i to Uw_i. We need the connection between the S-W classes and the Steenrod algebra ([Wu53]) given by $Sq^n U = Uw_n$ and

$$Sq^i(w_j) = \sum_{t=0}^{i} \binom{j+t-i-1}{t} w_{i-t} w_{j+t}.$$

Keep in mind that $Sq^n w_n = w_n^2$ and $Sq^i w_n = 0$ when $i > n$.

The cohomology, $H^*(BO)$, is a polynomial algebra on the S-W classes, [MS74]. We put an order on the monomials. We have $M < M'$ if the degree of M' is greater than that of M. Next, if they have the same degree, the one with the largest w_n is greater. If they have the same largest w_n, we go to the next largest and so on. To use Thom's examples from his paper: $w_4 < w_4 w_1^2 < w_4 w_2 w_1 < w_4 w_3$.

Lemma 2.1 (Thom, in the proof of II.8, [Thom54])**.** *For $I = (i_1, i_2, \dots, i_k)$ admissible, in $H^*(MO)$,*

$$Sq^I(U) = U(w_{i_1} w_{i_2} \cdots w_{i_k} + \Delta)$$

where Δ is a sum of monomials of lower order.

Remark 2.2. The filtration is not a filtration of $\mathcal{A}$-modules, but Thom's result allows us to distinguish between admissible Sq^I using the S-W classes in the Thom spectrum. Because it was 1954, Thom worked in the stable range of $MO(n)$ where his Thom class was w_n.

Proof of Proposition 1.3. It is enough to consider the case when $i_1 = b$. When $a \geq 2b$, there is nothing to prove because $Sq^a Sq^I$ is already admissible. When $2b > a \geq b$, this is no longer the case. If $Sq^a Sq^I$ is written in terms of admissible Sq^J, we need to determine what the maximum possibility is for j_1. We look at

$$\begin{aligned} Sq^a Sq^I(U) &= Sq^a\Big(U(w_b w_{i_2} \cdots w_{i_k} + \Delta)\Big) \\ &= \sum_{j=0}^{a} Sq^{a-j}(U) Sq^j(w_b w_{i_2} \cdots w_{i_k} + \Delta) \\ &= \sum_{j=0}^{a} U w_{a-j} Sq^j(w_b w_{i_2} \cdots w_{i_k} + \Delta). \end{aligned}$$

Since $a - j < 2b$, the largest possible new S-W class is given by $Sq^j(w_b)$, but the largest this can be is $Sq^{b-1}(w_b) = w_{2b-1}$ plus other terms with products. Similarly, if $a = 2b-1$, we could get w_{2b-1} when $j = 0$ in the formula. Not only is $n = 2b-1$ the largest possible, but it is realized.

Using the same formula when $b > a > 0$, the largest possible w_n is when $Sq^a w_b$ includes w_{a+b} and that is only realized when $\binom{b-1}{a} = 1 \pmod 2$. This concludes the proof. □

It is time to introduce one of our key tools, the Adem relations ([Ade52]):

$$Sq^a Sq^b = \sum_i^{[a/2]} \binom{b-1-i}{a-2i} Sq^{a+b-i} Sq^i.$$

These apply when $a < 2b$, that is, when $Sq^a Sq^b$ is not admissible. The resulting terms are admissible. The sum is from the maximum of 0 or $a - b + 1$.

Proof of Proposition 1.5. We induct on m. Let $m = 2^{k_1} + \cdots + 2^{k_\ell}$ with $k_1 > \cdots > k_\ell$ and $\ell \le n$. Let $s = \min\{i : k_i > k_{i+1} + 1\}$. If no such s, let $s = \ell$. If $s = \ell$ and $k_\ell = 0$, then $m + 1 = 2^\ell$, and we are done since $Sq^{2^\ell} = 0$ in $\mathcal{A}/\!/\mathcal{A}_n$. Otherwise, write $m = 2a + b$ with

$$a = 2^{k_1-1} + \cdots + 2^{k_s-1} \text{ and } b = 2^{k_{s+1}} + \cdots + 2^{k_\ell}.$$

Note that if $s = \ell$, then $b = 0$ and m is even, having already done the odd case. Then $Sq^{a+b+1} \in T_{a+b}$ by the induction hypothesis. Proposition 1.3 says $Sq^a T_{a+b} \subset T_{2a+b}$, so we have $Sq^a Sq^{a+b+1} \in T_{2a+b}$. Since $\binom{a+b}{a} = 1$, (This is because the binary expansion of $a+b$ includes the binary expansion of a in this case. We are always working mod 2), the Adem relation gives

$$Sq^a Sq^{a+b+1} = \sum \binom{a+b-i}{a-2i} Sq^{2a+b+1-i} Sq^i = Sq^{2a+b+1} + \Delta.$$

Here $\Delta \in T_{2a+b}$ and so is the left hand side, so $Sq^{m+1} = Sq^{2a+b+1} \in T_{2a+b} = T_m$. □

We now consider the obvious homomorphism from the span of admissible monomials described in Theorem 1.1 into $\mathcal{A}/\!/\mathcal{A}_1$. In Section 5, we do the deduction for $\mathcal{A}/\!/E_1$.

3. Injectivity

We wish to show that the Sq^I of Theorem 1.1 are linearly independent. This follows directly from Lemma 2.1 once the background is set up. For that we need the polynomial algebra from [Thomas62]

$$H^*(BSpin) = P[w_i] \qquad i \ge 4 \qquad i \ne 2^n + 1.$$

We give a quick proof of this because it involves techniques we need anyway.

Mod decomposables, we have the following easily verified formulas:

$$Sq^{2^k}(w_{2^k+1}) \equiv w_{2^{k+1}+1} \qquad Sq^{(2^n, 2^{n-1}, \ldots, 4, 2, 1)}(w_2) \equiv w_{2^{n+1}+1}.$$

The second follows immediately from the first. Because $w_2 = 0 \in H^*(BSpin)$ by definition, any Steenrod operations on it are zero as well. The formula tells us that $w_{2^{n+1}+1}$ is decomposable in $H^*(BSpin)$. Most are non-trivial, but we do have $w_3 = w_5 = w_9 = 0$.

We know $H^*(BSO) = P[w_i]$ with $i > 1$. We have a fibration $BSpin \to BSO \to K_2 = K(\mathbf{Z}_2, 2)$, where the last map is given by w_2. Note that $H^*(K_2)$ is a polynomial algebra on the $Sq^{(2^n,2^{n-1},\dots,4,2,1)}(\iota_2)$. The above computation shows the map $H^*(K_2) \longrightarrow H^*(BSO)$ is an injection giving us a short exact sequence of Hopf algebras $H^*(K_2) \longrightarrow H^*(BSO) \longrightarrow H^*(BSpin)$ from the Eilenberg-Moore (or Serre) spectral sequence. The collapse of the EM-s.s. is because we are working with Hopf algebras so the injection makes $H^*(BSO)$ free over $H^*(K_2)$. This gives $H^*(BSpin)$ and the decomposability of the w_{2^n+1}.

We are also going to look at the Thom spectrum, $MSpin$. Let U be the Thom class in $H^0(MSpin)$. The reason we are looking at the Thom spectrum is because as a module over the Steenrod algebra, $H^*(MSpin)$ is a sum of cyclic modules and the module generated by U is precisely $\mathcal{A}/\!/\mathcal{A}_1$, [ABP67].

Proof of injectivity for Theorem 1.1. If there were a relation in $\mathcal{A}/\!/\mathcal{A}_1$ among the admissible Sq^I with no $i_s = 2^n + 1$ and $i_k \geq 4$, Lemma 2.1 would imply a similar relations among the S-W classes in $H^*(MSpin)$. But because we are not using the w_{2^n+1}, these are linearly independent. □

4. Surjectivity

All that is left to do with our Theorem 1.1 is to show that any admissible Sq^I with some $i_s = 2^n + 1$ can be written in terms of admissible Sq^J with no $i_s = 2^t + 1$.

We specialize Proposition 1.3 to $Sq^{2^n+j}T_{2^n} \subset T_{2^{n+1}}$ when $2^n > j \geq 0$. Proposition 1.3 actually tells us $T_{2^{n+1}-1}$ but we don't need that little extra bit.

Lemma 4.1. *In $\mathcal{A}/\!/\mathcal{A}_1$, if $J = (2^{n+1}+1, i_0, \dots, i_k)$ is admissible, then $Sq^J \in T_{2^{n+1}}$, that is, $T_{2^{n+1}+1} \subset T_{2^{n+1}}$.*

Proof of Theorem 1.1 for $H^*(bo)$ from Lemma 4.1. If we have an I admissible with some $i_s = 2^n + 1$ with $n \geq 1$, we want to show that Sq^I can be replaced without this $i_s = 2^n + 1$. Write $I = LJ$ where J is the shortest possible as in Lemma 4.1. Lemma 4.1 tells us that Sq^J can be written in terms of Sq^K admissible with $k_1 < 2^{n+1} + 1$. When this sum replaces Sq^J in Sq^I, LK is still admissible. By induction, we do not have to worry about smaller J like this showing up. Since I is finite, this process of replacement is also finite. We have shown that every Sq^I, I admissible, can be replaced with one of the desired form, and we have shown that the Sq^I of this form are linearly independent. This concludes the proof of Theorem 1.1 from Lemma 4.1. □

Proof of Lemma 4.1. When $J = (2^{n+1} + 1)$, we can use the $\mathcal{A}/\!/\mathcal{A}_1$ case of Proposition 1.5.

To start our induction on k, we need the $J = (2^{n+1} + 1, i_0)$ case. We begin with $Sq^{2^n+1} = \Delta \in T_{2^n}$ from Proposition 1.5 and apply $Sq^{2^n+i_0}$. For J to be admissible, we have $i_0 \le 2^n$. From Proposition 1.3, $Sq^{2^n+i_0}T_{2^n} \subset T_{2^{n+1}}$ so $Sq^{2^n+i_0}Sq^{2^n+1} \in T_{2^{n+1}}$ since $Sq^{2^n+i_0}\Delta \in T_{2^{n+1}}$. All we need now is:

$$\begin{aligned} Sq^{2^n+i_0}Sq^{2^n+1} &= \sum_{s\ge i_0}\binom{2^n-s}{2^n+i_0-2s}Sq^{2^{n+1}+1+i_0-s}Sq^s \\ &= Sq^{2^{n+1}+1}Sq^{i_0}+\sum_{s>i_0}\binom{2^n-s}{2^n+i_0-2s}Sq^{2^{n+1}+1+i_0-s}Sq^s. \end{aligned}$$

The terms in the sum are all also in $T_{2^{n+1}}$ so the same is true for $Sq^{2^{n+1}+1}Sq^{i_0}$.

The following induction proves our Lemma 4.1 because the two terms in $T_{2^{n+1}}$ force the third term to be there as well. The induction is started above as it can be rephrased in the format of our induction below as the $k = 0$ case.

Induction 4.2. In $\mathcal{A}/\!/\mathcal{A}_1$, with $(2^{n+1}+1, i_0, \dots, i_k)$ admissible,

$$Sq^{2^n+i_0}Sq^{2^{n-1}+i_1}\cdots Sq^{2^{n-k}+i_k}Sq^{2^{n-k}+1} \in T_{2^{n+1}}$$

and is equal in $\mathcal{A}/\!/\mathcal{A}_1$ to

$$Sq^{2^{n+1}+1}Sq^{i_0}Sq^{i_1}\cdots Sq^{i_k}+\Delta_{n+1} \text{ with } \Delta_{n+1}\in T_{2^{n+1}}.$$

Proof of our Induction 4.2. By induction on k, we can write

$$Sq^{2^{n-1}+i_1}\cdots Sq^{2^{n-k}+i_k}Sq^{2^{n-k}+1}\in T_{2^n}$$

and it is equal to

$$Sq^{2^n+1}Sq^{i_1}\cdots Sq^{i_k}+\Delta_n \text{ with } \Delta_n\in T_{2^n}.$$

Now we take $Sq^{2^n+i_0}$ times everything. Since $Sq^{2^n+i_0}T_{2^n}\subset T_{2^{n+1}}$, we have $Sq^{2^n+i_0}\Delta_n=\Delta_{n+1}\in T_{2^{n+1}}$ and

$$Sq^{2^n+i_0}Sq^{2^{n-1}+i_1}\cdots Sq^{2^{n-k}+i_k}Sq^{2^{n-k}+1}\in T_{2^{n+1}}$$

and is equal in $\mathcal{A}/\!/\mathcal{A}_1$ to

$$Sq^{2^n+i_0}Sq^{2^n+1}Sq^{i_1}\cdots Sq^{i_k}+\Delta_{n+1}.$$

So, the term

$$Sq^{2^n+i_0}Sq^{2^n+1}Sq^{i_1}\cdots Sq^{i_k}$$

is also in $T_{2^{n+1}}$. It is equal to

$$\begin{aligned} &= \Big(\sum_{s\ge i_0}\binom{2^n-s}{2^n+i_0-2s}Sq^{2^{n+1}+1+i_0-s}Sq^s\Big)Sq^{i_1}\cdots Sq^{i_k} \\ &= Sq^{2^{n+1}+1}Sq^{i_0}Sq^{i_1}\cdots Sq^{i_k} \\ &\quad +\Big(\sum_{s>i_0}\binom{2^n-s}{2^n+i_0-2s}Sq^{2^{n+1}+1+i_0-s}Sq^s\Big)Sq^{i_1}\cdots Sq^{i_k}. \end{aligned}$$

Since $s > i_0$, the elements in the sum are admissible and in $T_{2^{n+1}}$. They can now be incorporated into Δ_{n+1}. We are left with $Sq^J = Sq^{2^{n+1}+1}Sq^{i_0}Sq^{i_1}\cdots Sq^{i_k}$ from Lemma 4.1 which is therefore also in $T_{2^{n+1}}$. □

Lemma 4.1 follows. □

Although we don't need this next Lemma, it is interesting in its own right. Let E_r be spanned by all Sq^I, I admissible, $e(I) \leq r$. Let $K(\mathbf{Z}_2, r) = K_r$ be the Eilenberg-MacLane space with $\iota_r \in H^r(K_r)$ the fundamental class. The significance of excess is that the $Sq^I\iota_r$ with $Sq^I \in E_r$ are linearly independent in $H^*(K_r)$ and $Sq^I\iota_r = 0$ for $e(I) > r$.

Lemma 4.3. *In $\mathcal{A}/\!/\mathcal{A}_1$, if $J = (2^{n+1}+1, i_0, \ldots, i_k)$ is admissible, then $Sq^J \in E_{e(J)-2}$.*

Proof. In $\mathcal{A}/\!/\mathcal{A}_1$, write $Sq^J = \sum Sq^K$ with K admissible and, from Lemma 4.1, $k_1 \leq 2^{n+1}$. If $Sq^J = 0$, there is nothing to prove. We have $|J| = |K|$. For I admissible, recall $|I| = i_1 + \cdots + i_k$ and $e(I) = i_1 - i_2 - \cdots - i_k$. We can connect them with $2i_1 - |I| = e(I)$. Now

$$e(K) = 2k_1 - |K| = 2k_1 - |J| \leq 2^{n+2} - |J| = 2(2^{n+1}+1) - |J| - 2 = e(J) - 2.$$

□

5. $H^*(bu)$

We give a quick derivation of $H^*(bu)$ from $H^*(bo)$. We have the standard fibration

$$bo \longrightarrow bu \longrightarrow \Sigma^2 bo.$$

This gives a short exact sequence of $\mathcal{A}$ modules. The map from $H^*(\Sigma^2 bo)$ takes 1 to Sq^2 and is injective so must hit all $Sq^I Sq^2$ with Sq^I a basis for $H^*(bo)$. The surjection $H^*(bu) \longrightarrow H^*(bo)$ must hit the Sq^I for a basis for $H^*(bo)$. This is the stated answer for $H^*(bu)$ in Theorem 1.1.

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