Periodic solutions for a class of nonautonomous Hamiltonian systems

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1. Introduction

In this paper, we study the existence of periodic solutions for a Hamiltonian system

\[-Jz - B(t)z = \nabla H(t, z), \quad z \in \mathbb{R}^{2N}, \quad t \in \mathbb{R},\]

(1)

where \(B(t)\) is a given \(T\)-periodic and symmetric \(2N \times 2N\)-matrix function of \(C^1\) class in \(t\), \(H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})\) is \(T\)-periodic in \(t\), \(\nabla H := \nabla_z H \in C(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R}^{2N})\) and

\[J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}\]

is the standard symplectic matrix. The main results of this paper are the following:

**Theorem 1.1.** For \(T > 0\), suppose that \(H\) satisfies the following conditions:

(H1) \(H \in C^1(S_T \times \mathbb{R}^{2N}, \mathbb{R}), \quad S_T = \mathbb{R}/(T\mathbb{Z}).\)
(H2) There are constants $\mu > 2$ and $r > 0$ such that
\[ 0 < \mu H(t, z) - z \nabla H(t, z), \quad \forall |z| \geq r. \]

(H3) $H(t, z) = o(|z|^2)$, uniformly in $t$ as $z \to 0$.

(H4) There exists a constant $\tilde{a}$ such that
\[ \lim_{|z| \to \infty} \frac{H_t(t, z)}{H(t, z)} \geq \tilde{a} > -\frac{2}{T}, \quad \text{uniformly in } t. \]

Then Eq. (1) has a nontrivial $T$-periodic solution in each of the following two cases:

(i) The boundary value problem
\[ -J \dot{z} = B(t)z, \quad z(0) = z(T), \quad (2) \]
has only the trivial solution.

(ii) There is a constant $\rho > 0$ such that $H(t, z) > 0$ (or $H(t, z) < 0$) for all $z$ satisfying $0 < |z| < \rho$.

**Theorem 1.2.** Suppose that $H$ satisfies (H1)–(H3) and the following (H5).
There are constants $c, d > 0$, such that $|\nabla H(t, z)| \leq c(\nabla H(t, z), z) + d$, $\forall z \in \mathbb{R}^{2N}$.
Then Eq. (1) has a nontrivial $T$-periodic solution in each of case (i) and (ii) in Theorem 1.1.

For the autonomous case, i.e. $H$ is independent of $t$, in his pioneering work [9] Rabinowitz first proved the existence of at least one periodic solution for Eq. (1). Many works have been done on this problem. For example, in [1, 2, 4–6, 9–12] some existence results of Eq. (1) are proved. We refer to [1, 12] for further references. These results have further restrictions on $\nabla H(t, z)$ in addition to (H1)–(H3). In this paper, we prove the existence of periodic solutions for Eq. (1) under a different and new condition (H4), which measures the difference of Eq. (1) from the autonomous systems. Define $H(t, z) = f(t)e^{\mu|z|^2}$ for large $|z|$, with $\mu > 0$ and $f \in C^1(S_T, \mathbb{R})$ satisfying $f'(t)/f(t) > -2/T$ for all $t$. Such kinds of functions as above satisfy the conditions of our Theorem 1.1, but are not contained in the above mentioned papers. Our Theorem 1.2 generalizes Theorem 2.1 of [2], where [2] requires $|\nabla H(t, z)|^p \leq c\nabla H(t, z)z + d$, for all $z \in \mathbb{R}^{2N}$, where $p > 1$. One may also compare our theorems with Theorem 1.4 of [10].

2. Proofs of main results

In this section, we consider the Hamiltonian system

\[ -J \dot{z} = B(t)z = \nabla H(t, z), \quad z \in \mathbb{R}^{2N}, \quad t \in \mathbb{R} \]

with $B(t)$ being a given continuous $T$-periodic and symmetric matrix function and $H$ being $T$-periodic in $t$. Let $X := W^{1/2, 2}(S_T, \mathbb{R}^{2N})$ be the Sobolev space of $T$-periodic
\[ R^{2N} \]-valued functions with inner product \((\cdot, \cdot)_X\) and norm \(\| \cdot \|_X\). Define two self-adjoint operators \(A, B \in \mathcal{L}(X)\) by extending the bilinear forms

\[
(Ax, y) = \int_0^T (Jx, y) \, dt, \quad (Bx, y) = \int_0^T (B(t)x, y) \, dt, \quad \forall x, y \in X.
\]

By [7] and standard spectral theory, \(B\) is compact on \(X\). Denote the eigenvalues of \(A - B\) on \(X\) by

\[
\cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0(= \lambda_0) < \lambda_1 \leq \lambda_2 \leq \cdots,
\]

where when \(\dim \ker(A - B) = 0\), \(\lambda_0 \notin \sigma(A - B)\). Let \(\{e_{\pm j}\}\) be the eigenvectors of \(A - B\) corresponding to \(\{\lambda_{\pm j}\}\), respectively. Define \(X_+ = \text{span}\{e_1, e_2, \ldots\}\), \(X_- = \text{span}\{e_{-1}, e_{-2}, \ldots\}\), \(X_0 = \ker(A - B)\). Hence there exists a decomposition \(X = X_+ \oplus X_0 \oplus X_-\) with \(\dim X_0 < \infty\), \(\dim X_+ = \dim X_- = \infty\) and an equivalent inner product in \(X\), denoted by \((\cdot, \cdot)\), for \(u = u^+ + u^0 + u^-\) and \(v = v^+ + v^0 + v^- \in X = X_+ \oplus X_0 \oplus X_-\), define

\[
(u, v) = ((A - B)u^+, v^+)_X - ((A - B)u^-, v^-)_X + (u^0, v^0)_X.
\]

Hence, we have

\[
\int_0^T (J\dot{u} - B(t)u) \, dt = ((A - B)u, u)_X = \|u^+\|^2 - \|u^-\|^2.
\]

Note that \(\dim X_0 > 0\) if and only if the boundary value problem

\[
-J\ddot{z} = B(t)z, \quad z(0) = z(T)
\]

has at least a nontrivial solution.

Set \(\alpha_0 = \min_{|z| = r_0, t \in S_T} |H(t, z)|, \beta_0 = \max_{|z| \leq r_0, t \in S_T} |H(t, z)|\). Conditions (H1) and (H2) imply that for some \(\beta_1 \geq 0\)

\[
\alpha_0 |z|^\mu \leq H(t, z), \quad \forall |z| \geq r_0,
\]

\[
\alpha_0 |z|^\mu \leq H(t, z) + \beta_0 \leq \frac{1}{\mu} \langle \nabla H(t, z)z + \beta_3 \rangle, \quad \forall z \in \mathbb{R}^{2N}.
\]

Modifying [5] (cf. appendix of [5]), choose \(\sigma \in (0, 1)\), such that \(\mu \sigma > 2\), we truncate \(H\) as in the following proposition:

**Proposition 2.1.** Assume conditions (H1) and (H2), then there exist two sequences \(\{K_n\}\) and \(\{K'_n\}\) in \(\mathbb{R}\) and a sequence of functions \(\{H_n\}\) such that

(i) \(0 < K_0 < K_n < K_{n+1}, \forall n \in \mathbb{N}\), and \(K_n \to \infty\), as \(n \to \infty\), where \(K_0 = \max\{1, r, \beta_0/\alpha_0(1 - \sigma)\}\); and \(K_n < K'_n, \forall n \in \mathbb{N}\).

(ii) \(H_n(t, \cdot) \in C(S_T \times \mathbb{R}^{2N}, \mathbb{R})\) and for any given \(t \in S_T\), \(H_n(t, \cdot) \in C^1(\mathbb{R}^{2N}, \mathbb{R})\), for every \(n \in \mathbb{N}\).

(iii) \(H_n(t, z) = H(t, z), \forall |z| \leq K_n, \forall n \in \mathbb{N}\); and \(H_n(t, z) = (\tau_n + 1)z|z|^\mu, \forall |z| \geq K'_n, \forall n \in \mathbb{N}\).
(iv) $H_n(t,z) \leq H_{n+1}(t,z) \leq H(t,z)$, $\forall (t,z) \in S_T \times \mathbb{R}^{2N}$.
(v) $0 < \mu \sigma H_n(t,z) \leq \nabla H_n(t,z) \cdot \frac{\partial}{\partial z}$, $\forall |z| \geq r_0$, for every $n \in \mathbb{N}$.

Note that in [5] the truncating functions are constructed for autonomous Hamiltonian functions. But the proof also works for time-dependent $H(t,z)$.

Now integrating (v) yields

$$H_n(t,z) \geq a|z|^p - b, \quad \forall z \in \mathbb{R}^{2N},$$

for some $n$-independent constants $a$ and $b$. Let $\Psi_n(u) = \int_0^T H_n(t,u) \, dt$. Define a functional $I_n : X \to \mathbb{R}$ by

$$I_n(u) = \frac{1}{2} \int_0^T (-Ju' - B(t)u)u \, dt - \int_0^T H_n(t,u) \, dt$$

$$= \frac{1}{2} \left( \|u^+\|^2 - \|u^-\|^2 \right) - \Psi_n(u).$$

It is well known that $I_n \in C^1(X, \mathbb{R})$, and

$$\langle I'_n(u), v \rangle = \int_0^T (-Ju' - B(t)u)v \, dt - \int_0^T \nabla H_n(t,u)v \, dt$$

$$= \langle u^+ - u^-, v \rangle - \langle \Psi'_n(u), v \rangle$$

and $\Psi'_n$ is compact as in [12]. So finding $T$-periodic solutions of Eq. (1) with $H$ replaced by $H_n$ is equivalent to finding critical points of $I_n$ in $X$.

We will use Theorem 1.3 of [2] to prove that $I_n$ has a critical point $u_n$ which is different from 0. Similarly to the proof of [2], it is easy to show that the functional $I_n$ satisfies (I2), (I3) and (I4) in Theorem 1.3 of [2] without using (H4) or (H5). Different from [2], we also prove (I1) without using (H4) or (H5) as the following.

**Lemma 2.1.** $I_n$ satisfies (PS)*.

**Proof.** Suppose $\{u_k\}$ is a sequence in $X$ such that

$$u_k \in X, \quad I_n(u_k) \leq C < \infty \quad \text{and} \quad P_k I'_n(u_k) \to 0 \quad \text{as} \quad k \to \infty.$$

Then for large $n$ and $v = u_k$,

$$C + \|u_k\|_{X} \geq I_n(u_k) - \frac{1}{2} \langle P_k I'_n(u_k), u_k \rangle$$

$$= \int_0^T \left( \frac{1}{2} \nabla H_n(t,u_k)u_k - H_n(t,u_k) \right) \, dt$$

$$\geq \left( \frac{1}{2} - \frac{1}{\mu \sigma} \right) \int_0^T \nabla H_n(t,u_k)u_k \, dt - c_1$$
\[
\geq \left( \frac{\mu \sigma}{2} - 1 \right) \int_0^T H_n(t, u_k) \, dt - c_2
\]
\[
\geq c_3 \| u_k \|_{L^{\mu \sigma}}^{\mu \sigma} - c_4
\]
via (H2) and the growth of \( H_n \) at infinity. Writing
\[
u_k = u_k^+ + u_k^- + u_0 \in X_+ \oplus X_- \oplus X_0.
\]
Because \( X_0 \) is a finite-dimensional space, it follows from Eq. (3) that
\[
\| u_0 \|_X \leq c_5 (1 + \| u_k \|_X^{1/\mu \sigma})^2
\]
Taking \( v = u_k^+ \) in the inequality \( |\langle P_k I'_k(u_k), v \rangle| \leq \| v \| \) (which holds for large \( n \)), we have
\[
\| u_k^+ \|_X^2 - \left| \int_0^T \nabla H_n(t, u_k) u_k^+ \, dt \right| \leq \| u_k \|_X^2
\]
Using the Hölder inequality and \( \| u \|_{L^{\mu \sigma}} \leq C_{\mu \sigma} \| u \|_X \), by Eq. (3) we have
\[
\| u_k^+ \|_X^2 \leq \left\{ \int_0^T |\nabla H_n(t, u_k)|^{\mu \sigma/(\mu \sigma - 1)} \, dt \right\}^{(\mu \sigma - 1)/\mu \sigma} \| u_k^+ \|_{L^{\mu \sigma}}^\mu + \| u_k^+ \|_X^\mu
\]
\[
= \left\{ \int_{|u| \leq \kappa_{n+1}} + \int_{|u| > \kappa_{n+1}} \left| \nabla H_n(t, u_k) \right|^{\mu \sigma/(\mu \sigma - 1)} \, dt \right\}^{(\mu \sigma - 1)/\mu \sigma}
\]
\[
\times \| u_k^- \|_{L^{\mu \sigma}}^\mu + \| u_k^- \|_X^\mu
\]
\[
\leq \{ C_0(n) + (\mu \sigma R)^{\mu \sigma/(\mu \sigma - 1)} \} \| u_k \|_{L^{\mu \sigma}}^{(\mu \sigma - 1)/\mu \sigma} \| u_k^+ \|_{L^{\mu \sigma}}^\mu + \| u_k^+ \|_X^\mu
\]
\[
\leq C_4(n) (1 + \| u_k \|_{L^{\mu \sigma}}^{(\mu \sigma - 1)/\mu \sigma}) \| u_k^+ \|_X,
\]
i.e.,
\[
\| u_k^+ \|_X \leq C_4(n) (1 + \| u_k \|_{L^{\mu \sigma}}^{(\mu \sigma - 1)/\mu \sigma}) \leq C_2(n) (1 + \| u_k \|_X^{(\mu \sigma - 1)/\mu \sigma}),
\]
where \( C_i(n) \)’s are constants depending on \( n \). Similarly, for \( v = u_k^- \) we have
\[
\| u_k^- \|_X \leq C_3(n) (1 + \| u_k \|_X^{(\mu \sigma - 1)/\mu \sigma}),
\]
Hence,
\[
\| u_k \|_X \leq C_4(n) (1 + \| u_k \|_X^{(\mu \sigma - 1)/\mu \sigma})
\]
i.e., \( \{ u_k \} \) is bounded on \( X \). Since
\[
\| u_k^+ - u_k^- \|_{L^{\mu \sigma}} = P_k \Psi'_k(u_k) = P_k I'_k(u_k) \to 0 \quad \text{as} \quad k \to \infty,
\]
\(Y. \text{ Long, X. Xu / Nonlinear Analysis 41 (2000) 455–463}\)

\(\Psi_0\) is a compact operator, and \(\{u^0_k\} \subset X^0\) is bounded, \(\{u_k\}\) has a convergence subsequence, i.e., \((PS)^*\) holds.

**Proof of Theorem 1.1.** By our above discussions, \(I_n\) satisfies the hypotheses of Theorem 1.3 of [2]. So \(I_n\) possesses a nontrivial critical point \(u_n\). We shall prove \(\|u_n\|_C \leq K_n\) for large \(n\).

We first prove that there is a constant \(M > 0\) such that \(I_n(u_n) \leq M\), for every \(n \in \mathbb{N}\). If every one of \(\{u_n\}\) is gained in the first case in the proof of Theorem 1.3 of [2] (p. 228), \(I_n(u_n) < 0\) holds for every \(n \in \mathbb{N}\). Otherwise, there exists an \(n_0\) such that \(u_{n_0}\) is gained in the second case. Note that \(I_n \leq I_{n_0}\) for \(n > n_0\) (since \(H_n \geq H_{n_0}\) for \(n > n_0\)), we replace \(I_{n_0}\) by \(I_n\) only in the proof of the Theorem 1.3 of [2] (pp. 228–230), and use the same \(\Phi, \Gamma, \Phi', \mathcal{H}, m, q\) as gained for \(I_{n_0}\) and \(B^m_1\) for \(I_n\). Then we can gain a critical point \(u_n\) of \(I_n\) such that \(\|u_n\|_C \leq K_n\) for large \(n\).

Now we show that \(\|u_n\|_C \leq K_n\) for large \(n\). Since \(I_n(u_n) = 0\), similarly to Eq. (3) we have

\[
\int_0^T \nabla H_n(t, u_n) u_n \, dt \leq M_1, \quad \int_0^T H_n(t, u_n) \, dt \leq M_2
\]

for some constants \(M_1\) and \(M_2\) independent of \(n\).

Denote by \(\tilde{H}_n(t, z) = \frac{1}{2} \langle B(t) z, z \rangle + H_n(t, z)\). Then (H1), (H2) and (H4) also hold for \(\tilde{H}_n\) with some \(\tilde{\mu}, \tilde{r}\) independent of \(n\) and the same \(\tilde{a}\). Thus we can omit \(\langle B(t) z, z \rangle\) in the following proof.

Denote

\[A_n = \{t \in S_T \mid |u_n(t)| < K_n\}.\]

By Eq. (4) we have

\[M_2 \geq \int_0^T H_n(t, u_n) \, dt \geq \bar{c}_0 \|u_n\|_{L^m}^m + b\]

for some \(n\)-independent constant \(b\). Thus we know for large \(n\), \(A_n \neq \emptyset\) and \(\text{measure}(A_n) > T/2\). Since \(u_n \in C^1\), \(A_n\) is open. Let \(A_n = \bigcup_{l=1}^\infty (a_n, b_n, j)\). It suffices to prove \(A_n = S_T\).

We prove this indirectly by assuming that this claim fails in a subsequence of \(\{A_n\}\). Without loss generality, we still denote this subsequence by \(\{A_n\}\). By Eq. (4), \(H_n(t, u_n)\) is \(H(t, u_n)\) and \(K_n > r\), we have

\[M_2 \geq \int_0^T H_n(t, u_n) \, dt \geq \int_{A_n} H_n(t, u_n) \, dt = \sum_{j=1}^\infty \int_{a_n, j}^{b_n, j} H(t, u_n) \, dt.
\]

For every \((a_{n, j}, b_{n, j})\), let

\[B^n_j = \{t \in (a_{n, j}, b_{n, j}) \mid H(t, u_n(t)) < H(a_{n, j}, u_n(a_{n, j}))\} = \bigcup_{l=1}^\infty (c^n_{l, j}, d^n_{l, j}).\]
We have $H(a_{n,j}, u_n(a_{n,j})) = H(c_{l}^{j}, u_n(c_{l}^{j})) = H(d_{l}^{j}, u_n(d_{l}^{j}))$, $\forall l \in \mathbb{N}$. Thus,

$$
\int_{a_{n,j}}^{b_{n,j}} H(t, u_n) \, dt
\geq (b_{n,j} - a_{n,j}) H(a_{n,j}, u_n(a_{n,j})) + \int_{B_{r}^{n}} [H(t, u_n(t)) - H(a_{n,j}, u_n(a_{n,j}))] \, dt
$$

$$
= (b_{n,j} - a_{n,j}) H(a_{n,j}, u_n(a_{n,j})) + \sum_{j=1}^{\infty} \int_{c_{l}^{j}}^{d_{l}^{j}} \int_{c_{l}^{j}}^{d_{l}^{j}} H(s, u_n(s)) \, ds \, dt,
$$

the last equality holds since $\dot{u}_n = J \nabla H(t, u_n)$. By (H4) there exists $N > r$ independent of $n$ such that

$$
\frac{H(t, z)}{H(t, z)} > -\frac{1}{T} + \frac{\tilde{a}}{2}, \quad \forall |z| > N.
$$

When $|u_n(s)| \geq N$ and $H_s(s, u_n(s)) < 0$ for $s \in B_{r}^{n}$, we have

$$
\frac{H_s(s, u_n(s))}{H(a_{n,j}, u_n(a_{n,j}))} \leq \frac{H_s(s, u_n(s))}{H(s, u_n(s))} \leq -\frac{1}{T} + \frac{\tilde{a}}{2}.
$$

Let $\beta = \min_{s \in S, |z| \leq N} \{H_s(s, z), 0\}$, then $\beta$ is finite and independent of $n$. Hence we have

$$
\int_{a_{n,j}}^{b_{n,j}} H(t, u_n) \, dt \geq H(a_{n,j}, u_n(a_{n,j})) \left\{ (b_{n,j} - a_{n,j}) + \int_{Q_1}^{Q_2} \int_{Q_2}^{Q_3} H(s, u_n(s)) \, ds \, dt \right\}
$$

$$
+ \left( \int_{Q_1}^{Q_2} + \int_{Q_2}^{Q_3} \right) H_s(s, u_n(s)) \, ds \, dt,
$$

where

$$
Q_1 = \{ s \in B_{r}^{n} \mid |u_n(s)| > N, H_s(s, u_n(s)) < 0 \},
$$

$$
Q_2 = \{ s \in B_{r}^{n} \mid |u_n(s)| > N, H_s(s, u_n(s)) \geq 0 \},
$$

$$
Q_3 = \{ s \in B_{r}^{n} \mid |u_n(s)| \leq N \}.
$$

Then we have

$$
\int_{a_{n,j}}^{b_{n,j}} H(t, u_n) \, dt
\geq H(a_{n,j}, u_n(a_{n,j})) \left[ (b_{n,j} - a_{n,j}) - \sum_{j=1}^{\infty} \int_{c_{l}^{j}}^{d_{l}^{j}} \int_{c_{l}^{j}}^{d_{l}^{j}} \left( -\frac{1}{T} + \frac{\tilde{a}}{2} \right) \, ds \, dt \right] + \sum_{j=1}^{\infty} \int_{c_{l}^{j}}^{d_{l}^{j}} \int_{c_{l}^{j}}^{d_{l}^{j}} \beta \, ds \, dt
$$

$$
\geq \left[ (b_{n,j} - a_{n,j}) - \frac{1}{4} (b_{n,j} - a_{n,j}) \right] \left( \frac{2}{T} - \tilde{a} \right) H(a_{n,j}, u_n(a_{n,j})) + \frac{(b_{n,j} - a_{n,j})^2 \beta}{2}.
$$
\[
\geq \left[ (b_{n,j} - a_{n,j}) - \frac{1}{4} (b_{n,j} - a_{n,j})^2 \left( \frac{2}{T} - \tilde{a} \right) \right] (\alpha_0 |u_n(a_{n,j})|^\mu - b) + \frac{(b_{n,j} - a_{n,j})^2 \beta}{2} \\
\geq (b_{n,j} - a_{n,j}) \left[ \frac{2 + T\tilde{a}}{4} (\alpha_0 K_n^\mu - b) + \frac{T\beta}{2} \right].
\]

Thus, we have
\[
M_2 \geq \sum_{j=1}^{\infty} \int_{a_{n,j}}^{b_{n,j}} H(t, u_n) \, dt \geq \sum_{j=1}^{\infty} (b_{n,j} - a_{n,j}) \left[ \frac{2 + T\tilde{a}}{4} (\alpha_0 K_n^\mu - b) + \frac{T\beta}{2} \right] \\
\geq \frac{T}{2} \left[ \frac{2 + T\tilde{a}}{4} (\alpha_0 K_n^\mu - b) + \frac{T\beta}{2} \right].
\]

Since \(\mu > 2\), \(2/T + \tilde{a} > 0\) and \(K_n \to \infty\) as \(n \to \infty\), we have a contradiction. Hence \(\|u_n\|_C \leq K_n\) for large \(n\). Since \(H_n(t, u_n) = H(t, u_n)\) for \(\|u_n\|_C \leq K_n\), we have that \(u_n\) is a nontrivial solution of Eq. (1) for large \(n\). Hence Theorem 1.1 is proved. \(\Box\)

**Proof of Theorem 1.2.** As in the proof of Theorem 1.1, we have Eq. (4) and \(A_n = \bigcup_{j=1}^{\infty} (a_{n,j}, b_{n,j})\) for large \(n\). By passing a subsequence, assume \(A_n \neq S_T\), for \(n \in \mathbb{N}\). Otherwise, we have the conclusion. From Eq. (4) and (H5)
\[
M_1 \geq \int_0^T \nabla H_n(t, u_n) u_n \, dt \geq \int_{A_n} \nabla H(t, u_n) u_n \, dt \geq \frac{1}{c} \int_{A_n} (|\nabla H(t, u_n)| - d) \, dt.
\]

Thus, we have
\[
\sum_{j=1}^{\infty} \int_{a_{n,j}}^{b_{n,j}} |\dot{u}_n(t)| \, dt = \int_{A_n} |\nabla H(t, u_n)| \, dt \leq cM_1 + dT.
\]

For \(t \in (a_{n,j}, b_{n,j})\), we have
\[
|u_n(t)| - |u_n(a_{n,j})| \geq - \int_{a_{n,j}}^t |\dot{u}_n(s)| \, ds \geq -(cM_1 + dT),
\]
i.e., \(|u_n(t)| \geq K_n - (cM_1 + dT)\). By Eq. (4) we have
\[
M_2 \geq \int_0^T H_n(t, u_n) \, dt \geq \int_0^T (\alpha_0 |u_n|^\mu - b) \, dt
\]

\[ \geq \int_0^T \left[ z_0(K_n - cM_1 - dT)^{\mu T} - b \right] dt \]

\[ = T \left[ z_0(K_n - cM_1 - dT)^{\mu T} - b \right]. \]

Since \( z_0, b, c, d, \mu, M_1 \) are independent of \( n \) and \( K_n \to \infty \) as \( n \to \infty \), we have a contradiction. Hence \( A_n = S_T \) for large \( n \), i.e., \( \|u_n\|_C \leq K_n \). Since \( H_n(t, u_n) = H(t, u_n) \) for \( \|u_n\|_C \leq K_n \), we have that \( u_n \) is a nontrivial solution of Eq. (1) for large \( n \). Hence Theorem 1.2 is proved.

References