Spectral Expansions of Piecewise Smooth Functions on compact Riemannian manifolds with boundary

Xiangjin Xu*
Department of Mathematics
Johns Hopkins University
Baltimore, MD 21218

Abstract

The purpose of this note is to study conditions for the convergence and Riesz means of spectral expansions of piecewise smooth functions on compact Riemannian manifolds (M, g) of dimension $n \geq 2$ with boundary.

1 Introduction

The purpose of this note is to study conditions for the convergence and Riesz means of spectral expansions of piecewise smooth functions on compact Riemannian manifolds (M, g) of dimension $n \geq 2$ with boundary. Here we say that a function $f(x)$ on M is piecewise smooth if it is uniformly continuous in M, and has uniformly derivatives in M up to order $l \geq 0$.

Let $L = -\Delta_g$ be the Laplace-Beltrami operator associated to the Riemannian metric g. Recall that the spectrum of Laplacian is discrete and tends to infinity. Let $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ denote the eigenvalues, and let $\{e_j(x)\}$ be an associated real orthogonal normalized basis in $L^2(M)$, and define

$$e_j(f)(x) = e_j(x) \int_M f(y) e_j(y) dy,$$

and the unit band spectral projection operators,

$$\chi_\lambda f = \sum_{\sqrt{\lambda_j} \in [\lambda, \lambda+1)} e_j(f),$$

*email: xxu@math.jhu.edu

and E_λ is the corresponding expansion of the identity, then the spectral expansion of any function $f \in L^2(M)$ has the form

$$E_\lambda f(x) = \int_M e(x, y, \lambda) f(y) dy, \tag{1}$$

where

$$e(x, y, \lambda) = \sum_{\sqrt{\lambda_j} \leq \lambda} e_j(x) e_j(y).$$

is the spectral function of the Laplacian. For each $s > 0$, we introduce the Riesz means

$$S^s_\lambda f(x) = \int_0^\lambda (1 - \frac{t}{\lambda})^s dE_t f(x), \tag{2}$$

of the spectral expansion.

In [10], Sogge proved that for a fixed compact Riemannian manifold (M, g) with boundary, there is a uniform constant C so that

$$||\chi_\lambda f||_\infty \leq C\lambda^{(n-1)/2} ||f||_2, \quad \lambda \geq 1,$$

which is equivalent to

$$\sum_{\sqrt{\lambda_j} \in [\lambda, \lambda+1)} e_j(x)^2 \leq C\lambda^{n-1}, \quad \forall x \in M.$$

And he use this estimate to prove some new estimates for Riesz means of eigenfunctions on compact manifolds with boundary, for $s > (n-1)/2$, one has the uniformly bounds

$$||S^f_\lambda||_p \leq C||f||_p,$$

for every $1 \leq p \leq \infty$.

In [1], the author studies conditions for the convergence and Riesz summability of spectral expansions of piecewise smooth functions for self-adjoint elliptic operators on the compact subdomain of a n-dimensional domain. In [5] and [6], the authors obtained some necessary and sufficient conditions for the convergence of Fourier inversion and spectral expansion of the Laplace operator of a rotationally invariant Riemannian manifold by using the asymptotic properties of corresponding special functions. And in [7], the authors use a wave equation approach to study point-wise Fourier inversion and point-wise convergence or divergence of spectral expansion of Laplace operator of Riemannian manifolds with some symmetry, including spheres, hyperbolic spaces and other compact and noncompact rank-one symmetric space, and on strongly scattering manifolds.

Now for general compact Riemannian manifolds, one can't use the asymptotic properties of special functions to study the asymptotic behavior of spectral functions any more. Here we use Sogge's asymptotic L^∞ estimates on χ_λ and L^2 estimates on the normal derivative of eigenfunctions on the boundary of [2], instead of the asymptotic properties of special functions, to study the asymptotic behavior of spectral functions, and we obtain the following results,

Theorem 1.1 *Let f be a piecewise smooth function on a manifold M with boundary, $\dim M = n$, satisfying $f \in L^2(M)$, more precise, we inquire $f \in L^2_{n-1-2s}(M)$, we have*

(1). $n = 2$, on each compact subset of the smoothness domain of f, the spectral expansion (1) is uniformly bounded and the Riesz means (2) of any positive order $s > 0$ uniformly converge to f.

(2). $n > 2$, on each compact subset of the smoothness domain of f, the Riesz means (2) of any positive order $s \geq (n-1)/2$ uniformly converge to f.

Remark 1.1 *For $n > 2$, there are some simple examples, such as the characteristic function of unit ball χ_B in $\mathbf{R}^n$, see [5], show that the spectral expansion of a piecewise smooth function may diverge even at points far from the discontinuity surface, and, if $n > 3$, the divergence will be unbounded.*

Now applying the uniformly bounds for Riesz means on $L^p(M)$ in [10] and a density argument, from Theorem 1.1, we have the following almost everywhere convergence results for Riesz means on $L^p(M)$.

Theorem 1.2 *Fix a smooth compact Riemannian manifold with boundary of dimension $n \geq 2$, for any $s > (n-1)/2$, let $f \in L^p(M) \cap L^2_{n-1-2s}(M)$, $1 \leq p \leq \infty$, we have*

$$\lim_{\lambda\to\infty} S^s_\lambda f(x) = f(x), \quad \textit{almost everywhere for } x \in M.$$

And for any $s > (n-1)/2$, let $f \in L^p(M)$, $1 \leq p \leq \infty$, we have

$$\lim_{\lambda\to\infty} S^s_\lambda f(x) = f(x), \quad \textit{in measure for } x \in M.$$

In what follows we shall use the convention that C will denote a constant that is not necessarily the same at each occurrence.

2 Proof of Theorem 1.1 and Theorem 1.2

In this section, we shall prove Theorem 1.1. For each $\tau \geq 0$, we introduce the kernel

$$G_\tau(x,y) = \int_0^\infty \lambda^{-\tau} d_\lambda e(x,y,\lambda)$$

of fractional order. In this notation, we can see $G_1(x,y)$ is the Green function of the Laplacian for the eigenvalue problem. And for any smooth function defined on M, we have the following equality

$$\int_{\partial M} (\partial_\nu G_\tau(x,y)) f(y) ds(y) = \sum_{j=1}^{\infty} e_j(x) \lambda_j^{-\tau} \int_S (\partial_\nu (e_j(y)) f(y) d\sigma, \tag{3}$$

where σ is the area element on surface ∂M, and ν is the outward normal direction on the boundary. In [2], the authors have the following results: for the inequality

$$c\lambda_j \leq ||\partial_\nu e_j||^2_{L^2(\partial M)} \leq C\lambda_j,$$

the upper bound holds for some constant C independent of λ_j, and the lower bound holds provided that M can be embedded in the interior of a compact manifold with boundary, N, of the same dimension, such that every geodesic in M eventually meets the boundary of N. In particular, the lower boundary holds if M is a sub-domain of Euclidean space. Here we use the idea of proving the upper bounds in [2], we obtain the following estimates:

Lemma 2.1 *For any smooth function f on M, the estimates*

$$(a) \quad \sum_{\sqrt{\lambda_j} \in [\lambda, \lambda+1)} \left| \int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma \right|^2 \leq C\lambda^{n+1} ||f||^2_{L^2(\partial M)},$$

$$(b) \quad \sum_{\sqrt{\lambda_j} \in [\lambda, \lambda+1)} \left| \int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma \right|^2 \leq 2(\lambda+1)^4 ||\chi_\lambda f||^2_{L^2(M)} + 2||\chi_\lambda(\Delta f)||^2_{L^2(M)},$$

hold both as $\lambda \to \infty$.

Proof. For estimate (a), we use the upper bound for $||\partial_\nu e_j||_{L^2(\partial M)}$ in [2], and by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\sum_{\sqrt{\lambda_j} \in [\lambda, \lambda+1)} \left| \int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma \right|^2 &\leq \sum_{\sqrt{\lambda_j} \in [\lambda, \lambda+1)} \int_{\partial M} (\partial_\nu e_j(y))^2 d\sigma \int_{\partial M} f(y)^2 d\sigma \\
&\leq ||f||^2_{L^2(\partial M)} \sum_{\sqrt{\lambda_j} \in [\lambda, \lambda+1)} ||\partial_\nu e_j||^2_{L^2(\partial M)} \\
&\leq C\lambda^{n+1} ||f||^2_{L^2(\partial M)}
\end{aligned}$$

For the last inequality we use the Weyl formula

$$\sharp\{\lambda_j \ : \ \sqrt{\lambda_j} \in [\lambda, \lambda+1)\} = C\lambda^{n-1} + o(\lambda^{n-1}).$$

For estimate (b), by the Green's formula, we have

$$\begin{aligned}\int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma &= \int_M \Delta e_j(y) \cdot f(y) dy - \int_M e_j(y) \cdot \Delta f(y) dy \\ &= -\lambda_j \int_M e_j(y) \cdot f(y) dy - \int_M e_j(y) \cdot (\Delta f(y)) dy.\end{aligned}$$

Notice that $\int_M e_j(y) \cdot f(y) dy$ is the Fourier coefficient of f with respect to the spectral decomposition, then we have

$$\begin{aligned}&\sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} |\int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma|^2 \\ \leq \ & 2 \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} [\lambda_j^2 (\int_M e_j(y) \cdot f(y) dy)^2 + (\int_M e_j(y) \cdot (\Delta f(y)) dy)^2] \\ \leq \ & 2(\lambda+1)^4 ||\chi_\lambda f||^2_{L^2(\partial M)} + 2||\chi_\lambda (\Delta f)||^2_{L^2(\partial M)},\end{aligned}$$

Q.E.D.

Applying the L^∞ estimates on χ_λ in [10], we have the following Lemma:

Lemma 2.2 *For any given smooth function f on M, the estimates*

$$\begin{aligned}(a) \quad & \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} |e_j(x) \int_{\partial M} \partial_\nu e_j(y) f(y) ds(y)| \leq C\lambda^n ||f||_{L^2(\partial M)} \\ (b) \quad & \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} |e_j(x) \int_{\partial M} \partial_\nu e_j(y) f(y) ds(y)| \leq C(\lambda+1)^{(n+3)/2} ||\chi_\lambda f||_{L^2(M)} \\ & \qquad + C(\lambda+1)^{(n-1)/2} ||\chi_\lambda (\Delta f)||_{L^2(M)}\end{aligned}$$

hold both as $\lambda \to \infty$.

Proof. From [10], we have estimates

$$\sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} e_j(x)^2 \leq C\lambda^{n-1},$$

then by the Cauchy-Schwarz inequality, we get the results from Lemma 2.1.

Q.E.D.

Lemma 2.3 *Let $\tau > 0$, then for any smooth function f and any constant $h \in (0,1)$, the estimates*

$$\begin{aligned} (a) \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} |\lambda_j^{-2\tau} e_j(x) \int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma| &\leq C\lambda^{n-2\tau} ||f||_{L^2(\partial M)} \\ (b) \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} |\lambda_j^{-2\tau} e_j(x) \int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma| &\leq C(\lambda+1)^{\frac{n+3}{2}-2\tau} ||\chi_\lambda f||_{L^2(M)} \\ &\quad + C(\lambda+1)^{\frac{n-1}{2}-2\tau} ||\chi_\lambda(\Delta f)||_{L^2(M)} \end{aligned}$$

hold both as $\lambda \to \infty$.

Proof of Theorem 1.1 Given a piecewise smooth function f on M. Let us fix an arbitrary compact set $K \subset M - \partial M$ and consider a smooth function $f_\tau(x)$ with compact support in M such that in some neighborhood U of the compact subset K, we have

$$f_\tau(x) = \int_{\partial M} \partial_\nu G_\tau(x,y) f(y) d\sigma, \quad x \in U.$$

From (3), the right hand side has the spectral expansion

$$\int_{\partial M} \partial_\nu G_\tau(x,y) f(y) d\sigma = \sum_{j=1}^{\infty} \lambda_j^{-\tau} e_j(x) \int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma. \tag{4}$$

Consider the function

$$\phi_\tau(\lambda) = \phi_\tau(\lambda, x) = \sum_{\lambda_j \leq \lambda} \lambda_j^{-\tau} e_j(x) \int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma - E_\lambda f_\tau(x).$$

We can see that the Fourier-Stieltjes transform

$$\Phi(\xi) = \int_0^\infty e^{-it\xi} d\phi_\tau(t^2)$$

is bounded in a neighborhood of zero and vanishes at $\xi = 0$ together with all its derivatives of even order.

Lemma 2.3 implies that if $x \in K$, we have

$$\begin{aligned} &(a) \quad |\phi_\tau((t+h)^2) - \phi_\tau(t^2)| \leq C t^{n-2\tau} ||f||_{L^2(\partial M)}, \quad as\ t \to \infty, \\ &(b) \quad |\phi_\tau((t+h)^2) - \phi_\tau(t^2)| \leq C(t+1)^{\frac{n+3}{2}-2\tau} ||\chi_t f||_{L^2(M)} \\ &\qquad + C(t+1)^{\frac{n-1}{2}-2\tau} ||\chi_t(\Delta f)||_{L^2(M)}, \quad as\ t \to \infty, \end{aligned}$$

for any $h \in [0,1]$. According to the Tauberian Theorem of Hörmander (Lemma 17.5.6 in [4]), for Riesz means, we have the following estimates

$$
\begin{aligned}
(a) \quad & |\int_0^\lambda (1-\frac{t}{\lambda})^s d\phi_\tau(t)| \leq C\lambda^{n-2\tau-s}||f||_{L^2(\partial M)}, \quad as\ t \to \infty, \\
(b) \quad & |\int_0^\lambda (1-\frac{t}{\lambda})^s d\phi_\tau(t)| \leq C\lambda^{\frac{n+3}{2}-2\tau-s}||E_\lambda f||_{L^2(M)} \\
& +C\lambda^{\frac{n-1}{2}-2\tau-s}||E_\lambda(\Delta f)||_{L^2(M)}, \quad as\ \lambda \to \infty,
\end{aligned}
$$

for all $s \geq 0$.

Now we set $\tau = 1$, since $G_1(x,y)$ is the Green's function of the Laplacian for the eigenvalue problem, we have $f_1(x) = f(x)$ for all $x \in U$. The left term in the last inequalities (a) and (b) is the error term for the spectral expansion $E_\lambda f(x)$ and the Riesz means $S_\lambda^s f(x)$ to the $f(x)$.

For $n = 2$, the estimate (a) is better than the estimate (b), and implies the assertion (1) of Theorem 1.1.

For $n > 2$, the estimate (b) is better than the estimates (a), and implies the assertion (2) of Theorem 1.1., here we need use the properties, for $f \in L^2_{n-1-2s}(M)$, which ensures $\Delta f \in L^2_{n-5-2s}(M)$,

$$
\begin{aligned}
\lambda^{n-1-2s}||E_\lambda f||^2_{L^2(M)} &\to 0, \quad as\ \lambda \to \infty \\
\lambda^{n-5-2s}||E_\lambda(\Delta f)||^2_{L^2(M)} &\to 0, \quad as\ \lambda \to \infty.
\end{aligned}
$$

It follows from when $f \in L^2_{n-1-2s}(M)$, one has

$$
||f||^2_{L^{n-1-2s}(M)} = \sum_{k=1}^{\infty} k^{n-1-2s}||\chi_k f||^2_{L^2(M)} < \infty.
$$

The same reason for $\Delta f \in L^2_{n-5-2s}(M)$. Q.E.D.

For Theorem 1.2, we know that for smooth functions we have the almost everywhere convergence on M from Theorem 1.1. Notice that $C^\infty(M)$ is dense on $L^p(M)$ for any $1 \leq p \leq \infty$. Approximated any $f \in L^p(M)$ by smooth functions $\{f_k\}$, using the uniform bounds results of Riesz means on $L^p(M)$ in [10], we have convergence for Riesz means $S_\lambda^s f(x)$ for any $f \in L^p(M)$ in measure. When we further assume that $f \in L^2_{n-1-2s}(M)$, we can let smooth functions $\{f_k\}$ approximate f in $L^2_{n-1-2s}(M)$, the from proof Theorem 1.1, we know that for all f_k, the Riesz means $S_\lambda^s f_k(x)$ uniformly converge to $f_k(x)$ as $\lambda \to \infty$, in any compact subset of $M - \partial M$ for all $k \in \mathbf{N}$, which ensures the almost everywhere convergence for $S_\lambda^s f(x)$ to $f(x)$ in M.

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