Gradient estimates for the eigenfunctions on compact manifolds with boundary and Hörmander multiplier Theorem

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Abstract

On compact Riemannian manifolds \((M, g)\) of dimension \(n \geq 2\) with boundary, the gradient estimates for the eigenfunctions of the Dirichlet Laplacian are proved. Using the \(L^\infty\) estimates and gradient estimates, the Hörmander multiplier theorem is showed for the Dirichlet Laplacian on compact manifolds with boundary.

Keywords: gradient estimate, unit band spectral projection operator, Dirichlet Laplacian, Hörmander multiplier theorem
1 Introduction

Let \((M, g)\) be a compact Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) with smooth boundary \(\partial M\). The purpose of this paper is to give a proof of sharp gradient estimates for eigenfunctions of the Dirichlet Laplacian on \(M\) and then to use these estimates to prove Hörmander multiplier theorem in this setting. Thus, we shall consider the Dirichlet eigenvalue problem

\[
(\Delta + \lambda^2)u(x) = 0, \quad x \in M, \quad u(x) = 0, \quad x \in \partial M, \tag{1}
\]

with \(\Delta = \Delta_g\) being the Laplace-Beltrami operator associated to the Riemannian metric \(g\). Recall that the spectrum of \(-\Delta\) is discrete and tends to infinity. Let \(0 < \lambda_1^2 \leq \lambda_2^2 \leq \lambda_3^2 \leq \cdots\) denote the eigenvalues, so that \(\{\lambda_j\}\) is the spectrum of the first order operator \(P = \sqrt{-\Delta}\). Let \(\{e_j(x)\} \subset L^2(M)\) be an associated real orthonormal basis, and let

\[
e_j(f)(x) = e_j(x) \int_M f(y)e_j(y)dy
\]

be the projection onto the \(j\)-th eigenspace. Here and in what follows \(dy\) denotes the volume element associated with the metric \(g\). And define the unit band spectral projection operators,

\[
\chi_{\lambda}f = \sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(f).
\]

Then any function \(f \in L^2(M)\) can be written as

\[
f = \sum_{j=1}^{\infty} e_j(f) = \sum_{k=1}^{\infty} \chi_kf,
\]

where the partial sums converge in \(L^2(M)\), and, moreover,

\[
\|f\|_{L^2(M)}^2 = \sum \|e_j(f)\|_{L^2(M)}^2 = \sum \|\chi_kf\|_{L^2(M)}^2.
\]
The study for $L^p$ estimates for the eigenfunctions on compact manifolds has a long history. In the case of manifolds without boundary, the most general results of the form

$$||\chi_\lambda f||_p \leq C\lambda^{\sigma(p)}||f||_2, \quad \lambda \geq 1, \quad p \geq 2,$$

(2)

where

$$\sigma(p) = \max\{\frac{n-1}{2} - \frac{n}{p}, \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)\}$$

were proved in [13]. These estimates cannot be improved since one can show that the operator norm satisfy

$$\limsup_{\lambda \to \infty} \lambda^{-\sigma(p)}||\chi_\lambda||_{L^2 \to L^p} > 0. \text{ (see [14]).}$$

The special case of (2) where $p = \infty$ can be proved using the estimates of Hörmander [5] that proved the sharp Weyl formula for general self-adjoint elliptic operators on manifolds without boundary. Recently, in case of manifolds without boundary, Sogge and Zelditch [16] proved estimates that imply that for generic metrics on any manifold one has the bounds $||e_j||_\infty = o(\lambda_j^{(n-1)/2})$ for $L^2$-normalized eigenfunctions.

In case of manifolds with boundary, for the unit band spectral projection operators, Grieser [3], and Smith and Sogge [11] showed that the bounds (2) hold under the assumption that the manifold has geodesically concave boundary. The two dimensional case was handled in [3], and higher dimensional in [11]. Recently, without assuming the geodesically concave boundary, Grieser [4] proved the $L^\infty$ estimates for single eigenfunction

$$||e_j(f)||_\infty \leq C\lambda_j^{(n-1)/2}||f||_2,$$

and Sogge [15] proved the following $L^\infty$ estimate for $\chi_\lambda f$,

$$||\chi_\lambda f||_\infty \leq C\lambda^{(n-1)/2}||f||_2, \quad \lambda \geq 1,$$
for a uniform constant $C$, which is equivalent to

$$\sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(x)^2 \leq C\lambda^{n-1}, \quad \forall x \in M,$$

which are sharp for instance when $M$ is the upper hemisphere of $S^n$ with standard metric. Grieser [4] and Sogge [15] first showed that one has the uniform bounds for the interior point $x \in M$ when the distance from $x$ to $\partial M$ is bounded from below by $\lambda^{-1}$, using the proof of estimates for the local Weyl low which are due to Seeley [10], Pham The Lai [7] and Hörmander [6]. They then used those bounds and a form of the maximum principle ([8], Theorem 10, p73) for solutions of Dirichlet eigenvalue problem (1) to obtain the bounds in the $\lambda^{-1}$ neighborhood of the boundary.

One of our main results is on the gradient estimates on $\chi_\lambda f$ for $f \in L^2(M)$,

**Theorem 1.1** Fix a compact Riemannian manifold $(M, g)$ of dimension $n \geq 2$ with boundary, there is a uniform constant $C$ so that

$$||\nabla \chi_\lambda f||_\infty \leq C\lambda^{(n+1)/2}||f||_2, \quad \lambda \geq 1,$$

which is equivalent to

$$\sum_{\lambda_j \in [\lambda, \lambda+1)} |\nabla e_j(x)|^2 \leq C\lambda^{n+1}, \quad \forall x \in M.$$

The motivation to study the gradient estimates as Theorem 1.1 is that one cannot get the gradient estimates on the eigenfunctions of Dirichlet Laplacian from standard calculus of pseudo-differential operators as done for the Laplacian on the manifolds without boundary, since $P = \sqrt{-\Delta_g}$ for the Dirichlet Laplacian is not a pseudo-differential operator any more and one cannot get good estimates on $L^\infty$ bounds on
χλ and ∇χλ near the boundary only by studying the Hadamard parametrix of the wave kernel as for compact manifolds without boundary in [9] and [14]. Here we shall prove Theorem 1.1 using the ideas of interior and boundary gradient estimates for Poisson’s equation in [2] and a maximum principle argument as in [4] and [15].

Our other main result will be Hörmander Multiplier Theorem for eigenfunction expansions on compact Riemannian manifolds with boundary. Given a bounded function m(λ) ∈ L∞(R), we can define a multiplier operator, m(P), by

\[ m(P)f = \sum_{j=1}^{\infty} m(\lambda_j)e_j(f) \]  

(3)

such an operator is always bounded on L²(M). However, if one considers on any other space Lᵖ(M), it is known that some smoothness assumptions on the function m(λ) are needed to ensure the boundedness of

\[ m(P) : L^p(M) \to L^p(M). \]  

(4)

When m(λ) is C∞ and, moreover, in the symbol class S⁰, i.e.,

\[ \left| (\frac{d}{d\lambda})^\alpha m(\lambda) \right| \leq C_\alpha (1 + |\lambda|)^{-\alpha}, \quad \alpha = 0, 1, 2, \ldots \]

It has been known for some time that (4) holds for all 1 < p < ∞ on compact manifolds without boundary (see [18]). One assumes the following regularity assumption: Suppose that m ∈ L∞(R), let L²⁺(R) denote the usual Sobolev space and fix β ∈ C∞₀((1/2, 2)) satisfying \( \sum_{j=0}^{\infty} \beta(2^j t) = 1 \), t > 0, for s > n/2, there is

\[ \sup_{\lambda>0} \lambda^{-1+s} ||\beta(\cdot/\lambda)m(\cdot)||^2_{L²⁺} = \sup_{\lambda>0} ||\beta(\cdot)m(\lambda\cdot)||^2_{L²⁺} < \infty. \]  

(5)
Hörmander [6] first proved the boundedness of the multiplier operator $m(P)$ on $\mathbb{R}^n$ under the assumption (5), using the Calderón-Zygmund decomposition and the estimates on the kernel of the multiplier operator. Stein and Weiss [17] studied the Hörmander multiplier Theorem for multiple Fourier series, which can be regarded as the case on the flat torus $T^n$. Seeger and Sogge [9] and Sogge [14] proved the boundedness of $m(P)$ on $L^p(M)$ for compact manifolds without boundary under the assumption (5), where the authors used the paramatrix of the wave kernel of $m(P)$ to get the required estimates on the integral kernel of $m(P)$, and applying these estimates, proved the weak-type $(1,1)$ estimates on $m(P)$. As people know, the the paramatrix construction of the wave equation does not work well for general compact manifolds with boundary unless one assumes that the boundary is geodesically concave.

For an important class of special multiplier, Riesz means:

$$S^\delta_N f(x) = \sum_{\lambda_j \leq \lambda} (1 - \frac{\lambda_j^2}{\lambda^2})^\delta e_j(f)(x),$$  \hspace{1cm} (6)

of the spectral expansion. Stein and Weiss [17] studied the Riesz means for multiple Fourier series, which can be regarded as the case on the flat torus $T^n$. Sogge [12] and Christ and Sogge [1] proved the sharp results for manifolds without boundary, which the Riesz means (6) are uniformly bounded on all $L^p(M)$ spaces, provided that $\delta > \frac{n-1}{2}$, but no such result can hold when $\delta \leq \frac{n-1}{2}$. Recently, Sogge [15] proved the same results on Riesz means of the spectral expansion for Dirichlet Laplacian on compact manifolds with boundary. Since the multipliers $(1 - \frac{\lambda_j^2}{\lambda^2})^s$ satisfy (5) if $s < \delta + \frac{1}{2}$, but not when $s > \delta + \frac{1}{2}$, one can see the Hörmander multiplier Theorem generalizes the results for the Riesz means.
Using the $L^\infty$ estimates on $\chi_\lambda f$ and $\nabla \chi_\lambda f$ and the ideas in [9], [12]-[15], we have the following Hömander multiplier Theorem for compact manifolds with boundary:

**Theorem 1.2** Let $m \in L^\infty(\mathbb{R})$ satisfy (5), then there are constants $C_p$ such that

$$
||m(P)f||_{L^p(M)} \leq C_p||f||_{L^p(M)}, \quad 1 < p < \infty.
$$

(7)

Since the complex conjugate of $m$ satisfies the same hypotheses (5), we need only to prove Theorem 1.2 for exponents $1 < p \leq 2$. This will allow us to exploit orthogonality, and since $m(P)$ is bounded on $L^2(M)$, also reduce Theorem 1.2 to show that $m(P)$ is weak-type $(1,1)$ by the Marcinkiewicz Interpolation Theorem. The weak-type $(1,1)$ estimates of $m(P)$ will involve a splitting of $m(P)$ into two pieces $m(P) = \tilde{m}(P) + r(P)$. For the remainder $r(P)$, one can obtain the strong $(1,1)$ estimates by the $L^\infty$ estimates on $\chi_\lambda f$ as done for compact manifolds without boundary in [14]. For the main term $\tilde{m}(P)$, we make a second decomposition \{K_{\lambda,l}(x,y)\}_{l=-\infty}^{\infty} for each term $K_\lambda(x,y)$, which comes from the dyadic decomposition of the integral kernel $K(x,y)$ of $\tilde{m}(P)$

$$
K(x,y) = \sum_{k=1}^{\infty} K_{2^k}(x,y) + K_0(x,y), \quad (*)
$$

such that $K_\lambda(x,y) = \sum_{l=-\infty}^{\infty} K_{\lambda,l}(x,y)$ and

$$
\tilde{m}(P)f(x) = \sum_{k=0}^{\infty} \sum_{l=-\infty}^{\infty} T_{2^k,l}(P)f(x), \text{ where } T_{\lambda,l}(P)f(x) = \int_M K_{\lambda,l}(x,y)f(y)dy.
$$

We shall give the definitions of $K_{\lambda,l}(x,y)$ in the proof of Theorem 1.2. And for each operator $T_{\lambda,l}(P)$, we shall prove the $L^1 \to L^2$ estimates by a rescaling argument of the proof for the remainder $r(P)$ using the $L^\infty$ estimates and gradient estimates on $\chi_\lambda f$ which we obtained in Theorem 1.1. With the support properties and the finite
propagation speed properties, in each of the second sum of (*), there are uniform constants \( c, C > 0 \), such that

\[
c \lambda \text{dist}(x, y) \leq 2^l \leq C \lambda
\]

holds for each \( \lambda = 2^k \), which is one key observation when we apply the Calderón-Zygmund decomposition to show the weak-type \((1,1)\) estimates on \( \tilde{m}(P) \).

In what follows we shall use the convention that \( C \) will denote a constant that is not necessarily the same at each occurrence.

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## 2 Gradient estimates For Dirichlet Laplacian

In this section, as said in Introduction, we shall prove Theorem 1.1. by using maximum principle according to three different cases: for the interior points with \( \text{dist}(x, \partial M) \geq \epsilon \lambda^{-1} \), for the boundary points, and for the points on the strip near boundary with \( \text{dist}(x, \partial M) \leq \epsilon \lambda^{-1} \). For the unit band spectral projection operators \( \chi_\lambda \) of Dirichlet Laplacian on \((M, g)\) with smooth boundary, Sogge [15] proved the \( L^\infty \) estimates

\[
||\chi_\lambda f||_\infty \leq C \lambda^{(n-1)/2} ||f||_2, \quad \lambda \geq 1,
\]
for a uniform constant $C$, which is equivalent to

$$\sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(x)^2 \leq C\lambda^{n-1}, \quad \forall x \in M,$$

Here we need to show that one has the uniform bounds

$$|\nabla \chi_\lambda f(x)| \leq C\lambda^{(n+1)/2}\|f\|_2, \quad \lambda \geq 1,$$  \hspace{1cm} (8)

Note that

$$\nabla \chi_\lambda f(x) = \int_M \sum_{\lambda_j \in [\lambda, \lambda+1)} \nabla e_j(x)e_j(y)f(y)dy,$$

therefore, by the converse to Schwarz’s inequality and orthogonality, one has the bounds (8) at a given point $x$ if and only if

$$\sum_{\lambda_j \in [\lambda, \lambda+1)} (\nabla e_j(x))^2 \leq C^2\lambda^{n+1}.$$  

First we show the gradient estimate for the interior points with $\text{dist}(x, \partial M)$ bounded below by $\lambda^{-1}$,

**Lemma 2.1** For the Riemannian manifold $(M, g)$ with boundary $\partial M$, we have the gradient estimate

$$|\nabla \chi_\lambda f(x)| \leq C\epsilon \lambda^{(n+1)/2}\|\chi_\lambda f\|_2, \quad \text{for } \text{dist}(x, \partial M) \geq \epsilon(\lambda + 1)^{-1}$$

where the $0 < \epsilon < 1$ will be determined later.

**Proof.** We shall show this Lemma following the ideas for the following interior gradient estimates for Poisson’s equation $\Delta u = f$,

$$|\nabla u(x_0)| \leq \frac{C}{d} \sup_{B} |u| + Cd \sup_{B} |f|,$$
by using maximum principle in a cube $B$ centered at $x_0$ with length $d$.

Now we fix an $\epsilon$ and $x_0 \in M$ with $\text{dist}(x_0, \partial M) \geq \epsilon(\lambda + 1)^{-1}$. We shall use the maximum principle in the cube centered at $x_0$ with length $d = \epsilon(\lambda + 1)^{-1}/\sqrt{n}$ to prove the same gradient estimates for $\chi_\lambda f$ as above for Poisson’s equation.

Define the geodesic coordinates $x = (x_1, \cdots, x_n)$ centered at point $x_0$ as following, fixed an orthonormal basis $\{v_i\}_{i=1}^n \subset T_{x_0}M$, identity $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ with the point $\exp|_{x_0}(\sum_{i=1}^n x_i v_i) \in M$. In some small neighborhood of $x_0$, we can write the metric $g$ with the form
\[ \sum_{i,j=1}^n g_{ij}(x)dx^i dx^j, \]
and the Laplacian can be written as
\[ \Delta_g = \sum_{i,j=1}^n g^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i}, \]
where $(g^{ij}(x))_{1 \leq i,j \leq n}$ is the inverse matrix of $(g_{ij}(x))_{1 \leq i,j \leq n}$, and $b_i(x)$ are in $C^\infty$.

Now define the cube
\[ Q = \{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n| |x_i| < d, \ i = 1, \cdots, n \} \subset M, \]
where we can choose $d = \epsilon(\lambda + 1)^{-1}/\sqrt{n} \leq \text{dist}(x_0, \partial M)/\sqrt{n}$.

Denote $u(x; f) = \chi_\lambda f(x)$, we have $u \in C^2(Q) \cap C^0(\bar{Q})$, and
\[ \Delta_g u(x; f) = - \sum_{\lambda_j \in [\lambda, \lambda + 1)} \lambda_j^2 e_j(f) := h(x; f). \]
From the $L^\infty$ estimate in [15], and Cauchy-Schwarz inequality, we have
\[ |h(x; f)|^2 = \left( \sum_{\lambda_j \in [\lambda, \lambda + 1)} (\lambda_j^2 e_j(x))(\int_M e_j(y)f(y)dy) \right)^2 \]
\[
\sum_{\lambda_j \in [\lambda, \lambda+1)} \sum_{x \in [\lambda, \lambda+1)} (\int_M e_j(y)f(y)dy)^2
\]
\[
\leq (\lambda + 1)^4 \left( \sum_{\lambda_j \in [\lambda, \lambda+1)} e_j^2(x) \right) ||\chi f||^2_{L^2(M)}
\]
\[
\leq C(\lambda + 1)^{n+3} ||\chi f||^2_{L^2(M)}
\]

We estimate \(|D_n u(0; f)| = |\frac{\partial}{\partial x_n} u(0; f)| first, and the same estimate holds for \(|D_i u(0; f)| with i = 1, \cdots, n - 1 also. Now in the half-cube \(Q' = \{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n | x_i < d, i = 1, \cdots, n - 1, 0 < x_n < d. \} \subset M, \)

Consider the function
\[
\varphi(x', x_n; f) = \frac{1}{2} [u(x', x_n; f) - u(x', -x_n; f)],
\]
where we write \(x = (x', x_n) = (x_1, \cdots, x_{n-1}, x_n). \) One sees that \(\varphi(x', 0; f) = 0,\)
\[\sup_{\partial Q'} |\varphi| \leq \sup_{\partial Q} |u| := A, \text{ and } |\Delta_g \varphi| \leq \sup_Q |h| := N \text{ in } Q'. \] For function
\[
\psi(x', x_n) = \frac{A}{d^2}[|x'|^2 + \alpha x_n(nd - (n - 1)x_n)] + \beta N x_n(d - x_n)
\]
defined on the half-cube \(Q', \) where \(\alpha \geq 1 \) and \(\beta \geq 1 \) will be determined below, one has \(\psi(x', x_n) \geq 0 \) on \(x_n = 0 \) and \(\psi(x', x_n) \geq A \) in the remaining portion of \(\partial Q', \) and
\[
\Delta_g \psi(x) = \frac{A}{d^2}[2\text{tr}(g^{ij}(x)) - (2n\alpha - 2\alpha + 1) + 2 \sum_{i=1}^n b_i(x)x_i + b_n(x) \\
\times (nd - (2n\alpha - 2\alpha + 1)x_n)] + N\beta[-2g^{nn}(x) + b_n(x)(d - 2x_n)]
\]

Since in \(M, \) \(\text{tr}(g^{ij}(x)) \) and \(b_i(x) \) are uniformly bounded and \(g^{nn}(x) \) is positive, then for a large \(\alpha, \) one has
\[
2\text{tr}(g^{ij}(x)) - (2n\alpha - 2\alpha + 1) \leq -1,
\]
Fix such a $\alpha$, since $d = \epsilon(\lambda + 1)^{-1}/\sqrt{n}$, for large $\lambda$, one has

$$2 \sum_{i=1}^{n} b_1(x)x_i + b_n(x)(n\alpha d - (2n\alpha - 2\alpha + 1)x_n) < 1.$$  

Then the first term in $\Delta_g \psi(x)$ is negative, and for the second term, let $\beta$ large enough, one has

$$\beta[-2g^{nm}(x) + b_n(x)(d - 2x_n)] < -1.$$  

Hence one has $\Delta_g \psi(x) \leq -N$ in $Q'$.

Now one has $\Delta_g (\psi \pm \varphi) \leq 0$ in $Q'$ and $\psi \pm \varphi \geq 0$ on $\partial Q'$, from which it follows by the maximum principle that $|\varphi(x', x_n; f)| \leq |\psi(x', x_n)|$ in $Q'$. Hence one has

$$|D_n u(0; f)| = \lim_{x_n \to 0} |\frac{\varphi(0, x_n; f)}{x_n}| \leq \frac{\alpha n A}{d} + \beta d N.$$  

Note that $d = \epsilon(\lambda + 1)^{-1}/\sqrt{n}$, $A \leq C(\lambda + 1)^{(n-1)/2}\|\chi \lambda f\|_2$, and $N \leq C(\lambda + 1)^{(n+3)/2}\|\chi \lambda f\|_2$, then one has the estimate

$$|D_n u(0; f)| \leq C_\epsilon(\lambda + 1)^{(n+1)/2}\|\chi \lambda f\|_2.$$  

The same estimate holds for $|D_i u(0)|$, $i = 1, \cdots, n - 1$. Hence we have

$$|\nabla u(0)| \leq C_\epsilon(\lambda + 1)^{(n+1)/2}\|\chi \lambda f\|_2.$$  

Since the estimate is true for any $x_0 \in M$ with $\text{dist}(x_0, \partial M) \geq \epsilon(\lambda + 1)^{-1}$ and the constant $C_\epsilon$ is independent of the choice of $x_0$, the Lemma is proved. Q.E.D.

As done in proof of Proposition 2.2 in [15], we shall use the geodesic coordinates with respect to the boundary below to study the gradient estimates on the the $\epsilon\lambda^{-1}$ strip of the boundary. We can find a small constant $c > 0$ so that the map $(x', x_n) \in$
\( \partial M \times [0, c) \to M \), sending \((x', x_n)\) to the endpoint, \(x\), of the geodesic of length \(x_n\) which starts a \(x' \in \partial M\) and is perpendicular to \(\partial M\) is a local diffeomorphism. In this local coordinates \(x = (x_1, \cdots, x_{n-1}, x_n)\), the metric \(g\) has the form

\[
\sum_{i,j=1}^{n} g_{ij}(x) dx^i dx^j = (dx_n)^2 + \sum_{i,j=1}^{n} g_{ij}(x) dx'^i dx'^j,
\]

and the Laplacian can be written as

\[
\Delta_g = \sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i},
\]

where \((g^{ij}(x))_{1 \leq i,j \leq n}\) is the inverse matrix of \((g_{ij}(x))_{1 \leq i,j \leq n}\), and \(g^{nn} = 1\), and \(g^{nk} = g^{kn} = 0\) for \(k \neq n\), and \(b_i(x)\) are in \(C^\infty\) and real valued.

Now we show the following gradient estimates for the points on boundary \(\partial M\).

**Lemma 2.2** Assume \(|u(x)| \leq C_1 \lambda^{(n-1)/2}\) in \(M\) and \(u(x) = 0\) on \(\partial M\), and

\[
\Delta_g u(x) \geq -C_2 \lambda^{(n+3)/2},
\]

then we have

\[
|\nabla u(x)| \leq C\lambda^{(n+1)/2}, \quad \forall x \in \partial M.
\]

**Proof.** Fix a point \(x_0 \in \partial M\), choose a local coordinate so that \(x_0 = (0, \cdots, 0, R)\), with \(R = \lambda^{-1}\). Without loss the generality, we may assume that at the point \(x_0\), \(\Delta_{g(x_0)} = \Delta\), the Euclidean Laplacian, since we can transfer \(\Delta_{g(x_0)}\) to \(\Delta\) by a suitable nonsingular linear transformation as was done in Chapter 6 at \([2]\). Since \(g^{ij}(x)\) and \(b_i(x)\) are \(C^\infty\), we have a constant \(\Lambda > 0\) such that \(|\nabla g^{ij}(x)| < \Lambda\) and \(|\nabla b_i(x)| < \Lambda\) hold for all \(x\) in the \(\lambda^{-1}\) strip of the boundary.
We first assume $n = \dim M \geq 3$. For $n = 2$, we need define another comparing function $v(x)$. Define a function 

$$v(x) = \alpha \lambda^{(3-n)/2} \left( \frac{1}{R^{n-2}} - \frac{1}{r^{n-2}} \right) + \beta \lambda^{(n+3)/2} (R^2 - r^2),$$

on $A_R = B_{2R}(0) - B_R(0)$, where $r = ||x||_{\mathbb{R}^n}$, $B_r(0) = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n | |x| < r \}$, and $\alpha \geq 1$ and $\beta \geq 1$ will be determined below. Here we assume $B_{2R}(0)$ tangent to $\partial M$ at point $x_0$ from outside. We shall compare $v(x)$ and $u(x)$ in $A_R \cap M$. For any $x \in A_R \cap M$, we have

$$\Delta_g v(x) = \frac{(n - 2) \text{tr}(g^{ij}(x))}{r^n} - \frac{n(n - 2) \sum g^{ij}(x)x_i x_j}{r^{n+2}} + \frac{(n - 2) \sum b_i(x)x_i}{r^n} \alpha \lambda^{(3-n)/2} - \text{tr}(g^{ij}(x)) + 2 \sum b_i(x)x_i \beta \lambda^{(n+3)/2},$$

Since $M$ is a compact manifold, we have $0 < \theta < \Theta < \infty$ such that $|b_i(x)| < \Theta$, $\theta < \text{tr}(g^{ij}(x)) < \Theta$ and $\theta |y|^2 < \sum g^{ij}(x)y_i y_j < \Theta |y|^2$ hold for all $x \in M$. And since $x \in A_R \cap M$, we have $\lambda^{-1} < r < 2\lambda^{-1}$ and $|\sum b_i(x)x_i| < \theta/4$ for large $\lambda$. Hence

$$\Delta_g v(x) \leq \frac{n(n + 1) \Lambda \text{dist}(x, x_0)}{r^n} \alpha \lambda^{(3-n)/2} - (\theta - 2 \cdot \frac{\theta}{4}) \beta \lambda^{(n+3)/2}$$

$$\leq \frac{3n(n + 1) \Lambda R}{r^n} \alpha \lambda^{(3-n)/2} - \frac{\theta}{2} \beta \lambda^{(n+3)/2}$$

$$\leq 3n(n + 1) \Lambda \alpha \lambda^{(n+1)/2} - \frac{\theta}{2} \beta \lambda^{(n+3)/2}$$

where the first inequality comes from that one looks $\Delta_g(x)$ as the perturbation of $\Delta_{g(x_0)} = \Delta$, the Euclidean Laplacian, and $\Delta(1/|x|^{n-2}) = 0$, and the second inequality comes from $\text{dist}(x, x_0) \leq 3R = 3\lambda^{-1}$. Now let 

$$3n(n + 1) \Lambda \alpha \lambda^{-1} - \frac{\theta}{2} \beta \leq -C_2, \quad (9)$$

where
where $C_2$ is the constant in the assumptions of this Lemma, then we have $\Delta_g v(x) \leq \Delta u(x)$ in $A_R \cap M$.

Next we compare the values of $v$ and $u$ on $\partial (A_R \cap M)$.

Case I, $x \in \partial (A_R \cap M) \cap \partial A_R$.

\[
v(x) = \alpha \lambda^{(3-n)/2} \left( \frac{1}{R^{n-2}} - \frac{1}{(2R)^{n-2}} \right) + \beta \lambda^{(n+3)/2} \left( R^2 - (2R)^2 \right) \\
\geq \left( \frac{\alpha}{2} - 3\beta \right) \lambda^{(n-1)/2},
\]

Now let

\[
\frac{\alpha}{2} - 3\beta \geq C_1, \tag{10}
\]

then we have $v(x) \geq u(x)$ for $x \in \partial (A_R \cap M) \cap \partial A_R$.

Case II, $x \in \partial (A_R \cap M) \cap \partial M$.

\[
v(x) = \alpha \lambda^{(3-n)/2} \left( \frac{1}{R^{n-2}} - \frac{1}{r^{n-2}} \right) + \beta \lambda^{(n+3)/2} \left( R^2 - r^2 \right) := h(r),
\]

Since $B_R(0)$ is tangent to $\partial M$ at point $x_0$ from outside, we know the range of $h(r)$ is $[R, 2R]$ and $h(R) = 0$,

\[
h'(r) = (n - 2) \alpha \lambda^{(3-n)/2} \frac{1}{r^{n-1}} - 2\beta \lambda^{(n+3)/2} r \\
\geq (2^{1-n}(n - 2)\alpha - 2\beta) \lambda^{(n+1)/2}.
\]

Now let

\[
2^{1-n}(n - 2)\alpha - 2\beta > 0, \tag{11}
\]

Then $h(r) \geq h(R) = 0$, and we have $v(x) \geq u(x)$ for $x \in \partial (A_R \cap M) \cap \partial M$.

Finally we need determine the values of $\alpha$ and $\beta$. Let $\beta = 4C_2/\theta$, then from (9), (10), (11), we have the range of $\alpha$ as

\[
\max \{ \frac{2^n \beta}{n - 2}, 6\beta + 2C_1 \} \leq \alpha \leq \frac{C_2 \lambda}{3n(n + 1)\Lambda}.
\]
For large $\lambda$, the range of $\alpha$ is not empty set.

Hence for large $\lambda$, we can find a function $v(x)$ such that $\Delta_g(v(x) \pm u(x)) \leq 0$ in $A_R \cap M$ and $v(x) \pm u(x) \geq 0$ on $\partial(A_R \cap M)$, from which it follows by the maximum principle that $v(x) \pm u(x) \geq 0$ in $A_R \cap M$. On the other hand, $v(x_0) - u(x_0) = 0$. Hence we have

$$|\nabla u(x_0)| \leq \frac{\partial v}{\partial r}(x_0) = [(n - 2)\alpha - 2\beta] \lambda^{(n+1)/2} := C' \lambda^{(n+1)/2}$$

Since $x_0$ is an arbitrary point on $\partial M$, the Lemma is proved for $n \geq 3$.

For $n = 2$, we define the function

$$v(x) = \alpha \lambda^{1/2}(lnr - lnR) + \beta \lambda^{5/2}(R^2 - r^2),$$

for $x \in A_R \cap M$. By the same computation as above, we can show the Lemma holds also.

Q.E.D.

Note that $\chi \lambda f(x)$ satisfies the conditions of above Lemma, since

$$|\chi \lambda f(x)|^2 = |\sum_{\lambda_j \in [\lambda, \lambda + 1]} e_j(f)|^2 \leq (\sum_{\lambda_j \in [\lambda, \lambda + 1]} e_j(x)^2)^{2/2} ||\chi \lambda f||_{L^2}^2 \leq C \lambda^{n-1} ||\chi \lambda f||_{L^2}^2,$$

and

$$\Delta_g \chi \lambda f(x) \geq - (\sum_{\lambda_j \in [\lambda, \lambda + 1]} \lambda_j^2 e_j(x)^2)^{1/2} ||\chi \lambda f||_2 \geq -C \lambda^{(n+3)/2} ||\chi \lambda f||_2.$$

From Lemma 2.2, we have the gradient estimates for $\chi \lambda f$ on the boundary points.

**Lemma 2.3** For the compact Riemannian manifold $(M, g)$ with smooth boundary, we have the gradient estimates

$$|\nabla \chi \lambda f(x)| \leq C \lambda^{(n+1)/2} ||\chi \lambda f||_2, \quad \forall x \in \partial M.$$
Now we deal with the gradient estimates on the $\epsilon\lambda^{-1}$ strip of the boundary.

**Lemma 2.4** For the Riemannian manifold $(M, g)$ with boundary $\partial M \in C^2$, we have the gradient estimate

$$|\nabla \chi \lambda f(x)| \leq C\epsilon \lambda^{\frac{n+1}{2}} ||\chi \lambda f||_2, \text{ for } 0 \leq \text{dist}(x, \partial M) \leq \epsilon(\lambda + 1)^{-1}.$$ 

where $\epsilon$ is the same as in Lemma 2.1 and is determined its value in the Proof.

**Proof.** Here we shall apply the maximum principle to $\sum_{\lambda \in [\lambda, \lambda + 1)} |\nabla e_j(x)|^2$ on the $\lambda^{-1}$ boundary strip. We use the geodesic coordinates with respect to the boundary. First from the Bochner formulas, we have

$$\frac{1}{2} \Delta_g \sum_{\lambda \in [\lambda, \lambda + 1)} |\nabla e_j(x)|^2 = \sum_{\lambda \in [\lambda, \lambda + 1)} [\text{Hessian}(e_j)]^2 + (\nabla e_j(x), \nabla (\Delta e_j(x))) + \text{Ric}(\nabla e_j(x), \nabla e_j(x))$$

Since $M$ is a compact manifold, the Ricci curvature is bounded below in whole $M$. Then for large $\lambda$, we have

$$\frac{1}{2} \Delta_g \sum_{\lambda \in [\lambda, \lambda + 1)} |\nabla e_j(x)|^2 \geq - \sum_{\lambda \in [\lambda, \lambda + 1)} [\lambda^2 |\nabla e_j(x)|^2 + \text{Ric}(\nabla e_j(x), \nabla e_j(x))]$$

$$\geq - (\lambda + 2)^2 \sum_{\lambda \in [\lambda, \lambda + 1)} |\nabla e_j(x)|^2$$

Now define a function $w(x) = 1 - a(\lambda + 1)^2 x_n^2$ for the strip $\{x \in M \mid 0 \leq x_n \leq \epsilon(\lambda + 1)^{-1}\}$, and the constants $a$ and $\epsilon$ will be determined below. We have

$$\frac{1}{2} \leq 1 - a\epsilon^2 \leq w(x) \leq 1,$$

and

$$\Delta w(x) = -2a(\lambda + 1)^2 - 2ab_n(x)x_n(\lambda + 1)^2 \leq -a(\lambda + 1)^2,$$
for all point in the strip, here assuming that \( \lambda \) is large enough so that \( |\beta_n(x_n)| \leq 1/2 \).

Define \( h(x) = \sum_{\lambda \in [\lambda, \lambda+1]} |\nabla e_j(x)|^2 / w(x) \), we have

\[
\Delta_g \sum_{\lambda \in [\lambda, \lambda+1]} |\nabla e_j(x)|^2 = w(x) \Delta_g h(x) + h(x) \Delta w(x) + 2(\nabla h(x), \nabla w(x)) \\
\geq -2(\lambda + 2)^2 \sum_{\lambda \in [\lambda, \lambda+1]} |\nabla e_j(x)|^2 \\
= -2(\lambda + 2)^2 w(x) h(x).
\]

Divide by \( w(x) \) both sides, and apply the estimate on \( \Delta_g w(x) \), we have

\[
\Delta h(x) + 2(\nabla h(x), \frac{\nabla w(x)}{w(x)}) + (4 - a)(\lambda + 1)^2 h(x) \geq 0.
\]

If we assume \( a > 4 \), by maximum principle, \( h(x) \) achieves its maximum on \( \partial\{x \in M | 0 \leq x_n \leq \epsilon(\lambda + 1)^{-1}\} \). Since we have the gradient estimates both boundary point and interior points, we have

\[
\sup_{\{x \in M | 0 \leq x_n \leq \epsilon(\lambda + 1)^{-1}\}} \sum_{\lambda \in [\lambda, \lambda+1]} |\nabla e_j(x)|^2 \leq C\lambda^{n+1}.
\]

Finally we shall determine the constants \( a \) and \( \epsilon \). From proof, we need \( a > 4 \) and \( a \epsilon^2 \leq 1/2 \), which is easy to satisfy, for example, we may let \( a = 8 \) and \( \epsilon = 1/4 \). Q.E.D.

If we combine above Lemmas, we get the gradient estimates for eigenfunctions of compact Riemannian manifolds with boundary.

Since here our proof only involve the \( L^\infty \) estimate of the eigenfunctions. For compact Riemannian manifolds without boundary, by Lemma 2.1 and the \( L^\infty \) estimates on \( \chi_\lambda \) for a second-order elliptic differential operator on compact manifolds without boundary in [14], we have the following gradient estimates
Theorem 2.1 Fix a compact Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) without boundary, there is a uniform constant \(C\) so that

\[ ||\nabla \chi_\lambda f||_\infty \leq C \lambda^{(n+1)/2} ||f||_2, \quad \lambda \geq 1.\]

And the bounds are uniform if there is a uniformly bound on the norm of \(\text{tr}(g^{ij}(x))\) for a class metrics \(g\) on \(M\).

For Riemannian manifolds without boundary, in [16], the authors proved that for generic metrics on any manifold one has the bounds \(||e_j||_{L^\infty(M)} = o(\lambda_j^{(n-1)/2})\) for \(L^2\) normalized eigenfunctions. For Laplace-Beltrami operator \(\Delta_g\), there are eigenvalues \(\{-\lambda_j^2\}\), where \(0 \leq \lambda_0^2 \leq \lambda_1^2 \leq \cdots \to \infty\) are counted with multiplicity. Let \(\{e_j(x)\}\) be an associated orthogonal basis of \(L^2\) normalized eigenfunctions. If \(\lambda^2\) is in the spectrum of \(-\Delta_g\), let \(V_\lambda = \{u \mid \Delta_g u = -\lambda^2 u\}\) denote the corresponding eigenspace.

We define the eigenfunction growth rate in term of

\[ L^\infty(\lambda, g) = \sup_{u \in V_\lambda; ||u||_{L^2} = 1} ||u||_{L^\infty}, \]

and the gradient growth rate in term of

\[ L^\infty(\nabla, \lambda, g) = \sup_{u \in V_\lambda; ||u||_{L^2} = 1} ||\nabla u||_{L^\infty}. \]

In [16], Sogge and Zelditch proved the following results

\[ L^\infty(\lambda, g) = o(\lambda_j^{(n-1)/2})\]

for a generic metric on any manifold without boundary. Here we have the following estimates on the gradient growth rate.
Theorem 2.2 \( L^\infty(\nabla, \lambda, g) = o(\lambda_j^{(n+1)/2}) \) for a generic metric on any manifold without boundary. And the bounds are uniform if there is a uniformly bound on the norm of \( \text{tr}(g^{ij}(x)) \) for \((M, g)\).

Proof. For a compact Riemannian manifold \((M, g)\) without boundary, we can apply our Lemma 2.1 to any point in \(M\). From Theorem 1.4 in [16], we have \( L^\infty(\lambda, g) = o(\lambda_j^{(n-1)/2}) \) for a generic metric on any compact manifold. Fix that metric on the manifold and a \( L^2 \) normalized eigenfunction \( u(x) \), apply our Lemma 2.1 to \( u(x) \) at each point \( x_0 \in M \), we have

\[
|\nabla u(x_0)| \leq \frac{\alpha n A}{d} + \beta d N.
\]

where \( \alpha \) and \( \beta \) are constants depending on the norm of \( \text{tr}(g^{ij}(x)) \) at \( M \) only, which can been seen in the proof of Lemma 2.1, and \( A = \sup_{\partial Q} |u| = o(\lambda_j^{(n-1)/2}) \), and \( N = \sup_{Q} |\lambda^2 u| = (\lambda_j^{(n+3)/2}) \), where the cube

\[
Q = \{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n \mid |x_i| < d, \ i = 1, \cdots, n \} \subset M,
\]

we choose \( d = (\lambda + 1)^{-1}/\sqrt{n} \). Hence we have

\[
|\nabla u(x_0)| = o(\lambda_j^{(n+1)/2})
\]

holds for all \( L^2 \) normalized eigenfunction \( u(x) \in V_\lambda \), furthermore, the bounds are uniform when those metrics of \((M, g)\) have a uniformly bound on the norm of \( \text{tr}(g^{ij}(x)) \) from the proof. Hence we have our Theorem. Q.E.D.
3 Hörmander Multiplier Theorem

In this section, we shall see how the $L^\infty$ estimates and gradients estimates for $\chi_\lambda$ imply Hörmander multiplier Theorem. As discussed in Introduction, we reduce Theorem 1.2 to show that $m(P)$ is weak-type (1, 1), i.e.,

$$\mu\{x : |m(P)f(x)| > \alpha\} \leq \alpha^{-1}||f||_{L^1}, \quad (12)$$

where $\mu(E)$ denotes the $dx$ measure of $E \subset M$. Since the all eigenvalues of Dirichlet Laplacian are positive, we may assume $m(t)$ is an even function on $\mathbb{R}$. Then we have

$$m(P)f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{m}(t)e^{itP}f(x)dt = \frac{1}{\pi} \int_{\mathbb{R}^+} \hat{m}(t)\cos(tP)f(x)dt,$$

Here $P = \sqrt{-\Delta}$, and the cosine transform $u(t, x) = \cos(tP)f(x)$ is the solution of the following Dirichlet-Cauchy problem:

$$\left(\frac{\partial^2}{\partial t^2} - L\right)u(t, x) = 0, \quad u(0, x) = f(x), \quad u_t(0, x) = 0.$$

We shall use the finite propagation speed of solutions of the wave equation in Part 2 of the proof to get the key observation (**).

**Proof of Theorem 1.2.** The proof of the weak-type (1,1) estimate will involve a splitting of $m(P)$ into two pieces: a main piece which one need carefully study, plus a remainder which has strong (1,1) estimate by using the $L^\infty$ estimates for the unit spectral projection operators as done in [14]. Specifically, define $\rho \in C_0^\infty(\mathbb{R})$ as

$$\rho(t) = 1, \quad for \ |t| \leq \frac{\epsilon}{2}, \quad \rho(t) = 0, \quad for \ |t| \geq \epsilon. \quad (13)$$
where $\epsilon > 0$ is a given small constant related to the manifold, which will be specified later. Write $m(P) = \hat{m}(P) + r(P)$, where

\[
\hat{m}(P) = (m \ast \hat{\rho})(P) = \frac{1}{2\pi} \int e^{itP} \rho(t) \hat{m}(t) dt
\]

\[
r(P) = (m \ast (1 - \rho))(P) = \frac{1}{2\pi} \int e^{itP} (1 - \rho(t)) \hat{m}(t) dt
\]

To estimate the main term and remainder, we define for $\lambda = 2^j, j = 1, 2, \ldots$,

\[
m_\lambda(\tau) = \beta(\frac{\tau}{\lambda}) m(\tau) \quad (14)
\]

**Part 1:** Estimate on the remainder

\[
||r(P)f||_{L^1} \leq C||f||_{L^1}.
\]

We first show

\[
||r(P)f||_{L^2} \leq C||f||_{L^1}.
\]

Here we follow the first part in proof of Theorem 5.3.1 in [14] to estimate the remainder. Define

\[
r_\lambda(P) = (m_\lambda \ast (1 - \rho))(P) = \frac{1}{2\pi} \int e^{itP} (1 - \rho(t)) \hat{m}_\lambda(t) dt
\]

Notice that $r_0(P) = r(P) - \sum_{j \geq 1} r_{2^j}(P)$ is a bounded and rapidly decreasing function of $P$. Hence $r_0(P)$ is bounded from $L^1$ to any $L^p$ space. We need only to show

\[
||r_\lambda(P)f||_{L^2} \leq C\lambda^{n/2-s}||f||_{L^1}, \quad \lambda = 2^j, j = 1, 2, \ldots
\]

Using the $L^\infty$ asymptotic estimate for the unit spectral projection operator $\chi_k$ on compact manifold $(M, g)$ with smooth boundary $\partial M$, see [15], we have

\[
||r_\lambda(P)f||_{L^2}^2 \leq \sum_{k=1}^{\infty} ||r_\lambda(P)\chi_k f||_{L^2}^2 \leq C \sum_{k=1}^{\infty} \sup_{\tau \in [k, k+1]} |r_\lambda(\tau)|^2 (1 + k)^{n-1} ||f||_{L^1}^2
\]
Hence we need only to show
\[ \sum_{k=1}^{\infty} \sup_{\tau \in [k,k+1]} |r_{\lambda}(\tau)|^2 (1 + k)^{-n} \leq C \lambda^{n-2s} \]

Notice since \( m_{\lambda}(\tau) = 0 \), for \( \tau \notin [\lambda/2, 2\lambda] \), we have
\[ \tilde{m}_{\lambda}(\tau) = O((1 + |\tau| + |\lambda|)^{-N}) \]
\[ r_{\lambda}(\tau) = O((1 + |\tau| + |\lambda|)^{-N}) \]
for any \( N \) when \( \tau \notin [\lambda/4, 4\lambda] \). Hence we need only to show
\[ \sum_{k=\lambda/4}^{\lambda/2} \sup_{\tau \in [k,k+1]} |r_{\lambda}(\tau)|^2 (1 + k)^{-n} \leq C \lambda^{n-2s} \]
that is
\[ \sum_{k=\lambda/4}^{\lambda/2} \sup_{\tau \in [k,k+1]} |r_{\lambda}(\tau)|^2 \leq C \lambda^{1-2s} \]

Using the fundamental theorem of calculus and the Cauchy-Schwartz inequality, we have
\[
\sum_{k=\lambda/4}^{\lambda/2} \sup_{\tau \in [k,k+1]} |r_{\lambda}(\tau)|^2 \leq C \int_R |\hat{m}_{\lambda}(t)(1 - \rho(t))|^2 dt + \int_R |t\hat{m}_{\lambda}(t)(1 - \rho(t))|^2 dt.
\]
Recall that \( \rho(t) = 1 \), for \( |t| \leq \frac{\epsilon}{2} \), by a change variables shows that this is dominated by
\[
\lambda^{1-2s} \int_R |t^s \hat{m}_{\lambda}(t/\lambda)|^2 dt
\]
\[
= \lambda^{1-2s} \| \lambda \beta(\cdot)m(\lambda \cdot) \|_{L^2_t}^2
\]
\[
= \lambda^{1-2s} \| \beta(\cdot)m(\lambda \cdot) \|_{L^2_t}^2
\]
\[
\leq C \lambda^{1-2s}
\]
Here the first equality comes from a change variables, the second equality comes from the definition of Sobolev norm of $L^2_s(M)$ and the third inequality comes from our condition (7).

Hence we have the estimate for the remainder

$$||r(P)f||_{L^2} \leq C||f||_{L^1}.$$  

And since our manifold is compact, we have

$$||r(P)f||_{L^1} \leq Vol(M)^{1/2} ||r(P)f||_{L^2} \leq C||f||_{L^1}.$$  

that is, we have the strong-type $(1,1)$ estimate on the remainder $r(P)$.

**Part 2:** weak-type $(1,1)$ estimate on the main term

$$\mu\{x : |\tilde{m}(P)f(x)| > \alpha\} \leq \alpha^{-1}||f||_{L^1}.$$  

In [9] and [14], for compact manifold without boundary, the above estimate on $\tilde{m}(P)$ could be estimated by computing its kernel explicitly via the Hadamard parametrix and then estimating the resulting integral operator using straightforward adaptations of the arguments for the Euclidean case. But now for manifolds with boundary, this approach does not seem to work since the known parametrix for the wave equation do not seem strong enough unless one assumes that the boundary is geodesically concave. Here we shall get around this fact by following the ideas in [15], where deals with the Riesz means on compact manifolds with smooth boundary by using the finite propagation speed of solutions of the Dirichlet wave equation.

Now if we argue as [9], [14] and [15], The weak-type $(1,1)$ estimate on $\tilde{m}(P)$ would follow from the integral operator

$$\tilde{m}(P)f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{m}(t)\rho(t)e^{itP}f(x)dt$$
\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{m}(t)\rho(t) \sum_{k \geq 1} e^{it\lambda_k} e_{\lambda_k}(x) \int_M e_{\lambda_k}(y) f(y) dy dt
\]

\[
= \frac{1}{2\pi} \int_{M} \{ \int_{\mathbb{R}} \hat{m}(t)\rho(t) \sum_{k \geq 1} e^{it\lambda_k} e_{\lambda_k}(x) e_{\lambda_k}(y) dt \} f(y) dy
\]

with the kernel

\[
K(x, y) = \int_{\mathbb{R}} \hat{m}(t)\rho(t) \sum_{k \geq 1} e^{it\lambda_k} e_{\lambda_k}(x) e_{\lambda_k}(y) dt
\]

\[
= \sum_{k \geq 1} (m \ast \hat{\rho})(\lambda_k) e_{\lambda_k}(x) e_{\lambda_k}(y)
\]

is weak-type \((1,1)\). Now define the dyadic decomposition

\[
K_\lambda(x, y) = \int_{\mathbb{R}} \hat{m}_\lambda(t)\rho(t) \sum_{k \geq 1} e^{it\lambda_k} e_{\lambda_k}(x) e_{\lambda_k}(y) dt
\]

We have

\[
K(x, y) = \sum_{j=1}^{\infty} K_{2^j}(x, y) + K_0(x, y)
\]

where \(K_0\) is bounded and vanishes when \(\text{dist}(x, y)\) is larger than a fixed constant. In order to estimate \(K_\lambda(x, y)\), we make a second dyadic decomposition as follows

\[
K_{\lambda,l}(x, y) = \int_{\mathbb{R}} \hat{m}_\lambda(t)\beta(2^{-l}|t|)\rho(t) \sum_{k \geq 1} e^{it\lambda_k} e_{\lambda_k}(x) e_{\lambda_k}(y) dt
\]

We have

\[
K_\lambda(x, y) = \sum_{l=-\infty}^{\infty} K_{\lambda,l}(x, y)
\]

Define

\[
T_{\lambda,l}(P)f(x) = \int_M K_{\lambda,l}(x, y) f(y) dy.
\]
From above two dyadic decompositions, we have
\[
\tilde{m}(P)f(x) = \sum_{k=0}^{\infty} \sum_{l=-\infty}^{\infty} T_{2^k,l}(P)f(x).
\] (15)

Note that, because of the support properties of \(\rho(t)\), \(K_{\lambda,l}(x,y)\) vanishes if \(l\) is larger than a fixed multiple of \(\log\lambda\). Now we exploit the fact that the finite propagation speed of the wave equation mentioned before implies that the kernels of the operators \(T_{\lambda,l}, K_{\lambda,l}\) must satisfy
\[
K_{\lambda,l}(x,y) = 0, \quad \text{if } \text{dist}(x,y) \geq C(2^l\lambda^{-1}),
\]
since \(\cos(tP)\) will have a kernel that vanishes on this set when \(t\) belongs to the support of the integral defining \(K_{\lambda,l}(x,y)\). Hence in each of the second sum of (15), there are uniform constants \(c, C > 0\) such that
\[
c\lambda \text{dist}(x,y) \leq 2^l \leq C\lambda
\] (16)
must be satisfied for each \(\lambda = 2^k\), we will use this key observation later.

Now for \(T_{\lambda,l}(P)\)'s, we have the following estimates:

(a). \[\|T_{\lambda,l}(P)f\|_{L^2(M)} \leq C(2^l)^{-s}\lambda^{n/2}\|f\|_{L^1(M)}\]

(b). \[\|T_{\lambda,l}(P)g\|_{L^2(M)} \leq C(2^l)^{-s_0}\lambda^{n/2}[\lambda \max_{y,y_0 \in \Omega} \text{dist}(y,y_0)]\|g\|_{L^1(\Omega)}\]

where \(\Omega = \text{support}(g), \int_{\Omega} g(y)dy = 0\) and \(n/2 < s_0 < \min\{s, n/2 + 1\}\).

Now we first show estimate (a). Notice that \(\beta(2^{-l}\lambda|t|)\rho(t) = 0\) when \(|t| \leq 2^{l-1}\lambda^{-1}\), we can use the same idea to prove estimate (a) as we prove the estimate on the remainder \(r(P)\) in Part 1. Now we use orthogonality of \(\chi_k\) for \(k \in \mathbb{N}\), and
the $L^\infty$ estimates on $\chi_k$ in [15], we have

$$||T_{\lambda,l}(P)f||_{L^2}^2 \leq \sum_{k=1}^{\infty} ||T_{\lambda,l}(P)\chi_k f||_{L^2}^2$$

$$\leq C \sum_{k=1}^{\infty} \sup_{\tau \in [k,k+1]} |T_{\lambda,l}(\tau)|^2 (1 + k)^{n-1} ||f||_{L^1}^2$$

Hence we need only to show

$$\sum_{k=1}^{\infty} \sup_{\tau \in [k,k+1]} |T_{\lambda,l}(\tau)|^2 (1 + k)^{n-1} \leq C(2^l)^{-2\lambda n}$$

Notice since $m_\lambda(\tau) = 0$, for $\tau \not\in [\lambda/2, 2\lambda]$, we have

$$T_{\lambda,l}(\tau) = O((1 + |\tau| + |\lambda|)^{-N})$$

for any $N$ when $\tau \not\in [\lambda/4, 4\lambda]$. Then we have

$$\sum_{k \not\in [\lambda/2, 4\lambda]} \sup_{\tau \in [k,k+1]} |T_{\lambda,l}(\tau)|^2 (1 + k)^{n-1} \leq C \sum_{k \not\in [\lambda/2, 4\lambda]} (1 + k + \lambda)^{-2N} (1 + k)^{n-1}$$

$$\leq C \int_{x > 1, x \not\in [\lambda/4, 4\lambda]} \frac{x^{n-1}}{(x + \lambda)^{2N}} dx$$

$$\leq C (1 + \lambda)^{n-2N}$$

Since $2^l \leq C\lambda$ from our observation (16) above, we need only to show

$$\sum_{k=\lambda/4}^{4\lambda} \sup_{\tau \in [k,k+1]} |T_{\lambda,l}(\tau)|^2 (1 + k)^{n-1} \leq C(2^l)^{-2\lambda n}$$

that is

$$\sum_{k=\lambda/4}^{4\lambda} \sup_{\tau \in [k,k+1]} |T_{\lambda,l}(\tau)|^2 \leq C(2^l)^{-2\lambda}$$

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Using the fundamental theorem of calculus and the Cauchy-Schwartz inequality, we have

\[
\sum_{k=\lambda/4}^{4\lambda} \sup_{\tau \in [k,k+1]} |T_{\lambda,l}(\tau)|^2 \\
\leq C(\int_{\mathbb{R}} |T_{\lambda,l}(\tau)|^2 d\tau + \int_{\mathbb{R}} |T'_{\lambda,l}(\tau)|^2 d\tau) \\
= C(\int_{\mathbb{R}} |\tilde{m}_{\lambda}(t)\beta(2^{-l}\lambda|t|)\rho(t)|^2 dt + \int_{\mathbb{R}} |t\tilde{m}_{\lambda}(t)\beta(2^{-l}\lambda|t|)\rho(t)|^2 dt)
\]

Recall that \( \beta(2^{-l}\lambda|t|)\rho(t) = 0 \) when \( |t| \leq 2^{l-1}\lambda^{-1} \), by a change variables shows that this is dominated by

\[
(2^l)^{-2s}\lambda^{-1} \int_{\mathbb{R}} |t^s\tilde{m}_{\lambda}(t/\lambda)|^2 dt + (2^l)^{-2s+2}\lambda^{-2} \int_{\mathbb{R}} |t^s\tilde{m}_{\lambda}(t/\lambda)|^2 dt \\
= (2^l)^{-2s}(1 + \lambda^{-2}2^l)\lambda^{-1}\|\lambda\beta(\cdot)m(\lambda\cdot)\|_{L^2}^2 \\
= (2^l)^{-2s}\lambda(1 + \lambda^{-2}2^l)\|\beta(\cdot)m(\lambda\cdot)\|_{L^2}^2 \\
\leq C(2^l)^{-2s}\lambda \\
\leq C(2^l)^{-2s}\lambda
\]

Here the first equality comes from a change variables, the second equality comes from the definition of Sobolev norm of \( L^2_s(M) \), the third inequality comes from our condition (7), and the last inequality comes from the observation (16). Hence we proved the estimate (a),

\[
\|T_{\lambda,l}(P)f\|_{L^2(M)} \leq C(2^l)^{-s} \lambda^{n/2} \|f\|_{L^1(M)}
\]

Next we prove the estimate (b). We will use the orthogonality of \( \{e_j\}_{j \in \mathbb{N}} \),

\[
\int_M e_{\lambda_k}(x)e_{\lambda_j}(x)dx = \delta_{kj},
\]
and the $L^\infty$ estimates on $\nabla \chi_k$ for all $k \in \mathbb{N}$ as in Theorem 1.1. Now for function $g \in L^1(M)$ such that $\Omega = \text{support}(g)$ and $\int_\Omega g(y)dy = 0$. For some fixed point $y_0 \in \Omega$, we have

$$
\|T_{\lambda,\tau}(P)g\|_{L^2}^2
$$

$$
= \int_M |\int_\Omega K_{\lambda,\tau}(x,y)g(y)dy|^2 dx
$$

$$
= \int_M |\int_\Omega [K_{\lambda,\tau}(x,y) - K_{\lambda,\tau}(x,y_0)]g(y)dy|^2 dx
$$

(here use the cancellation of $g$)

$$
= \int_M |\int_\Omega \sum_{k \geq 1} T_{\lambda,\tau}(\lambda_k)e_{\lambda_k}(x)[e_{\lambda_k}(y) - e_{\lambda_k}(y_0)]g(y)dy|^2 dx
$$

$$
= \sum_{k \geq 1} \int_M |\int_\Omega \sum_{\lambda_j \in [k,k+1]} \{T_{\lambda,\tau}(\lambda_j)e_{\lambda_j}(x)[e_{\lambda_j}(y) - e_{\lambda_j}(y_0)]\}g(y)dy|^2 dx
$$

(here use the orthogonality)

$$
\leq \sum_{k \geq 1} \int_M \max_{y \in \Omega} |\sum_{\lambda_j \in [k,k+1]} \{T_{\lambda,\tau}(\lambda_j)e_{\lambda_j}(x)[e_{\lambda_j}(y) - e_{\lambda_j}(y_0)]\}|^2 dx |\int_\Omega |g(y)|dy|^2
$$

$$
= ||g||_{L^1}^2 \sum_{k \geq 1} \int_M |\int_\Omega \sum_{\lambda_j \in [k,k+1]} \{T_{\lambda,\tau}(\lambda_j)e_{\lambda_j}(x)[e_{\lambda_j}(y_1) - e_{\lambda_j}(y_0)]\}|^2 dx
$$

(where the maximum achieves at $y_1$)

$$
= ||g||_{L^1}^2 \sum_{k \geq 1} \int_M |\nabla_y \sum_{\lambda_j \in [k,k+1]} T_{\lambda,\tau}(\lambda_j)e_{\lambda_j}(x)e_{\lambda_j}(\bar{y}), y_1 - y_0)|^2 dx
$$

$$
= ||g||_{L^1}^2 \sum_{k \geq 1} \int_M |\{ \sum_{\lambda_j \in [k,k+1]} T_{\lambda,\tau}(\lambda_j)e_{\lambda_j}(x)(\nabla e_{\lambda_j}(\bar{y}), y_1 - y_0)\}|^2 dx
$$

$$
= ||g||_{L^1}^2 \sum_{k \geq 1} \sum_{\lambda_j \in [k,k+1]} |T_{\lambda,\tau}(\lambda_j)(\nabla e_{\lambda_j}(\bar{y}), y_1 - y_0)|^2
$$

$$
\leq ||g||_{L^1}^2 \sum_{k \geq 1} \max_{\tau \in [k,k+1]} |T_{\lambda,\tau}(\tau)|^2 \sum_{\lambda_j \in [k,k+1]} |\nabla e_{\lambda_j}(\bar{y})|\text{dist}(y_1, y_0)^2
$$

(here use the orthogonality)

$$
\leq ||g||_{L^1}^2 \left[ \max_{y,y_0 \in \Omega} \text{dist}(y, y_0) \right]^2 \sum_{k \geq 1} \max_{\tau \in [k,k+1]} |T_{\lambda,\tau}(\tau)|^2 \sum_{\lambda_j \in [k,k+1]} |\nabla e_{\lambda_j}(\bar{y})|^2
$$

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\[ \leq C \|g\|^2_{L^1} \max_{y,y_0 \in \Omega} \text{dist}(y,y_0)^2 \sum_{k \geq 1} \max_{\tau \in [k,k+1]} |T_{\lambda,l}(\tau)|^2 (1 + k)^{n+1} \]

Now using the same computation as to the estimate (a), for some constant \( s_0 \) satisfying \( n/2 < s_0 < \min\{s, n/2 + 1\} \), we have

\[ \sum_{k \geq 1} \max_{\tau \in [k,k+1]} |T_{\lambda,l}(\tau)|^2 (1 + k)^{n+1} \leq C (2^l)^{-2s_0} \lambda^{n+2}. \]

Combine above two estimates, we proved the estimate (b),

\[ \|T_{\lambda,l}(P)g\|_{L^2(M)} \leq C (2^l)^{-s_0} \lambda^{n/2} \max_{y,y_0 \in \Omega} \text{dist}(y,y_0) \|g\|_{L^1(\Omega)} \]

Now we use the estimates (a) and (b) to show

\[ \tilde{m}(P)f(x) = \int_M K(x,y)f(y)dy \]

is weak-type (1,1). We let \( f(x) = g(x) + \sum_{k=1}^{\infty} b_k(x) := g(x) + b(x) \) be the Calderón-Zygmund decomposition of \( f \in L^1(M) \) at the level \( \alpha \) using the same idea as Lemma 0.2.7 in [14]. Let \( Q_k \supset \text{supp}(b_k) \) be the cube associated to \( b_k \) on \( M \), and we have

\[ \|g\|_{L^1} + \sum_{k=1}^{\infty} \|b_k\|_{L^1} \leq 3\|f\|_{L^1}, \]

\[ |g(x)| \leq 2^n \alpha \quad \text{almost everywhere}, \]

and for certain non-overlapping cubes \( Q_k \),

\[ b_k(x) = 0 \quad \text{for } x \notin Q_k \quad \text{and} \quad \int_M b_k(x) dx = 0. \]

\[ \sum_{k=1}^{\infty} \mu|Q_k| \leq \alpha^{-1} \|f\|_{L^1}. \]

Now we show the weak-type (1,1) estimate for \( \tilde{m}(P) \). Since

\[ \{x : |\tilde{m}(P)f(x)| > \alpha\} \subset \{x : |\tilde{m}(P)g(x)| > \alpha/2\} \cup \{x : |\tilde{m}(P)b(x)| > \alpha/2\} \]

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Notice
\[ \int_M |g|^2 dx \leq 2^n \alpha \int_M |g| dx. \]
Hence we use the $L^2$ boundedness of $\tilde{m}(P)$ and Tchebyshev’s inequality to get
\[ \mu\{x : |\tilde{m}(P)g(x)| > \alpha/2\} \leq C\alpha^{-2}||g||_2^2 \leq C'\alpha^{-1}||f||_{L^1}. \]

Let $Q_k^*$ be the cube with the same center as $Q_k$ but twice the side-length. After possibly making a translation, we may assume that
\[ Q_k = \{x : \max|x_j| \leq R\}. \]

Let $O^* = \cup Q_k^*$, we have
\[ \mu|O^*| \leq 2^n \alpha^{-1}||f||_{L^1}. \]
and
\[ \mu\{x \notin O^* : |\tilde{m}(P)b(x)| > \alpha/2\} \]
\[ \leq 2\alpha^{-1} \int_{x \notin O^*} |\tilde{m}(P)b(x)| dx \]
\[ \leq 2\alpha^{-1} \sum_{k=1}^\infty \int_{x \notin Q_k^*} |\tilde{m}(P)b_k(x)| dx \]
Hence we need only to show
\[ \int_{x \notin Q_k^*} |\tilde{m}(P)b_k(x)| dx \]
\[ = \int_{x \notin Q_k^*} |\int_{Q_k} K(x,y)b_k(y)dy| dx \]
\[ \leq C \int_M |b_k| dx. \]
From the double dyadic decomposition (15), we show two estimates of $T_{\lambda,l}(P)b_k(x)$ on set \( \{x \in M : x \not\in \mathcal{O}^*\} \),

\[
(I) \quad ||T_{\lambda,l}(P)b_k||_{L^1(\{x \not\in \mathcal{O}^*\} \cap B_{R_{\lambda,l}})} \leq C(2^l)^{n/2-s}||b_k||_{L^1(Q_k)}
\]

\[
(II) \quad ||T_{\lambda,l}(P)b_k||_{L^1(\{x \not\in \mathcal{O}^*\} \cap B_{R_{\lambda,l}})} \leq C(2^l)^{n/2-s_0}[\lambda \max_{y,y_0 \in Q_k} \text{dist}(y,y_0)]||b_k||_{L^1(Q_k)}
\]

Since our observation (16), as was done in [15], in order to prove \((I),(II)\), it suffices to show that for all geodesic balls $B_{R_{\lambda,l}}$ of radius $R_{\lambda,l} = 2^l\lambda^{-1}$, one has the bounds

\[
(I)' \quad ||T_{\lambda,l}(P)b_k||_{L^1(\{x \not\in \mathcal{O}^*\} \cap B_{R_{\lambda,l}})} \leq C(2^l)^{n/2-s}||b_k||_{L^1(Q_k)}
\]

\[
(II)' \quad ||T_{\lambda,l}(P)b_k||_{L^1(\{x \not\in \mathcal{O}^*\} \cap B_{R_{\lambda,l}})} \leq C(2^l)^{n/2-s_0}[\lambda \max_{y,y_0 \in Q_k} \text{dist}(y,y_0)]||b_k||_{L^1(Q_k)}
\]

To show \((I)'\), using the estimate \((a)\), and Hölder inequality, we get

\[
||T_{\lambda,l}(P)b_k||_{L^1(\{x \not\in \mathcal{O}^*\} \cap B_{R_{\lambda,l}})} \leq Vol(B_{R_{\lambda,l}})^{1/2}||T_{\lambda,l}(P)b_k||_{L^2} \leq C(2^l\lambda^{-1})^{n/2}(2^l)^{-s}\lambda^{n/2}||b_k||_{L^1} = C(2^l)^{n/2-s}||b_k||_{L^1}
\]

To show \((II)'\), using the cancellation property $\int_{Q_k} b_k(y)dy = 0$, the estimate \((b)\), and Hölder inequality, we have

\[
||T_{\lambda,l}(P)b_k||_{L^1(\{x \not\in \mathcal{O}^*\} \cap B_{R_{\lambda,l}})} \leq Vol(B_{R_{\lambda,l}})^{1/2}||T_{\lambda,l}(P)b_k||_{L^2} \leq C(2^l\lambda^{-1})^{n/2}(2^l)^{-s_0}\lambda^{n/2}[\lambda \max_{y,y_0 \in Q_k} \text{dist}(y,y_0)]||b_k||_{L^1(Q_k)} = C(2^l)^{n/2-s_0}[\lambda \max_{y,y_0 \in Q_k} \text{dist}(y,y_0)]||b_k||_{L^1(Q_k)}
\]
From our observation (16), and estimates (I), we have
\[
\sum_{l=-\infty}^{\infty} ||T_{\lambda,l}(P)b_k||_{L^1(x \notin O^*)}
\leq C \sum_{2^l \geq \lambda \text{dist}(x,y)} (2^l)^{n/2-s}||b_k||_{L^1(Q_k)}
\leq C_s (\lambda \text{dist}(x,y))^{n/2-s}||b_k||_{L^1(Q_k)}
\leq C_s (\lambda R)^{n/2-s}||b_k||_{L^1(Q_k)},
\]
and from \(\max_{y,y_0 \in Q_k} \text{dist}(y,y_0) \leq CR\), estimate (II), and \(n/2 < s_0 < \min\{s,n/2 + 1\}\), we have
\[
\sum_{l=-\infty}^{\infty} ||T_{\lambda,l}(P)b_k||_{L^1(x \notin O^*)}
\leq C \sum_{2^l \geq \lambda \text{dist}(x,y)} (2^l)^{n/2-s} [\lambda \max_{y,y_0 \in Q_k} \text{dist}(y,y_0)]||b_k||_{L^1(Q_k)}
\leq C_s (\lambda \text{dist}(x,y))^{n/2-s} [\lambda \max_{y,y_0 \in Q_k} \text{dist}(y,y_0)]||b_k||_{L^1(Q_k)}
\leq C_s (\lambda R)^{n/2+1-s_0}||b_k||_{L^1(Q_k)}.
\]
Therefore, we combine the above two estimate we conclude that
\[
\int_{x \notin Q^*_k} |\tilde{m}(P)b_k(x)| dx
\leq \sum_{j=0}^{\infty} \sum_{l=-\infty}^{\infty} ||T_{2^j,l}(P)b_k||_{L^1(x \notin O^*)}
\leq C_s (\lambda R)^{n/2-s}||b_k||_{L^1} + C_{s_0} \sum_{2^j R \geq 1} (\lambda R)^{n/2+1-s_0}||b_k||_{L^1}
\leq C_s ||b_k||_{L^1}.
\]
Hence we have the weak-type \((1,1)\) estimate on the main term
\[
\mu\{x : |\tilde{m}(P)f(x)| > \alpha\} \leq \alpha^{-1}||f||_{L^1}.
\]
Combine Case 1 and Case 2, we have the weak-type estimate of $m(P)$ and we finish the proof of Theorem 1.2. Q.E.D.

**Remark 3.1** In [20], for a second-order elliptic differential operator on compact Riemannian manifolds without boundary, by the $L^\infty$ estimates on $\chi_\lambda$ in [14] and the gradient estimates on $\chi_\lambda$ from Theorem 2.1, using the same ideas in this section, we shall give a new proof of Hörmander Multiplier Theorem on Compact manifolds without boundary, which is first proved in [9].

**References**


[19] Xiangjin Xu, Gradient estimate for eigenfunction of Riemannian manifolds with boundary. (preprint)
