1.1. The distance between point $(3,2,1)$ and (x,y,z) is

$$\sqrt{(x-3)^2+(y-2)^2+(z-1)^2}$$

So we want to minimize the function:

$$f(x,y,z) = (x-3)^2+(y-2)^2+(z-1)^2 \quad \text{under constraint}$$

$$g(x,y,z) = x-2y+2z+5 = 0$$

by Lagrange Multiplier: $\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$

that is: $\begin{cases} 2x-6 = \lambda & ① \\ 2y-4 = -2\lambda & ② \\ 2z-2 = 2\lambda & ③ \\ x-2y+2z = -5 & ④ \end{cases}$

From ①: $x = \frac{6+\lambda}{2}$, From ②: $y = 2-\lambda$, From ③: $z = \lambda+1$

put these in ④:

$$\frac{6+\lambda}{2} - 2(2-\lambda) + 2(\lambda+1) = -5$$

$$\Rightarrow \lambda = -\frac{4}{3}$$

put value of λ back: $x = \frac{7}{3}$, $y = \frac{10}{3}$, $z = -\frac{1}{3}$

$f(\frac{7}{3}, \frac{10}{3}, -\frac{1}{3}) = 4$,

pick a point $(0, \frac{5}{2}, 0)$ in the plane, $f(0,\frac{5}{2},0) = \frac{41}{4} > 4$

this means $(\frac{7}{3}, \frac{10}{3}, -\frac{1}{3})$ is minimal point, and minimal distance is $\sqrt{4} = 2$ #

1.2. Interier part:

$$\begin{cases} \nabla f = 0 & \Rightarrow y=0, x=-z \\ x^2+y^2+z^2<1 \end{cases}$$

and for all points $y=0, x=-z$ $f(x,0,-x) = 0$

Boundary part:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = x^2+y^2+z^2 = 1 \end{cases}$$

$$\begin{cases} y = 2\lambda x & ① \\ x + z = 2\lambda y & ② \\ y = 2\lambda z & ③ \\ x^2 + y^2 + z^2 = 1 & ④ \end{cases}$$

[Basic idea: from ①②③, get x, y, z in terms of λ, then plug in ④, get value of λ, the plug back to get values of x, y, z]

by ①: $x = \frac{y}{2\lambda}$, by ③: $z = \frac{y}{2\lambda}$, put x, z in ②.

$$\frac{y}{2\lambda} + \frac{y}{2\lambda} = 2\lambda y \Rightarrow y = 2\lambda^2 y$$

case 1: $y = 0$, then $x = z = \frac{y}{2\lambda} = 0$ contradict with ④

case 2: $2\lambda^2 = 1 \Rightarrow \lambda = \pm\sqrt{\frac{1}{2}}$

then we get z, y in terms of x:

$$z = x, \quad y = 2\lambda x = \pm 2\sqrt{\tfrac{1}{2}}\, x \quad — ⑤$$

put ⑤ in ④: $x^2 + 2x^2 + x^2 = 1$

$$x = \pm\frac{1}{2}$$

so we have points: $\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

Find values of f at these pts and $(x, 0, -x)$, and compare.

$\boxed{\text{Max}: \frac{1}{\sqrt{2}}}$ #

1.3 • Use Lagrange Multiplier for two constraints.

$$\begin{cases} \nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h & ① \\ g = x + y + z = 1 & ② \\ h = x^2 + y^2 + (z-1)^2 = 1 & ③ \end{cases}$$

idea to solve the system of eq'ns.

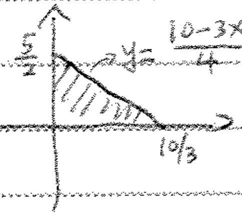
get x, y, z in terms of λ_1 and λ_2 from ①, then plug into ② and ③, we will get two eq'ns about λ_1 and λ_2, solve it, get values of λ_1, λ_2 and then plug back, get x, y, z.

$\lambda_1 = \pm\frac{1}{\sqrt{6}}$, $\lambda_2 = \frac{1}{3}$

$\boxed{\text{Max}: \frac{2+2\sqrt{6}}{3}}$ #

2.1 Talk about it in class

2.3



$3x+4y=10$

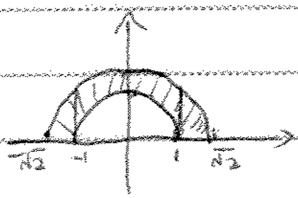
write down this elementary region.

$$\begin{cases} 0 \le x \le \frac{10}{3} \\ 0 \le y \le \frac{10-3x}{4} \end{cases} \quad \text{or} \quad \begin{cases} 0 \le y \le \frac{5}{2} \\ 0 \le x \le \frac{10-4y}{3} \end{cases}$$

So integral is: $\displaystyle\int_0^{\frac{10}{3}} \int_0^{\frac{10-3x}{4}} x^2+y^2 \, dy\, dx$

or $\displaystyle\int_0^{\frac{5}{2}} \int_0^{\frac{10-4y}{3}} x^2+y^2 \, dx\, dy$

2.4



method 1: $\displaystyle\iint_D 1+xy \, dA = \underbrace{\iint_D 1 \, dA}_{= \text{Area of } D} + \underbrace{\iint_D xy \, dA}_{= 0}$ → by symmetry of x, y

$= \text{Area of } D + 0$

$= \frac{\pi}{2}\left((\sqrt{2})^2 - 1^2\right) = \frac{\pi}{2}$

method 2: $\displaystyle\iint_D 1+xy \, dA = \left(\iint_{y\ge0,\, x^2+y^2 \le 2} - \iint_{x^2+y^2\le 1,\, y\ge0} \right)(1+xy)\, dA$

$\displaystyle = \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} (1+xy)\, dy\, dx - \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} (1+xy)\, dy\, dx$

or $\displaystyle\iint_D (1+xy)\, dA = \int_{-\sqrt{2}}^{-1} \int_0^{\sqrt{2-x^2}} 1+xy \, dy\, dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} 1+xy \, dy\, dx$

$\displaystyle + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} 1+xy \, dy\, dx$

method 3, use polar coordinates. $dy\,dx = r\, dr\, d\theta$

$D: \begin{cases} 1 \le r \le \sqrt{2} \\ 0 \le \theta \le \pi \end{cases} \qquad \begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{bmatrix}$

$\displaystyle\iint_D (1+xy)\, dA = \int_1^{\sqrt{2}} \int_0^{\pi} (1 + r^2 \cos\theta \sin\theta)\, d\theta\, dr$