Problem 1

Determine if the integral is convergent and, if it is, determine its value:

\[ \int_{1}^{\infty} \frac{1}{x^{1/3}} \, dx \]

The given integral is improper because it is on an unbounded interval. Thus, we must take a limit as \( a \to \infty \):

\[
\lim_{a \to \infty} \int_{1}^{a} \frac{1}{x^{1/3}} \, dx = \lim_{a \to \infty} \left[ \frac{3}{2} x^{2/3} \right]_{1}^{a} = \lim_{a \to \infty} \left( \frac{a^{2/3}}{2} - \frac{1}{2} \right)
\]

but this last limit diverges to infinity, therefore the integral is not convergent.

Problem 2

Determine if the integral is convergent and, if it is, determine its value:

\[ \int_{0}^{4} \frac{1}{x^{4}} \, dx \]

This integral is improper because the integrand is unbounded near 0, so we take a limit as \( a \to 0^+ \):

\[
\lim_{a \to 0^+} \int_{a}^{4} \frac{1}{x^{4}} \, dx = \lim_{a \to 0^+} -\frac{1}{3} \left[ \frac{1}{x^{3}} \right]_{a}^{4} = \left( \frac{1}{3 \cdot 64} + \frac{1}{3} \lim_{a \to 0^+} \frac{1}{a^3} \right)
\]

but the limit in the last expression diverges to infinity, therefore the integral is not convergent.

Problem 3

Determine if the integral is convergent and, if it is, determine its value:

\[ \int_{1}^{e} \frac{1}{x \ln(x)} \, dx \]

This integral is improper because the integrand is unbounded near 1, so we take a limit as \( a \to 1^+ \):

\[
\lim_{a \to 1^+} \int_{a}^{e} \frac{1}{x \ln(x)} \, dx
\]

Making the \( u \)-sub \( u = \ln(x) \), and performing appropriate substitutions at the bounds, recalling that \( \lim_{a \to 1^+} \ln(a) = 0 \), and we will still be approaching 0 from the right:

\[
\lim_{a \to 1^+} \int_{\ln(a)}^{1} \frac{1}{u} \, du = \lim_{b \to 0^+} \int_{b}^{1} \frac{1}{u} \, du = \ln |u| \bigg|_{b}^{1} = (0 - \lim_{b \to 0^+} \ln(b))
\]

and then we recognize that the limit in this last expression is \(-\infty\) and therefore this integral does not converge.
Problem 4

Determine whether the following integral is convergent:

\[
\int_{-\infty}^{\infty} \frac{1}{x^2 - 1} \, dx
\]

This integral is improper because the interval is unbounded on both the left and the right, and because it is unbounded near ±1. Thus, to really do this in extreme formality we will need to decompose this quite a lot, as follows (other places to break these up can, of course, be chosen—my choice to use ±2, 0 as boundaries was arbitrary):

\[
\int_{-\infty}^{\infty} \frac{1}{x^2 - 1} \, dx = \lim_{a \to -\infty} \int_{a}^{-2} \frac{1}{x^2 - 1} \, dx + \lim_{b \to -1} \int_{-2}^{b} \frac{1}{x^2 - 1} \, dx + \lim_{c \to -1^+} \int_{c}^{0} \frac{1}{x^2 - 1} \, dx
\]

\[
+ \lim_{d \to 1^-} \int_{d}^{0} \frac{1}{x^2 - 1} \, dx + \lim_{e \to 1} \int_{e}^{2} \frac{1}{x^2 - 1} \, dx + \lim_{f \to \infty} \int_{f}^{\infty} \frac{1}{x^2 - 1} \, dx
\]

We will now inspect the second summand on the right, and we will notice that because \(x^2 - 1 = (x+1)(x-1)\), we can decompose this with partial fractions:

\[
\lim_{b \to -1^-} \int_{-2}^{b} \frac{1}{x^2 - 1} \, dx = \lim_{b \to -1^-} \left( \frac{1}{2} \int_{-2}^{b} \frac{1}{x-1} \, dx - \frac{1}{2} \int_{-2}^{b} \frac{1}{x+1} \, dx \right) = \lim_{b \to -1^-} \left( \frac{1}{2} \ln|x-1|^{b} \right)_{-2}^{b} - \frac{1}{2} \ln|x+1|^{b} \right)_{-2}
\]

\[
= \frac{1}{2} \left( \ln(2) - \ln(3) - \lim_{b \to -1^-} \ln|b+1| + \ln(1) \right)
\]

and we notice that the limit in the last expression is not finite: this is the same as \(\lim_{a \to 0^+} \ln(a) = -\infty\). Therefore, the one integral we inspected was not convergent and therefore the overall integral is not convergent either.

Problem 5

Show that if \(p \geq 1\), then the following integral is divergent:

\[
\int_{0}^{1} \frac{1}{x^p} \, dx
\]

First, notice that for \(p = 1\), then the integral is improper because it is not defined at and is unbounded near \(x = 0\) so we take a limit and then we integrate,

\[
\lim_{a \to 0^+} \int_{a}^{1} \frac{1}{x} \, dx = \lim_{a \to 0^+} \ln|x| \bigg|_{a}^{1} = - \lim_{a \to 0^+} \ln|a|
\]

and we then notice that this last limit diverges.

Now, let \(p > 1\). The integral is still improper because it is unbounded near \(x = 0\), so we take a limit and integrate:

\[
\lim_{a \to 0^+} \int_{a}^{1} \frac{1}{x^p} \, dx = \lim_{a \to 0^+} \frac{-1}{p-1} \int_{a}^{1} x^{p-1} \, dx = \lim_{a \to 0^+} \left( \frac{-1}{p-1} + \frac{1}{p-1} \right)
\]

and we notice that this last limit fails to converge as \(a \to 0^+\), so this integral diverges.
Problem 6

Solve the initial-value problem with $y(0) = 1$:

$$\frac{dy}{dx} = \frac{1}{x^2 + 1}$$

The equation is separable, so we separate and integrate:

$$dy = \frac{dx}{x^2 + 1}$$

$$\int dy = \int \frac{dx}{x^2 + 1}$$

$$y = \tan^{-1}(x) + C$$

and then apply the initial value condition to solve for $C$:

$$1 = \tan^{-1}(0) + C$$

$$1 = 0 + C = C$$

Thus, the solution to this initial-value problem is:

$$y = \tan^{-1}(x) + 1$$

Problem 7

Solve the initial-value problem with $y(0) = 1$:

$$\frac{dy}{dx} = \sqrt{3x + 1}$$

The equation is separable, so we separate and integrate:

$$dy = \sqrt{3x + 1}dx$$

$$\int dy = \int \sqrt{3x + 1}dx$$

Let $u = 3x + 1$, then $du = 3dx$:

$$y + C = \frac{1}{3} \int \sqrt{u}du$$

$$y = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} - C$$

$$y = \frac{2}{9} \sqrt{(3x + 1)^3} - C$$

Applying the initial value condition:

$$1 = \frac{2}{9} \sqrt{(3(0) + 1)^3} - C = \frac{2}{9} - C$$

$$C = \frac{-7}{9}$$
Thus, the solution to this initial-value problem is:

\[ y = \frac{2}{9} \sqrt{(3x + 1)^3} + \frac{7}{9} \]

**Problem 8**

Solve the initial-value problem with \( y(0) = 2 \):

\[ \frac{dy}{dx} = 2(1 - y) \]

The equation is separable, so we separate and integrate:

\[ \frac{dy}{1 - y} = 2dx \]

\[ \int \frac{dy}{1 - y} = \int 2dx \]

\[ - \ln |1 - y| = 2x + C \]

Letting \( k = \pm e^C \):

\[ 1 - y = ke^{-2x} \]

\[ y = 1 - ke^{2x} \]

Applying the initial value condition:

\[ 2 = 1 - ke^0 \]

\[ k = -1 \]

Thus, the solution to this initial-value problem is:

\[ y = 1 + e^{-2x} \]

**Problem 9**

Solve the initial-value problem with \( y(0) = 2 \):

\[ \frac{dy}{dx} = (y + 1)e^{-x} \]

The equation is separable, so we separate and integrate:

\[ \frac{dy}{y + 1} = e^{-x}dx \]

\[ \int \frac{dy}{y + 1} = \int e^{-x}dx \]

\[ \ln |y + 1| = -e^{-x} + C \]

\[ y + 1 = \exp \left( -e^{-x} + C \right) = k \exp \left( -e^{-x} \right) \]

(Recall that \( \exp(x) = e^x \); I use this notation mostly for legibility here; above \( k = \pm e^C \))
Apply the initial-value condition:

\[ 2 + 1 = k \exp(-e^0) \]
\[ 3 = ke^{-1} \]
\[ k = 3e \]

Thus, the solution to this initial-value problem is:
\[ y = 3e \cdot \exp(-e^{-x}) - 1 \]

**Problem 10**

Solve the differential equation:
\[ \frac{dy}{dx} = y(1 + y) \]

The equation is separable so we separate and integrate:

\[ \frac{dy}{1(y + 1)} = dx \]
\[ \int \frac{dy}{1(y + 1)} = \int dx \]

Using partial fractions to decompose the left hand side:
\[ \frac{1}{(y + 1)(y)} = \frac{A}{y + 1} + \frac{B}{y} = \frac{Ay + By + A}{(y + 1)(y)} \]

so that \( A = 1 \) and \( B = -1 \), then we apply the decomposition and integrate:

\[ \int \frac{dy}{y} - \int \frac{dy}{y + 1} = \int dx \]
\[ \ln|y| - \ln|y + 1| = x + C \]
\[ \ln \left| \frac{y}{y + 1} \right| = x + C \]

Let \( k = \pm e^C \):
\[ \frac{y}{y + 1} = ke^x \]
\[ y = (y + 1)ke^x = yke^x + ke^x \]
\[ y(1 - ke^x) = ke^x \]
\[ y = \frac{ke^x}{1 - ke^x} \]
Problem 11

Solve the initial-value problem with \( y(1) = 5 \):

\[
\frac{dy}{dx} = 2y(3 - y)
\]

The equation is separable so we separate and integrate:

\[
\frac{dy}{(y)(3 - y)} = 2dx
\]

\[
\int \frac{dy}{(y)(3 - y)} = \int 2dx
\]

Applying partial fraction decomposition to the left will yield:

\[
\frac{1}{3} \left( \int \frac{dy}{y} + \int \frac{dy}{3 - y} \right) = 2x + C
\]

\[
\ln |y| - \ln |3 - y| = 6x + C
\]

\[
\ln \left| \frac{y}{3 - y} \right| = 6x + C
\]

Let \( k = \pm e^C \):

\[
\frac{y}{3 - y} = ke^{6x}
\]

Plugging in the initial value constraints:

\[
\frac{5}{3 - 5} = ke^6
\]

\[
k = \frac{-5}{2e^6}
\]

Plugging that in and finishing solving for \( y \):

\[
\frac{y}{3 - y} = \frac{-5e^{6x}}{2e^6}
\]

\[
y = \frac{-15e^{6x}}{2e^6} + \frac{5e^{6x}}{2e^6} \cdot y
\]

\[
y \left( 1 - \frac{5e^{6x}}{2e^6} \right) = \frac{-15e^{6x}}{2e^6}
\]

\[
y = \frac{-15e^{6x}}{2e^6 \cdot \left( 1 - \frac{5e^{6x}}{2e^6} \right)}
\]

\[
y = \frac{-15e^{6x}}{2e^6 - 5e^{6x}}
\]