Problem 1

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ be two vectors. Compute the lengths of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_1 + 2\mathbf{x}_2$, and $-2\mathbf{x}_1$.

For a vector $\mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$, $\|\mathbf{v}\| = \sqrt{(v_x)^2 + (v_y)^2}$ by definition. Thus:

$$\|\mathbf{x}_1\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\|\mathbf{x}_2\| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$$

$$\mathbf{x}_1 + 2\mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

$$\|\mathbf{x}_1 + 2\mathbf{x}_2\| = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$$

$$-2\mathbf{x}_1 = -2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

$$\| -2\mathbf{x}_1\| = \sqrt{(-2)^2 + (-6)^2} = \sqrt{40} = 2\sqrt{10}$$

Problem 2

Use a rotation matrix to rotate the vector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ counterclockwise by the angle $\frac{\pi}{3}$.

The rotation matrix to rotate a vector counterclockwise by the angle $\theta$ is simply $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Thus, the rotation matrix that rotates a vector by an angle of $\frac{\pi}{3}$ counterclockwise is

$$\begin{bmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

To rotate $\mathbf{x}_1$ by this angle, then, we simply multiply by our rotation matrix:

$$\frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 - 3\sqrt{3} \\ \sqrt{3} + 3 \end{bmatrix}$$

and this last vector is then the result of rotating $\mathbf{x}_1$ counterclockwise by the angle $\frac{\pi}{3}$.

Problem 3

Find the counterclockwise angle(s) of rotation $\theta$ so that $R_\theta(\mathbf{x}_1)$ where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is parallel to $\mathbf{x}_2 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

To solve this, we will find the counterclockwise angles of rotation from the positive $x_1$ axis (i.e. the horizontal) to each of these vectors. The desired angle will then be the difference of these. The angle from the positive horizontal axis to the first vector is simply $\tan^{-1}\left(\frac{3}{1}\right)$, and then we can compute angle from the negative horizontal axis to the second vector in a similar way—this is $\tan^{-1}\left(\frac{-3}{-2}\right)$—and then the angle from the positive horizontal axis is simply $\tan^{-1}\left(\frac{3}{1}\right) + \pi$.

Finally, the difference of these angles is $\tan^{-1}\left(\frac{3}{1}\right) + \pi - \tan^{-1}\left(\frac{3}{1}\right)$, which is about 2.875 radians.

Problem 3 continued on next page...
Another way to see this is to compute the dot product of these two vectors, \( x_1 \cdot x_2 = 1 \cdot -2 + 3 \cdot -3 = -11 \), then divide this by the products of their lengths (\( |x_1| = \sqrt{10} \), \( |x_2| = \sqrt{13} \)). This is the cosine of the angle between them, and taking the inverse cosine, we again find that the angle \( \cos^{-1} \left( \frac{-11}{\sqrt{130}} \right) \) is about 2.875 radians. However, either of the exact forms given here suffices as an answer—it is not necessary to compute the decimal approximation.

Problem 4

Find the eigenvalues of the matrices:

\[
A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}
\]

and

\[
B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}
\]

and find at least one eigenvector for each eigenvalue.

\( A \) is upper-triangular, so its eigenvalues are its diagonal terms, \( \lambda_1 = -1 \) and \( \lambda_2 = 2 \). Now, we find eigenvectors for these eigenvalues:

\[
0 = (A - \lambda I)v = \begin{bmatrix} 2 + 1 & 3 \\ 0 & 0 \end{bmatrix}v = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}v
\]

so that an eigenvector for \( \lambda_1 \) must satisfy \( 3v_1 + 3v_2 = 0 \), or \( v_1 = -v_2 \). Thus, one eigenvector for this eigenvalue is \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

\[
0 = (A - \lambda_2 I)v = \begin{bmatrix} 2 - 2 & 3 \\ 0 & -1 \end{bmatrix}v = \begin{bmatrix} 0 & 3 \\ 0 & -1 \end{bmatrix}v
\]

so that an eigenvector for \( \lambda_2 \) must satisfy \( v_1 = -v_2 \). Thus, one eigenvector for this eigenvalue is \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

To find the eigenvalues of \( B \), we compute its characteristic polynomial:

\[
\det(\lambda I - B) = \det \begin{vmatrix} \lambda - 2 & -1 \\ -2 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 3) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)
\]

so that the eigenvalues of \( B \) are \( \lambda_1 = 1 \) and \( \lambda_2 = 4 \).

\[
0 = (B - \lambda_1 I)v = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}v
\]

so that an eigenvector for \( \lambda_1 \) must satisfy \( v_1 + v_2 = 0 \) so that one eigenvector for this eigenvalue is \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

\[
0 = (B - \lambda_2 I)v = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}v
\]

so that an eigenvector for \( \lambda_2 \) must satisfy \( 2w_1 - w_2 = 0 \). Thus, one eigenvector for this eigenvalue is \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).
Problem 5

Represent \( x = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \) as a linear combination of \( u_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \) and \( u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

To do this, we must find \( a, b \in \mathbb{R} \) such that \( x = au_1 + bu_2 \), which is the same as solving \(-1 = a + b\) and \(2 = 4a + b\). We write this as a matrix equation:

\[
\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}
\]

and then we can solve this by e.g. row-reducing the corresponding augmented matrix, multiplying on the left by the inverse of our two by two matrix, or by working with the previous linear equations directly. In the end, we will find that \( a = 1 \) and \( b = -2 \), and we then can easily check that these satisfy the above equations.

Thus, \( x = u_1 - 2u_2 \).

Problem 6

Show that \( x_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \) and \( x_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \) are linearly independent.

For these to be linearly dependent, there would have to be a constant \( a \) such that \( ax_1 = x_2 \). In this case, \(-2a = 3\) and \( 1a = -1 \), which means (substituting the second equation into the first) that \( 2 = 3 \). This is not the case, so these vectors must be linearly independent.

Problem 7

Let

\[
A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}
\]

and \( x = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \). Find \( A^{20}x \). This problem was optional.

We do not want to compute \( A^{20} \) so instead we shall analyze the eigenvectors of \( A \). \( A \) is upper-triangular and therefore has \(-1, 2\) as its eigenvalues; \( A + I = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \) so that an eigenvector for the eigenvalue \(-1\) is \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) (reading off of the matrix, we will need \( v_2 = 0 \) but \( v_1 \) is free), while \( A - 2I = \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \) so that we need \( 3v_1 = v_2 \) in our eigenvector, so that \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) is an eigenvector for the eigenvalue 2.

Now, we shall represent \( x \) as a linear combination of these eigenvectors. Luckily, this is not difficult—we take \( 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) and verify that this is equal to \( x \), and then we apply \( A^{20} \) to this decomposition, using the linearity of \( A \) and the fact that these vectors in the decomposition are eigenvectors:

\[
A^{20}x = 2A^{20} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - A^{20} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2(-1)^{20} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2^{20} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1048576 \end{bmatrix} + \begin{bmatrix} 0 \\ -3145728 \end{bmatrix} = \begin{bmatrix} -1048574 \\ -3145728 \end{bmatrix}
\]
Problem 8

Find the lengths of the following vectors:

\[ x_1 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \quad x_2 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \]

Similarly to Problem 1 (although in 3 dimensions rather than 2), if \[ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \], then

\[ \|v\| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}. \]

Thus:

\[ \|x_1\| = \sqrt{(-2)^2 + 1^2 + (-3)^2} = \sqrt{14} \]
\[ \|x_2\| = \sqrt{(-1)^2 + 4^2 + 3^2} = \sqrt{26} \]
\[ \|x_3\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6} \]

Problem 9

Find the dot product of the following pairs of vectors:

(a) \[ x_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \]

(b) \[ x_1 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \text{ and } x_1 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \]

By the definition of the dot product:

(a) \[ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -3 \end{bmatrix} = 1 \cdot (-2) + 3 \cdot (-3) = -11 \]

(b) \[ \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = (-2) \cdot (-1) + 1 \cdot 4 + (-3) \cdot 3 = -3 \]

Problem 10

Find the angles between the following pairs of vectors:

(a) \[ x_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \]

(b) \[ x_1 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \text{ and } x_1 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \]

First, we recall that if the angle between two vectors \( v \) and \( w \) is \( \theta \), then \( \cos(\theta) = \frac{v \cdot w}{\|v\|\|w\|} \).

(a) Since we already know that the dot product of these vectors is \(-11\) (from Problem 9), we simply compute their lengths: \( \|x_1\| = \sqrt{10} \) and \( \|x_2\| = \sqrt{13} \). Then, the angle between these vectors is simply \( \cos^{-1}\left(\frac{-11}{\sqrt{10} \sqrt{13}}\right) \). If we wish to approximate this, it is about \( 2.875 \) radians, but the exact form given earlier, \( \cos^{-1}\left(\frac{-11}{\sqrt{130}}\right) \), suffices as an answer.
(b) Since we again know the dot product of these two vectors (it is \(-3\)) we check their lengths from Problem 8, finding that these are \(\sqrt{14}\) and \(\sqrt{26}\), respectively. Thus, the angle between these vectors is 
\[
\cos^{-1}\left( -\frac{3}{\sqrt{14}\sqrt{26}} \right).
\]
Again, this exact form suffices as an answer, but it is about 1.729 radians, if we wish to approximate it.

Problem 11

Let \(\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}\). Find a vector that is perpendicular to \(\mathbf{x}\).

A vector perpendicular to \(\mathbf{x}\) is a vector \(\mathbf{v}\) such that \(\mathbf{x} \cdot \mathbf{v} = 0\). If we let \(\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}\), then this equation tells us that \(-2a + b - 3c = 0\), so we can pick any vector \(\mathbf{v}\) for which this equation holds. In particular, \(a = 1\), \(b = 2\), and \(c = 0\) will satisfy this equation, so that \(\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\) is a vector that is perpendicular to \(\mathbf{x}\).

Problem 12

A triangle has vertices at coordinates \(P = (1, 2, 3)\), \(Q = (1, 5, 2)\), and \(R = (2, 4, 2)\).

(a) Compute the lengths of all three sides.

The lengths of these sides will be the lengths of the vectors obtained as differences of the vertices:

\[
\overrightarrow{PQ} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \quad \overrightarrow{PR} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \overrightarrow{QR} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}
\]

and we then compute the lengths of these sides in the standard way:

\[
\|\overrightarrow{PQ}\| = \sqrt{0^2 + 3^2 + (-1)^2} = \sqrt{10}
\]
\[
\|\overrightarrow{PR}\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}
\]
\[
\|\overrightarrow{QR}\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}
\]

Thus, the lengths of the three sides are \(\sqrt{10}\), \(\sqrt{6}\), and \(\sqrt{2}\).

(b) Compute all three angles (in radians).

To compute these angles, we take the dot products of the vectors above and recall the formula for the angle between vectors, as in Problem 10: if the angle between two vectors \(\mathbf{v}\) and \(\mathbf{w}\) is \(\theta\), then
\[
\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}.
\]
Since we already know all of these lengths, we first need to compute the dot products:

\[
\overrightarrow{PQ} \cdot \overrightarrow{PR} = 0 \cdot 1 + 3 \cdot 2 + (-1) \cdot (-1) = 7
\]
\[
\overrightarrow{QP} \cdot \overrightarrow{QR} = 0 \cdot 1 + (-3) \cdot (-1) + 1 \cdot 0 = 3
\]
\[
\overrightarrow{RP} \cdot \overrightarrow{RQ} = (-1) \cdot (-1) + (-2) \cdot 1 + 1 \cdot 0 = -1
\]
Thus, the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\cos^{-1}\left(\frac{7}{\sqrt{60}}\right)$, the angle between $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ is $\cos^{-1}\left(\frac{3}{\sqrt{20}}\right)$, and the angle between $\overrightarrow{PR}$ and $\overrightarrow{QR}$ is $\cos^{-1}\left(-\frac{1}{\sqrt{12}}\right)$. If we wish to, we may compute numerically to verify that they sum to $\pi$ (they do) but this is unnecessary; the exact forms given above are sufficient.

The following paragraph is somewhat technical and is only of interest if you wish to verify this computation on your own.

Note that in the computation of dot products above we made sure that our vector were both based at the same point—we used the vectors from $P$ to $Q$ and from $P$ to $R$, for example. This ensures that we measure the interior angle of the triangle—if instead we used the vector from $Q$ to $P$, and then from $P$ to $R$, we would instead have measured the exterior angle of the triangle at $P$. If you try to sum numerical approximations of the angles you calculated with a calculator, and they sum to something other than $\pi$, make sure that you’re summing the interior angles, and not some combination of interior and exterior angles! However, if you instead sum all three exterior angles, you should arrive at some familiar constant that is not $\pi$...

**Problem 13**

Find the equation of the line on the plane through $(2, 1)$ and perpendicular to the vector $\begin{bmatrix}1 \\ 2\end{bmatrix}$.

We have a point on this line, we must now determine its slope. Since it is perpendicular to the vector $\begin{bmatrix}1 \\ 2\end{bmatrix}$, it points in the direction of a vector perpendicular to this, i.e. a vector $\begin{bmatrix}a \\ b\end{bmatrix}$ such that $\begin{bmatrix}1 \\ 2\end{bmatrix} \cdot \begin{bmatrix}a \\ b\end{bmatrix} = 0$

This means that $a+2b = 0$ or $a = -2b$. Thus, one such vector is $\begin{bmatrix}-2 \\ 1\end{bmatrix}$. Thus, we may obtain parametric equations for our line in the standard method, by adding to our point $t$ times our vector:

$$x(t) = 2 - 2t$$
$$y(t) = 1 + t$$

Then, to solve this into a standard linear equation without the time parameter $t$, we solve for $t$ in both equations:

$$\frac{2-x}{2} = t \quad \text{and} \quad y - 1 = t$$

then set these equal to each other:

$$\frac{2-x}{2} = y - 1$$

solving for $y$:

$$y = \frac{-x}{2} + 2$$

and this is the equation of the specified line.
Problem 14

Find a parametric equation of the line in the plane that goes through the point \((-1, 4)\) in the direction of the vector \(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\).

The parametric equations for this line can be found by adding \(t\) times the direction vector to the specified point. This yields:

\[
\begin{align*}
  x(t) &= -1 + 2t \\
  y(t) &= 4 - 3t 
\end{align*}
\]

which is a parametric representation of the specified line.

Problem 15

Find a parametric equation of the line in space that goes through the point \((2, 0, 4)\) in the direction of the vector \(\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}\).

Just as in problem 14, we add \(t\) times the direction vector to the point that we want to be on the line to find a parametric representation of the line:

\[
\begin{align*}
  x(t) &= 2 + t \\
  y(t) &= 2t \\
  z(t) &= 4 - 3t 
\end{align*}
\]

which parametrically represent the desired line.

Problem 16

Find a parametric equation of the line in space that goes through the points \((2, -3, 1)\) and \((-5, 2, 1)\).

To reduce this to the form of the previous two questions, we find the direction vector \(\mathbf{v}\) from one point to the other:

\[
\mathbf{v} = \begin{bmatrix} 2 - (-5) \\ -3 - 2 \\ 1 - 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 0 \end{bmatrix}
\]

Now, we add \(t\) times this direction vector to our starting point, which we choose to be the second point arbitrarily:

\[
\begin{align*}
  x(t) &= -5 + 7t \\
  y(t) &= 2 - 5t \\
  z(t) &= 1 
\end{align*}
\]

and we observe that at \(t = 0\) the line passes through the second point, and at time \(t = 1\), it passes through the first. So this is a parametric representation of a line through these two points, as desired.
Problem 17

Suppose $P$ is a plane in space through $(1, -1, 2)$ perpendicular to the vector $[1, 2, 1]$; suppose $L$ is a line through the points $(0, -3, 2)$ and $(-1, -2, 3)$. Where do the plane $P$ and the line $L$ intersect?

First, we will compute equations for the plane and the line. Since the plane is perpendicular to $[1, 2, 1]$, any vector in the plane must be perpendicular to this given vector. To find a vector in the plane, let $(x, y, z)$ be in the plane, and consider $[x-1, y+1, z-2]$, the vector between $(x, y, z)$ and the given point in the plane, $(1, -1, 2)$. Since we have two perpendicular vectors, we compute their dot product, and this will give us our equation for the plane:

\[(x-1) + 2(y+1) + 1(z-2) = 0\]

\[x - 1 + 2y + 2 + z - 2 = 0\]

\[x + 2y + z = 1\]

and this is an equation for the plane.

To get a parametric equation for the line, we begin by finding a direction vector, as in Problem 16, one possible direction vector is $[1, -1, -1]$, and this line passes through $(-1, -2, 3)$ so one parametrization of it is:

\[x(t) = -1 + t\]

\[y(t) = -2 - t\]

\[z(t) = 3 - t\]

and it is easy to see that this passes through the second point at $t = 0$ and the first point at $t = 1$. Now, to combine this information, we plug these equations into the equation for the plane, and solve for $t$:

\[(-1 + t) + 2(-2 - t) + (3 - t) = 1\]

\[-1 + t - 4 - 2t + 3 - t = 1\]

\[-2t = 3\]

\[t = -\frac{3}{2}\]

and now we plug this back into the equations for our line to find $(x, y, z)$:

\[x\left(-\frac{3}{2}\right) = -1 - \frac{3}{2} = -\frac{5}{2}\]

\[y\left(-\frac{3}{2}\right) = -2 + \frac{3}{2} = -\frac{1}{2}\]

\[z\left(-\frac{3}{2}\right) = 3 + \frac{3}{2} = \frac{9}{2}\]
And then we find that the point of intersection is \((-\frac{5}{2}, \frac{-1}{2}, \frac{9}{2})\) and it is easy to check that this point also lies on the plane.