Exercise 1 (2.1.31). Given a function $f: \mathbb{R}^2 \to \mathbb{R}$, can two level curves of $f$ intersect?

Proof. No. Suppose for the sake of contradiction that the level curves $f(x, y) = c$ and $f(x, y) = c'$ ($c \neq c'$) of $f$ intersect. Then there is a point $(x_0, y_0)$, so that $f(x_0, y_0) = c$ and $f(x_0, y_0) = c'$. This contradicts the assumption that $f$ is a function.

Exercise 2 (2.2.6). Determine whether the set

$$S := \left\{ (x, y, z) \in \mathbb{R}^3; 1 < x^2 + y^2 < 4 \right\}$$

is open, closed, both open and closed, or neither open nor closed.

Proof. This set is open, but not closed. This is easiest to see if we convert to cylindrical coordinates. Using the standard conversion formulae, and using the convention $r \geq 0$, we observe that

$$S = \left\{ (r, \theta, z) \in \mathbb{R}^3; 1 < r < 2 \right\}.$$

Now let $s_0 := (r_0, \theta_0, z_0) \in S$. By definition of $S$, we know that $1 < r_0 < 2$. Set $\rho := \min\{r_0 - 1, 2 - r_0\}/2$. By construction, we clearly have that $B_{\rho}(s_0) \subset S$. We conclude that $S$ is open.

To see that $S$ is not closed, consider the complement $S^c$ of $S$. Now take a point $s_1 := (r_1, \theta_1, z_1)$ with $r_1 = 1$. Then if $B$ is any ball centered at $s_1$, we have $B \cap S \neq \emptyset$. Hence $B \not\subset S^c$, and we conclude that $S^c$ is not open. This completes the proof that $S$ is not closed.

Exercise 3 (2.2.30). Evaluate

$$\lim_{(x, y) \to (0, 0)} \frac{x^2 + xy + y^2}{x^2 + y^2}$$

by converting to polar coordinates.

Solution. In polar coordinates the limit above becomes

$$\lim_{(r, \theta) \to (0, 0)} \frac{r^2 + r^2 \sin \theta \cos \theta}{r^2} = 1 + \lim_{\theta \to 0} \sin \theta \cos \theta = 1.$$

Exercise 4 (2.3.34). Explain why the function

$$f(x, y) = xy - 7x^8 y^2 + \cos x$$

is differentiable in every point of its domain.

Proof. Polynomial and trigonometric functions are differentiable at every point of their domain, and the sum of differentiable functions is differentiable.
Exercise 5 (2.3.40). Find equations for the planes tangent to \( f(x, y) = x^2 - 6x + y^3 \) that are parallel to the plane \( P : 4x - 12y + z = 7 \).

Solution. Since the planes we are looking for are parallel to \( P \), they have the same tangent vector and are thus of the form

\[
4x - 12y + z = C
\]

for some \( C \in \mathbb{R} \). On the other hand, tangent planes to \( f \) are given by

\[
z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

Therefore we are looking for points \((x_0, y_0)\) \( \in \mathbb{R}^2 \) so that \( f_x(x_0, y_0) = -4 \) and \( f_y(x_0, y_0) = 12 \). In other words we need to solve the following system of equations

\[
\begin{aligned}
2x - 6 &= -4, \\
3y^2 &= 12.
\end{aligned}
\]

Solving gives the points \((1, 2)\) and \((1, -2)\). Hence the equations of the planes we want are

\[
z - 3 = -4(x - 1) + 12(y - 2)
\]

and

\[
z + 13 = -4(x - 1) + 12(y + 2).
\]