Linear system: Applications

1. Compartment models.

\[
\begin{align*}
\frac{dx_1}{dt} &= I - (a+c)x_1 + bx_2 \\
\frac{dx_2}{dt} &= ax_1 - (b+d)x_2
\end{align*}
\]

I: constant, at least one of a, b, c and d is positive.

Case 1. \( I > 0 \), then the system is inhomogeneous

(and we don’t know how to solve yet).

Case 2. \( I = 0 \), we are reduced to a linear system

\[
\frac{d\vec{x}}{dt} = A \vec{x}(t)
\]

where

\[
A = \begin{bmatrix}
-(a+c) & b \\
\ a & -(b+d)
\end{bmatrix}
\]
To find the eigenvalues of $A$, we compute

$$\det \begin{bmatrix} -(a+c) - \lambda & b \\ a & -(b+d) - \lambda \end{bmatrix}$$

$$= (a+c+\lambda)(b+d+\lambda) - ab$$

$$= \lambda^2 + (a+b+c+d) \lambda + (a+c)(b+d) - ab = 0$$

First of all, note that the discriminant is

$$(a+b+c+d)^2 - 4 \left[ (a+c)(b+d) - ab \right]$$

$$= \left[ (c+b) - (c+d) \right]^2 + 4ab = 0$$

So both eigenvalues are real.

Let's investigate a bit more into the eigenvalues:

recall that the sum of the two eigenvalues is the trace of $A$, namely $-(a+b+c+d)$, whereas the product of them is $\det(A)$, the determinant of $A$. 
Since \(-(a+b+c+d)\leq 0\) and \(\det(A) > 0\),

\(\Rightarrow\) both eigenvalues are non-positive.

There are two cases to consider:

Case 1. \(\Delta = \det(A) = (a+c)(b+d) - ab = ad + cd + bc > 0\)

so at least one term is positive.

\(\Rightarrow\) either \(a, d > 0\), or \(b, c > 0\), or \(c, d > 0\).

In all the 3 situations, all matter will eventually leave the system s.t. \((0, 0)\) is a stable equilibrium.

Case 2. \(\Delta = 0\), then \(\lambda_1 = -(a+b+c+d)\), \(\lambda_2 = 0\).

\(\Rightarrow\) \(c = a = 0\) or \(d = b = 0\) or \(c = d = 0\).

In all 3 situations, the total amount \(x(t) + y(t)\) of matter in the compartments is constant.

(called a conserved quantity)
Solving the system when \( \alpha = \beta = 0 \).

Now \( A = \begin{bmatrix} -a & b \\ a & -b \end{bmatrix} \).

\( u = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) is an eigenvector of \( \lambda_1 = -a - b \).

\( v = \begin{bmatrix} 1 \\ a \end{bmatrix} \) is an eigenvector of \( \lambda_2 = 0 \).

\( X(t) = c_1 e^{-(a+b)t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} b \\ a \end{bmatrix} \)

and \( \lim_{t \to \infty} X(t) = c_2 \begin{bmatrix} b \\ a \end{bmatrix} \)

At \( t = 0 \), \( \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} b \\ a \end{bmatrix} \)

\( c_1 + bc_2 = x_1(0) \)

\( -c_1 + ac_2 = x_2(0) \)

\( c_2 = \frac{x_1(0) + x_2(0)}{a + b} \).

Since \( x_1(t) + x_2(t) \) is constant for \( t \geq 0 \),
divided by \( k \), then \( \lim_{t \to \infty} 10x_1(t) = k \cdot \frac{b}{a + b} \)

\( \lim_{t \to \infty} \frac{x_2(t)}{x_1(t)} = k \cdot \frac{a}{a + b} \).
Assumption: A particle is moving along the $x$-$axis$, the acceleration is proportional to the distance to the origin and the direction of the acceleration points toward the origin.

Let $x(t)$ be the location of the particle at time $t$, so we find

$$\frac{d^2 x(t)}{dt^2} = -k x(t).$$

for some constant $k > 0$. → Second order equation

We can in fact solve it! (by breaking it into a system of first order equations).

Consider its velocity $v(t)$, which is the derivative of the location $x(t)$, and the antiderivative of the acceleration of the particle.
that is:

\[
\frac{dx}{dt} = v(t)
\]

\[
\frac{dv}{dt} = \frac{d^2x}{dt^2} = -kx(t).
\]

As such, we get a system consisting of:

\[
\frac{dx}{dt} = v
\]

\[
\frac{dv}{dt} = -kx
\]

In matrix notation:

\[
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dv}{dt}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-k & 0
\end{bmatrix} \begin{bmatrix}
x \\
v
\end{bmatrix}
\]

Note that \(\text{tr} A = 0\) and \(\det A = -k > 0\).

\(\rightarrow\) the eigenvalues of \(A\) are complex (pure imaginary).

\(\rightarrow\) \(\det (A - \lambda I) = \lambda^2 + k < 0\).

\(\rightarrow\) \(\lambda_1 = \sqrt{k} + i\) and \(\lambda_2 = -\sqrt{k} - i\)
The equilibrium (0,0) should be a neutral spiral.

To solve this system, we recognize sine and cosine are a pair of functions with being the mutual derivative (up to a sign).

\[ \cos(\omega t) \quad \text{and} \quad \sin(\omega t) \]

will solve the system.