Eigenvectors II

E.g. Without calculation:

\[ A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \]

We see \( \det (A - \lambda I) = \det \begin{bmatrix} -2 - \lambda & 1 \\ 0 & -1 - \lambda \end{bmatrix} \)

\[ = (-2 - \lambda)(-1 - \lambda) \]

\[ \Rightarrow \lambda_1 = -2, \lambda_2 = -1 \]

These are precisely the diagonal entries of \( A \).

Observation. If \( A \) is a 2x2 matrix where either or both of the off-diagonal entries is zero, then the diagonal entries are the eigenvalues.

E.g. Eigenvectors can be complex numbers.

\[ A = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \]
Trace.

Let $A$ be a square matrix. The trace of $A$, denoted by $\text{tr}(A)$, is the sum of all diagonal entries:

$$\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}. $$

(Here assuming $A$ is $n \times n$.)

Fact. $\text{tr}(A)$ is also the "sum" of its eigenvalues.

(not exactly)

e.g. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

By definition, $\text{tr}(A) = a + d$.

Let's try to find its eigenvalues.

Computing $\det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

\[= (a-\lambda)(d-\lambda) - bc \]

\[= \lambda^2 - (a+d)\lambda + ad-bc. \]

Observation: $\lambda_1 + \lambda_2 = a+d = \text{tr}(A)$

$\lambda_1 \cdot \lambda_2 = ad-bc = \det(A)$. 
Linear algebra

Linear independence

Def. Easy to define. For 2 vectors, let $\vec{v}_1$ and $\vec{v}_2$ ($\neq \vec{0}$) be two vectors (on the plane, in space, or even in higher dimensional spaces). They are said to be linearly independent if one is not a scalar multiple of the other. Conversely, if there is a number $a$ such that $a \vec{v}_1 = a \vec{v}_2$, they are said to be linearly dependent.

Some properties:

1. If $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent, $\vec{v}_2$ and $\vec{v}_3$ are linearly dependent, then $\vec{v}_1$ and $\vec{v}_3$ are linearly dependent.

Remark: The number $a$ can be negative as well as positive.

It's not so easy to generalize this concept to more than 2 vectors.
**Upshot:** Let $A$ be a $2 \times 2$ matrix with eigenvalues $\lambda_1$ and $\lambda_2$. Let $\vec{v}_1$ (resp., $\vec{v}_2$) be an eigenvector of $\lambda_1$ (resp., $\lambda_2$). If $\lambda_1 \neq \lambda_2$, then $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

**Ex.** On the plane, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

From geometry we know any point on the plane is uniquely determined by its coordinates. Say the point $P = (2, 3)$, the vector $\overrightarrow{OP}$ is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. By splitting the two components of the coordinates of $P$, the vector $\overrightarrow{OP}$ can be written as

$$\overrightarrow{OP} = 2 \vec{v}_1 + 3 \vec{v}_2 \quad (\ast)$$

where 2 and 3 are precisely the coefficients of the point $P$. Clearly, this procedure ($\ast$) can be done
For any vector on the plane.

Alternatively, take \( \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

They are linearly independent.

Question, can you express the vector \( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) as a combination of \( \vec{u}_1 \) and \( \vec{u}_2 \) (instead of \( \vec{v}_1 \) and \( \vec{v}_2 \))?

Let's start by assuming we can do it:

\[
\begin{bmatrix} 2 \\ 3 \end{bmatrix} = a \vec{u}_1 + b \vec{u}_2 \quad (**)
\]

It's enough to find the numbers \( a \) and \( b \) above.

Plugging in \( \vec{u}_1 \) and \( \vec{u}_2 \):

\[
\begin{bmatrix} 2 \\ 3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}
\]

\[
\Rightarrow \quad \begin{cases} 2 = -b \\ 3 = a \end{cases} \quad \Rightarrow \quad \begin{cases} a = 3 \\ b = -2 \end{cases} \quad \text{Done!}
\]
Think about it: this can also be done for any vector on the plane. 

Given two vectors \( \vec{u}_1 \) and \( \vec{u}_2 \). For any numbers \( a \) and \( b \), 
\[ a \vec{u}_1 + b \vec{u}_2 \] 
is called a linear combination of \( \vec{u}_1 \) and \( \vec{u}_2 \).

If \( \vec{u}_1 \) and \( \vec{u}_2 \) are linearly independent, then any vector on the plane can be written as a linearly combination of \( \vec{u}_1 \) and \( \vec{u}_2 \).

**Question.** How to generalize this statement to space?

**An application. (Iteration of linear maps)**

Suppose \( A \) is 2x2 with eigenvalues \( \lambda_1 \) and \( \lambda_2 \) and associated eigenvectors \( \vec{v}_1 \) and \( \vec{v}_2 \). Assume \( \vec{u}_1 \) and \( \vec{u}_2 \) are linearly independent.

Let \( \vec{x} \) be any vector. By definition, we can write \( \vec{x} \) as a linear combination of \( \vec{u}_1 \) and \( \vec{u}_2 \).
say \( \vec{x} = a \vec{u}_1 + b \vec{u}_2 \) for some \( a \) and \( b \).

This gives us an easy way of calculating results such as

\[
A^2 \vec{x} = A (A \vec{x}) , \quad A^3 \vec{x} = A (A (A \vec{x}))
\]

and so on.

E.g. \( A^2 \vec{x} = A (A \vec{x}) \)

\[
= A \left( A \left( a \vec{u}_1 + b \vec{u}_2 \right) \right)
\]

\[
= A \left( a A \vec{u}_1 + b A \vec{u}_2 \right)
\]

\[
= A \left( a \lambda_1 \vec{u}_1 + b \lambda_2 \vec{u}_2 \right)
\]

\[
= a \lambda_1 A \vec{u}_1 + b \lambda_2 A \vec{u}_2
\]

\[
= a \lambda_1^2 \vec{u}_1 + b \lambda_2^2 \vec{u}_2
\]

What about \( A^3 \vec{x} \), \( A^4 \vec{x} \), \ldots, \( A^n \vec{x} \)?

\[
A^3 \vec{x} = a \lambda_1^3 \vec{u}_1 + b \lambda_2^3 \vec{u}_2
\]

In general,

\[
A^n \vec{x} = a \lambda_1^n \vec{u}_1 + b \lambda_2^n \vec{u}_2.
\]