Mean & Variance

1. The expected value of a random variable $X$ (or mean) denoted by $E(X)$ is

$$E(X) = \sum_{x} xP(X = x)$$

where the sum takes over all possible values of $X$.

Note: Expected values should be thought as some statistical tendency, and they indicate the value for which the random variables have the highest probability, or the average value of them.

E.g. Tossing a fair die once. What is $E(X)$?

The experiment has a sample space

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

and $P(X = n) = \frac{1}{6}$ for $n = 1, 2, 3, 4, 5, 6$.

Hence

$$E(X) = \sum_{x=1}^{6} x \cdot \frac{1}{6} = (1 + 2 + 3 + 4 + 5 + 6) \cdot \frac{1}{6}$$

$$= 3.5$$
Example 2: Tossing a fair die twice. Let $X$ be the random variable of the sum of the numbers facing up from the two tosses. Find $E(X)$.

Idea: First, the possible sums are 2, 3, ..., 11, 12. Now, this time not every number has the same equal probability. For example, there is only one way of having sum equal to 2, $(2=1+1)$ and there are many ways of summing to 6, $(6 = 1+5 = 2+4 = 3+3 = 4+2 = 5+1)$.

We can list them in a table:

<table>
<thead>
<tr>
<th>$X$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X=x)$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{2}{36}$</td>
<td>$\frac{3}{36}$</td>
<td>$\frac{4}{36}$</td>
<td>$\frac{5}{36}$</td>
<td>$\frac{6}{36}$</td>
<td>$\frac{5}{36}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$X$</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X=x)$</td>
<td>$\frac{4}{36}$</td>
<td>$\frac{3}{36}$</td>
<td>$\frac{2}{36}$</td>
<td>$\frac{1}{36}$</td>
</tr>
</tbody>
</table>
Therefore,

\[ E(x) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + \cdots + 12 \cdot \frac{12}{36} \]

**Key Properties of the expected value**

1. Let \( a \) and \( b \) be constants. Then
   
   \[ E(ax + b) = a[E(x)] + b \]
   
   \[ E(x + y) = E(x) + E(y) \]

2. What is \( x + y \)?
   
   For two random variables \( x \) and \( y \) having the same sample space, \( x + y \) is another random variable.

**Eq 3. The Binomial distribution \( Sn \).**

Recall: \( Sn \) = random variable counting \# successes in \( n \) independent trials, where each has probability \( p \).

Let’s calculate \( E(Sn) \).

Idea: Let’s first recall the probability mass function of \( Sn \).
Within n trials, the probability of winning k trials is:

\[ P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k} \]

So by definition of the expected value,

\[ E(S_n) = 0 \cdot P(S_n = 0) + 1 \cdot P(S_n = 1) + 2 \cdot P(S_n = 2) + \cdots + n \cdot P(S_n = n) \]

If we were trying to proceed in this way, we need to calculate this sum \((*)\), which would be really complicated (wanna buy?)

Instead, we take the random variable \(S_n\) and interpret it alternatively. We introduce a sequence of \(n\) random variables \(X_1, X_2, \ldots, X_n\). Each \(X_i\) only takes values either 0 or 1.

\[ X_i = \begin{cases} 1 & \text{if } i\text{-th trial is successful} \\ 0 & \text{otherwise} \end{cases} \]

Since trials are independent, the \(X_i\)'s are independent.
Since $S_n$ counts the total number of successful trials, we recognize that

$$S_n = X_1 + X_2 + \ldots + X_n$$

(because if a trial is successful, it returns a "1", otherwise a "0").

In this way, we see

$$E(S_n) = E(X_1) + E(X_2) + \ldots + E(X_n).$$

And the calculation of $E(S_n)$ is reduced to calculating $E(X_1), \ldots, E(X_n)$. These are much easier:

$$E(X_i) = 1 \cdot p + 0 \cdot (1-p) = p.$$

for $i=1,2,\ldots,n$.

Therefore, $E(S_n) = n \cdot p$. \(\Box\)

Recall the scenario: play a Bernoulli trial repeatedly and stop the game at the first success, where each trial has a probability of success $p$.

We see that possible values of this random variable (the number of trials) are 1, 2, 3, ... with their probabilities:

$$P(X = k) = (1-p)^{k-1}p, \quad k = 1, 2, ...$$

Find $E(X)$.

Idea. By definition, infinite sum!

$$E(X) = \sum_{k=1}^{\infty} k \cdot P(X = k)$$

$$= 1 \cdot (1-p)^0 p + 2 \cdot (1-p)^1 p + 3 \cdot (1-p)^2 p + ...$$

This calculation is a bit involved, but still manageable.

(Not required, see p 708-709.)
2. **Variance**

Another quantity that measures a distribution is the variance, which reflects the spreading of the random variable.

Eq. Two micro towns have 6 households each.

The annual income of them are as follows:

Town A: 200,000, 200,000, 200,000, 200,000, 200,000, 2,000,000

Town B: 500,000, 500,000, 500,000, 500,000, 500,000, 500,000

We can easily see that the average annual household income of town A and town B is equal, but town B has higher income uniformity, or town A has higher income inequality. Such income inequality is reflected by a bigger variance.

**Def.** For a random variable $X$ with $E(X) = \mu$, the variance of $X$ is defined as

$$\text{Var}(X) = E[(X - \mu)^2]$$
In particular, if $X$ is discrete, then
\[ \text{var}(X) = \sum_{x} (x - \mu)^2 \, P(X = x) \]

where the sum takes over all possible values of $X$.

Another related quantity is the **standard deviation**, denoted by $\text{s.d.}$, or $\sigma$.

\[ \text{s.d.} = \sigma = \sqrt{\text{Var}(X)} \]

**Properties of the variance**

\[ \text{var}(aX + b) = a^2 \, \text{var}(X) \]
\[ \text{var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \]
\[ \text{var}(X + Y) = \text{var}(X) + \text{var}(Y) \]

**Two examples:**

*Example 1: Binomial distribution* $(S_n, p)$
\[ \text{var}(S_n) = np(1-p) \]

(This is not required.)
Eq 2. Geometric distribution

\[ P(X = k) = (1-p)^{k-1} \cdot p \]

\[ \text{Var}(X) = \frac{1-p}{p^2} \]

(This is not required.)

Eq 3. Uniform distribution. We've seen it many times, so recap for convenience.

Say \( \Omega = \{1, 2, \ldots, n\} \) (for tossing a die, \( n=6 \).

For flipping a coin, \( n=2 \).

Suppose \( P(X = k) = \frac{1}{n} \) for \( k=1, \ldots, n \).

(i.e., each outcome has equal probability \( \frac{1}{n} \).)

We can find \( E(X) \) and \( \text{Var}(X) \).

\[
E(X) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + \ldots + n \cdot \frac{1}{n}
= \left(1+2+\ldots+n\right) \cdot \frac{1}{n} = \frac{n(n+1)}{2} \cdot \frac{1}{n} = \frac{n+1}{2}
\]

\[
\text{Var}(X) = E(X^2) - [E(X)]^2
= \left(1^2 + 2^2 + \ldots + n^2\right) \frac{1}{n} - \left(\frac{n+1}{2}\right)^2
\]

Can you check this step?..