Partial derivatives

**Definition**: Suppose \( f(x, y) \) is a function of \( x \) and \( y \).

The **partial derivative** of \( f \) with respect to \( x \) is:

\[
\frac{\partial f(x, y)}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}
\]

Likewise, the partial derivative of \( f \) with respect to \( y \) is

\[
\frac{\partial f(x, y)}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}
\]

**Notation**: \( f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y} \).

In words, the partial derivative of \( f(x, y) \) with respect to \( x \) is obtained as follows:

1. Think of \( y \) as a constant, then \( f(x, y) \) is thought of as a function solely in \( x \).
2. Compute the usual derivative of this function with respect to \( x \).

**Example**:

Find \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) when \( f(x, y) = xe^{x+y} - y^2 \).

**Solution**: If we look at \( f(x, y) \) and regard it as a function only in \( x \), we see

\[
\frac{\partial f}{\partial x} = e^{x+y} + xe^{x+y}
\]
Also, thinking of $f(x,y)$ as a function in $y$,

$$\frac{df}{dy} = xe^{x+y} - 2y$$

**Geometric interpretation**

<table>
<thead>
<tr>
<th>$y = f(x)$</th>
<th>$z = f(x,y)$</th>
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<tbody>
<tr>
<td>$f'(x)$ at $x_0$ is the slope of the line tangent to the graph at $(x_0, f(x_0))$</td>
<td>( \frac{df(x,y)}{dx}, \frac{df(x,y)}{dy} )</td>
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To fill in the "?", we need to look for the analogy of "tangent line" in the one-variable case.

It turns out, there is such a concept, to be introduced in the next class, called "tangent plane", in 2-variable case.
We have regarded partial derivatives as the derivatives of the 1-variable functions obtained from \( f(x, y) \) if we think of the other variable as "fixed".

Q: What does it mean by fixing a variable geometrically?

That means we are slicing the graph of \( f(x, y) \) (which is a surface) by a plane perpendicular to \( xy \)-plane.

\[
\begin{align*}
\text{(this curve is the graph of the function } \ f(x_0, y) \text{ )} \\
\text{as a function of } y \text{ with } x \text{ fixed at } x_0.
\end{align*}
\]

So, if we calculate \( \frac{df(x, y)}{dy} \) at \((x_0, y_0)\), this value is the slope of the line tangent to the graph of \( f(x, y) \) at \((x_0, y_0)\).
\[ f(x, y) = xe^{x+y} - y^2. \]

Let's take some value \( x_0 = 1 \).

Plugging in \( f(x, y) \), we have

\[ f(1, y) = (1) e^{1+y} - y^2 = e^{1+y} - y^2. \]

It is a function in \( y \).

Now if we calculate its derivative, we get

\[ f'(1, y) = e^{1+y} - 2y. \]

Look at the graph of \( f(1, y) \). If we choose a point on the graph, say \( y = 0 \). Then we can evaluate the slope of the line tangent to the graph of \( f(1, y) \) at this point \( y = 0 \), which is precisely as before,

\[ f'(1, 0) = e^{1+0} - 2(0) = e. \]
There are two directions to go after our discussion:

1. Functions in more than 2 variables.
   Can you define the partial derivatives of those?


Let's look at the first question first.

If \( F(x, y, z) \) is a function in \( x, y, z \), there will be
3 partial derivatives: \( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \).

All can be calculated exactly in the same way:

If you want to calculate \( \frac{\partial F}{\partial x} \), think of \( y \) and \( z \) as constants and calculate the usual derivative of \( F \) as a function of \( x \).

E.g., find \( \frac{\partial F}{\partial y} \) for \( F(x, y, z, w) = xz - y^4 e^z + \ln(y^2 z) \)

\[
\frac{\partial F}{\partial y} = 4y^3 e^z + \frac{1}{y^2 z} \cdot 2yz
\]

\[
= 4y^3 e^z + \frac{2}{y}
\]
Higher order partial derivatives.

Let's look at the 2-variable case, more variables is the same.

Suppose \( f(x, y) \) is a function. Then \( \frac{\partial f(x, y)}{\partial x} \) and \( \frac{\partial f(x, y)}{\partial y} \) will be typically functions of \( x \) and \( y \) as well. So we can take further partial derivatives of them.

In particular, we can take 4 of them:

\[
\frac{\partial}{\partial x} \left( \frac{\partial f(x, y)}{\partial x} \right) \quad \text{and} \quad \frac{\partial}{\partial y} \left( \frac{\partial f(x, y)}{\partial y} \right)
\]

Mostly, the notations are simplified:

\[
\frac{\partial}{\partial x} \left( \frac{\partial f(x, y)}{\partial x} \right) = \frac{\partial^2 f(x, y)}{\partial x^2} \quad \text{\textit{(notation)}}
\]

\[
= f_{xx} \quad \text{\textit{(other books)}}
\]

Similarly, \( \frac{\partial}{\partial y} \left( \frac{\partial f(x, y)}{\partial x} \right) = \frac{\partial^2 f(x, y)}{\partial y \partial x} \)

\[
\uparrow \quad y \text{ comes first if it is the outer layer}
\]

\[
= f_{xy}
\]
Also \[
\frac{d}{dx} \left( \frac{df(x,y)}{dy} \right) = \frac{\delta^2 f(x,y)}{\delta x \delta y}
\]

= \frac{\delta f(x,y)}{\delta y}

= f_{yx}

finally, \[
\frac{d}{dx} \left( \frac{df(x,y)}{dy} \right) = \frac{\delta^2 f(x,y)}{\delta x \delta y}
\]

= \frac{\delta f(x,y)}{\delta y}

= f_{yy}

\text{Eg. } f(x,y) = \cos (x^2 - 2y)

\text{Find } f_x, f_y, f_{xx}, f_{xy}, f_{yx}, \text{ and } f_{yy}

\text{Solution. Notice that } f(x,y) \text{ is a composite function, so chain rule is necessary to find the partial derivatives.}

f_x = -\sin (x^2 - 2y) \cdot (2x)

f_y = -\sin (x^2 - 2y) \cdot (-2)

\text{Furthermore, product rule + chain rule}

f_{xx} = -\cos (x^2 - 2y) \cdot (2x) \cdot (2x) + (-\sin (x^2 - 2y)) \cdot (2)

f_{xy} = -\cos (x^2 - 2y) \cdot (2x) \cdot (-2)

f_{yx} = -\cos (x^2 - 2y) \cdot (-2) \cdot (2x)

f_{yy} = -\cos (x^2 - 2y) \cdot (-2) \cdot (-2)
Key observation

\[ f_{xy} = f_{yx} \]

The mixed-derivative theorem. If \( f(x,y) \), \( f_x \), \( f_y \), \( f_{xy} \) and \( f_{yx} \) are all continuous, then

\[ f_{xy} = f_{yx} \]