Math 211, Calculus III
Spring 2018
Final Exam
May 14, 2018
Time Limit: 180 Minutes

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The following instruction only applies to those who were not seeking for special accommodations.

This exam contains 16 pages (including this cover page) and 15 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your name on the top of every page, in case the pages become separated.

- You may *not* use your books, notes, or any calculator on this exam.
- You are required to show your work on each problem on this exam. The following rules apply:
- **Organize your work,** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
11	20	
12	20	
13	20	
14	20	
15	20	
Total:	300	

1. (20 points) (a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$; then $\det A = 2$ and $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & a & b \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}$. Find a and b.

(b) Solve the system $AX = B$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

(c) In the matrix A, replace the entry 2 in the upper-right corner by c. Find a value of c for which the resulting matrix M is not invertible.

Solution:

$\boxed{8 \text{ pts}}$ (a). Minors: $\begin{bmatrix} 1 & 1 & 2 \\ -2 & -2 & -2 \\ -3 & -5 & -6 \end{bmatrix}$. Cofactors: $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 2 \\ -3 & 5 & -6 \end{bmatrix}$

Inverse: $\frac{1}{2}\begin{bmatrix} 1 & \boxed{2} & \boxed{-3} \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}$

$\boxed{5 \text{ pts}}$ (b). $X = A^{-1}B = \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}$

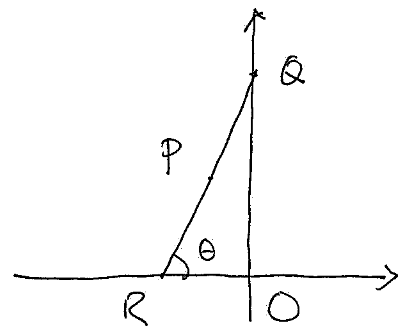
$\boxed{7 \text{ pts}}$ (c). $M = \begin{bmatrix} 1 & 3 & c \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

$$\det M = c \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow c = 1.$

2. (20 points) The top extremity of a ladder of length L rests against a vertical wall (think of the wall as the vertical axis), while the bottom is being pulled away (along the negative horizontal axis to the left). Find parametric equations for the midpoint P of the ladder, using as parameter the angle θ between the ladder and the horizontal ground.

Solution:



Q = top of ladder.

5 pts $\quad \vec{OQ} = (0, L\sin\theta)$

R = bottom of ladder

5 pts $\quad \vec{OR} = (-L\cos\theta, 0)$

Midpoint: $\vec{OP} = \frac{1}{2}(\vec{OQ} + \vec{OR})$

$= \left(-\frac{1}{2}L\cos\theta, \frac{1}{2}L\sin\theta\right)$

$\Rightarrow$ parametric equations:

10 pts
$$x = -\frac{1}{2}L\cos\theta$$
$$y = \frac{1}{2}L\sin\theta$$

3. (20 points) A rectangular box is placed in the first octant in space, with one corner at the origin and the three adjacent faces in the coordinate planes. The opposite point $P : (x, y, z)$ is constrained to lie on the paraboloid $x^2 + y^2 + z = 1$. Which P gives the box of greatest volume?

   (a) Show that the problem leads one to maximize $f(x, y) = xy - x^3y - xy^3$, and white down the equations for the critical points of f.

   (b) Find a critical point of f which lies in the first quadrant ($x > 0, y > 0$).

   (c) Determine the nature of this critical point.

   (d) Find the maximum of f in the first quadrant and justify your answer.

Solution: (a). The volume of the box B

5 pts

$$V = xyz = xy(1-x^2-y^2) = xy - x^3y - xy^3.$$

Critical points:

$$f_x = y - 3x^2y - y^3 = 0$$

$$f_y = x - x^3 - 3xy^2 = 0$$

5 pts

(b). Assuming $x>0$ and $y>0$, the equations from part (a) become $1-3x^2-y^2=0$ & $1-x^2-3y^2=0$

Solution: $(x,y) = (\frac{1}{2}, \frac{1}{2})$

5 pts

(c) $f_{xx} = -6xy = -\frac{3}{2}$, $f_{yy} = -6xy = -\frac{3}{2}$

$$f_{xy} = 1 - 3x^2 - 3y^2 = -\frac{1}{2}.$$

$$\Rightarrow f_{xx}f_{yy} - f_{xy}^2 > 0. \quad f_{xx} < 0 \Rightarrow \text{local max.}$$

5 pts

(d). The max of f in the first quadrant is either at $(\frac{1}{2}, \frac{1}{2})$ or does not exist (if on the boundary)

Note $f = 0$ if $x \to 0$ or $y \to 0$ and $f \to -\infty$ if $x \to \infty$ or $y \to \infty$. So max at $(x, y) = (\frac{1}{2}, \frac{1}{2})$. $f = \frac{1}{8}$

4. (20 points) In Problem 3 above, instead of substituting for z, one could also use Lagrange multiplier to maximize the volume $V = xyz$ with the same constraint $x^2 + y^2 + z = 1$.

   (a) Write down the Lagrange multiplier equations for this problem.

   (b) Solve the equations (still assuming $x > 0, y > 0$).

Solution. (a). $f(x,y,z) = xyz$

$\boxed{10 \text{ pts}}$

$$g(x,y,z) = x^2+y^2+z = 1.$$

Lagrange equations : $\nabla f = \lambda \nabla g$

i.e.,
$$\begin{cases} yz = 2\lambda x & (1) \\ xz = 2\lambda y & (2) \\ xy = \lambda & (3) \end{cases}$$

and the constraint equation $g(x,y,z) = 1$ (4)

$\boxed{10 \text{ pts}}$

(b) Dividing (1) and (2) by each other,

we get $\frac{y}{x} = \frac{x}{y}$. so $x^2 = y^2$

Since $x>0$, $y>0$. we get $y = x$.

Sub this into (1) – (3), we get:

$$z = 2\lambda \quad \text{and} \quad x^2 = \lambda$$

Hence $z = 2x^2$. and $g(x,y,z) = 4x^2 = 1$.

$\Rightarrow$ $x = \frac{1}{2}$, $y = \frac{1}{2}$, $z = \frac{1}{2}$. $f_{max} = \frac{1}{8}$

5. (20 points) Let $(\bar{x}, \bar{y})$ be the center of mass of the triangle with vertices at $(-2,0),(0,1),(2,0)$ and uniform density $\delta = 1$.

   (a) Write an integral formula for $\bar{y}$. Do not evaluate the integral(s), but write explicitly the integrand and limits of integration.

   (b) Find $\bar{x}$.

Solution:

10 pts

(a). The area of the triangle is 2.

so $$\bar{y} = \frac{1}{2} \int_0^1 \int_{2y-2}^{2-2y} y \, dx \, dy$$

(b). By symmetry $\bar{x} = 0$.

10 pts

6. (20 points) Let $\mathbf{F} = (ax^2y + y^3 + 1)\mathbf{i} + (2x^3 + bxy^2 + 2)\mathbf{j}$ be a vector field on the plane, where a and b are constants.

(a) Find the values of a and b for which $\mathbf{F}$ is conservative.

(b) For these values of a and b from part (a), find $f(x, y)$ such that $\mathbf{F} = \nabla f$.

(c) Still for these values of a and b from part (a), compute $\int_C \mathbf{F} \cdot d\mathbf{s}$ along the curve C such that $x = e^t \cos t, y = e^t \sin t, 0 \le t \le \pi$.

Solution: (a). Write $F = Mi + Nj$

$\boxed{7 \text{ pts}}$

with $M = ax^2y + y^3 + 1$ & $N = 2x^3 + bxy^2 + 2$.

Then $N_x = 6x^2 + by^2$

$M_y = ax^2 + 3y^2$.

So $N_x = M_y \iff a = 6$ & $b = 3$

$\boxed{8 \text{ pts}}$

(b). $f_x = M \Rightarrow f = 2x^3y + xy^3 + c(y)$

for some function $c(y)$ of y.

$\Rightarrow f_y = 2x^3 + 3xy^2 + c'(y)$.

Setting $f_y = N$. $\Rightarrow 2x^3 + 3xy^2 + c'(y)$

$= 2x^3 + 3xy^2 + 2$.

$\Rightarrow c'(y) = 2$ and $c(y) = 2y$.

So $f = 2x^3y + xy^3 + 2y + (\text{Constant})$

$\boxed{5 \text{ pts}}$

(c). C starts at $(1, 0)$ and ends at $(-e^{\pi}, 0)$.

So $\int_C \vec{F} \cdot d\mathbf{s} = f(-e^{\pi}, 0) - f(1, 0) = -e^{-\pi} - 1$

7. (20 points) Consider the region R in the first quadrant bounded by the curves $y = x^2, y = x^2/5, xy = 2$, and $xy = 4$.

   (a) Compute $dxdy$ in terms of $dudv$ if $u = x^2/y$ and $v = xy$.

   (b) Find a double integral for the area of R in uv coordinates and evaluate it.

Solution:

10 pts

(a). $$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ y & x \end{vmatrix} = \frac{3x^2}{y}.$$

Therefore,

$$du\,dv = (3x^2/y)\,dx\,dy = 3u\,dx\,dy$$

$$\Rightarrow \quad dx\,dy = \frac{1}{3u}\,du\,dv.$$

10 pts

(b). $$\int_2^4 \int_1^5 \frac{1}{3u}\,du\,dv = \int_2^4 \frac{1}{3}\ln 5 \; dv$$

$$= \frac{2}{3}\ln 5$$

8. (20 points) (a) Let C be a simple closed curve going counterclockwise around a region R. Let $M = M(x, y)$. Express $\oint_C M dx$ as a double integral over R.

(b) Fine M so that $\oint_C M dx$ is the mass of R with density $\delta(x, y) = (x + y)^2$.

Solution. (a).

$\boxed{10 \text{ pts}}$

$$\oint_C M \, dx = \iint_R -M_y \, dA$$

(b). We want M s.t.

$\boxed{10 \text{ pts}}$

$$-M_y = (x+y)^2$$

Can Use $M = -\frac{1}{3}(x+y)^3$.

9. (20 points) (a) A solid sphere S of radius a is placed above the xy-plane so it is tangent at the origin and its diameter lies along the z-axis. Give its equation in *spherical coordinates.*

   (b) Give the equation of the horizontal plane $z = a$ in spherical coordinates.

   (c) Set up a triple integral in spherical coordinates which gives the volume of the portion of the sphere S lying *above* the plane $z = a$. (Give integrand and limits of integration, but do *not* evaluate.)

Solution:

$\boxed{5 \text{ pts}}$ (a). sphere : $\rho \leq 2a\cos\phi$

$\boxed{5 \text{ pts}}$ (b). plane : $\rho = a\sec\phi$

$\boxed{10 \text{ pts}}$ (c). $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{a\sec\phi}^{2a\cos\phi} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta.$

10. (20 points) Let $\mathbf{F} = (2xy + z^3)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 3xz^2 - 1)\mathbf{k}$.

(a) Show that $\mathbf{F}$ is conservative.

(b) Find a potential function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

Solution:

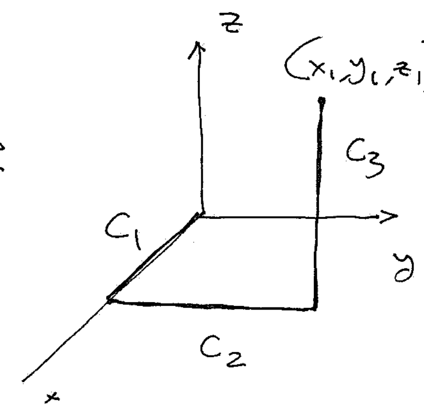
5 pts

(a) $$\frac{\partial}{\partial y}(2xy + z^3) = 2x = \frac{\partial}{\partial x}(x^2 + 2yz)$$

$$\frac{\partial}{\partial z}(2xy + z^3) = 3z^2 = \frac{\partial}{\partial x}(y^2 + 3xz^2 - 1)$$

$$\frac{\partial}{\partial z}(x^2 + 2yz) = 2y = \frac{\partial}{\partial y}(y^2 + 3xz^2 - 1)$$

$\Rightarrow$ $\vec{F}$ is conservative



15 pt

(b) Method 1. $$f(x,y,z) = \int_{C_1 + C_2 + C_3} \vec{F} \cdot d\vec{r}$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{x_1} (2xy + z^3)\, dx$$

$$= \int_0^{x_1} 0\, dx = 0 \quad (y = z = 0)$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{y_1} (x^2 + 2yz)\, dy = \int_0^{y_1} x_1^2\, dy = x_1^2 y_1 \quad (x = x_1, z = 0)$$

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_0^{z_1} (y^2 + 3xz^2 - 1)\, dz = \int_0^{z_1} (y_1^2 + 3x_1 z^2 - 1)\, dz$$

$$= y_1^2 z_1 + x_1 z_1^3 - z_1 \quad (x = x_1, y = y_1)$$

So $f(x,y,z) = x^2 y + y^2 z + z^3 x - z + C$.

Method 2. $\frac{\partial f}{\partial x} = 2xy + z^3$. $\Rightarrow f(x,y,z) = x^2 y + xz^3 + g(y,z)$.

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 + 2yz \Rightarrow \frac{\partial g}{\partial y} = 2yz$$

$$\Rightarrow g(y,z) = y^2 z + h(z). \qquad \frac{\partial f}{\partial z} = 3xz^2 + y^2 + h'(z) \Rightarrow h'(z) = -$$

11. (20 points) Let $\mathbf{F} = -2xz\mathbf{i} + y^2\mathbf{k}$.

(a) Calculate $\operatorname{curl}\mathbf{F}$.

(b) Show that $\iint_R \operatorname{curl}\mathbf{F}\cdot\mathbf{n}dS = 0$ for any finite portion R of the unit sphere $x^2+y^2+z^2=1$.

(c) Show that $\oint_C \mathbf{F}\cdot dr = 0$ for any simple closed curve C on the unit sphere $x^2+y^2+z^2=1$.

Solution. (a). $\boxed{8 \text{ pts}}$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -2xz & 0 & y^2 \end{vmatrix} = 2y\hat{i} - 2x\hat{j}$$

(b). $\boxed{7 \text{ pts}}$ On the unit sphere, $\vec{n} = x\hat{i} + y\hat{j} + z\hat{k}$

so $\operatorname{curl}\vec{F}\cdot\vec{n} = (2y, -2x, 0)\cdot(x, y, z)$

$$= 2xy - 2xy = 0.$$

Therefore, $\iint_R \operatorname{curl}\vec{F}\cdot\vec{n}\,dS = 0.$

(c). $\boxed{5 \text{ pts}}$ By Stokes, $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\cdot\vec{n}\,dS$

where R is the region bounded by C on the unit sphere. Using result of (b).

$$\iint_R \operatorname{curl}\vec{F}\cdot\vec{n}\,dS = \oint_C \vec{F}\cdot d\vec{r} = 0$$

12. (20 points) The surface given parametrically by $\mathbf{X}(s,t) = (st, t, s^2)$ is known as the **Whitney umbrella**.

(a) Verify that this surface may also be described by the xyz-coordinate equation $y^2z = x^2$. (You would need to verify two directions: every point of $\mathbf{X}(s,t)$ satisfies the xyz-coordinate equation; every solution to the xyz-coordinate equation lies on $\mathbf{X}(s,t)$.)

(b) Some points (x, y, z) of the Whitney umbrella do not correspond to a single parameter point (s,t). Find all such pairs of parameter points.

(c) Give an equation of the plane tangent to the Whitney umbrella at the point $(2, 1, 4)$, if exists.

Solution: (a).

5 pts

Denote by $X(s,t) = (x(s,t), y(s,t), z(s,t))$

For any $(s,t) \in \mathbb{R}^2$, $X(s,t)$ satisfies $y^2 z = x^2$.

Conversely, give a parametrization of the equation $y^2 z = x^2$: let $y = t$.

If $t=0$, then $x=0$. z is arbitrary.

If $t \neq 0$, then $z = \frac{x^2}{y^2} \geq 0$.

let $s = \frac{x}{y}$. then $z = s^2$, and $x = sy = st$.

Note that $(0,0,z)$ also satisfies the parametrization (st, t, s^2)

8 pts

(b). If $X(s_1,t_1) = X(s_2,t_2)$ for $(s_1,t_1) \neq (s_2,t_2)$.

then $t_1 = t_2$ from $y(s,t)$.

If $t_1 = t_2 \neq 0$, then $s_1 = s_2$. $\Rightarrow (s_1,t_1) = (s_2,t_2)$.

So $t_1 = t_2 = 0$. and $s_1 = -s_2$ from $z(s,t)$.

7 pts

(c). $4x - 8y - z + 4 = 0$.

13. (20 points) The surface given by $\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t))$, where

$$\begin{cases} x = \left(a + \cos\frac{s}{2}\sin t - \sin\frac{s}{2}\sin 2t\right)\cos s \\ y = \left(a + \cos\frac{s}{2}\sin t - \sin\frac{s}{2}\sin 2t\right)\sin s \\ z = \sin\frac{s}{2}\sin t + \cos\frac{s}{2}\sin 2t \end{cases}$$

for a positive constant $a > 0$, and $0 \le s \le 2\pi, 0 \le t \le 2\pi$, is known as a **Klein bottle**.

(a) Determine (and describe) the s-coordinate curve at $t = 0$.

(b) Calculate the standard normal vector $\mathbf{N}$ along the s-coordinate curve at $t = 0$ (i.e., find $\mathbf{N}(s, 0)$). Note that $\mathbf{X}(0,0) = \mathbf{X}(2\pi, 0)$. By comparing $\mathbf{N}(0,0)$ and $\mathbf{N}(2\pi, 0)$, comment regarding the orientability of the Klein bottle.

Solution. (a). At $t=0$,

$$\begin{cases} x = a\cos s \\ y = a\sin s \\ z = 0. \end{cases}$$

$\boxed{7 \text{ pts}}$

So the s-coordinate curve at $t=0$ is a circle of radius a centered at O lying on the xy-plane

$\boxed{13 \text{ pts}}$

(b). $$N(s,0) = \left(a\cos s\left(2\cos\tfrac{s}{2} + \sin\tfrac{s}{2}\right),\ a\sin s\left(2\cos\tfrac{s}{2} + \sin\tfrac{s}{2}\right),\ a\left(2\sin\tfrac{s}{2} - \cos\tfrac{s}{2}\right)\right)$$

From this we see that $N(0,0) = (2a, 0, -a)$ while $N(2\pi, 0) = (-2a, 0, a)$. Therefore, the Klein bottle cannot be orientable, since the normal vector along the s-coordinate curve at $t=0$ changes direction.

14. (20 points) (a) Show that the path $\mathbf{x}(t) = (\cos t, \sin t, \sin 2t)$ lies on the surface $z = 2xy$.

(b) Evaluate

$$\oint_C (y^3 + \cos x)dx + (\sin y + z^2)dy + xdz,$$

where C is the closed curve parametrized and oriented by the path $\mathbf{x}$ in part (a).

Solution: (a). By the double angle formula,

$\boxed{5 \text{ pts}}$ for any $t \in \mathbb{R}$, $2\cos t \sin t = \sin 2t$,

therefore, the point $X(t)$ lies on $z = 2xy$

(b). $\cancel{\oint_C \vec{F} \cdot d\vec{s} = \int_0^{2\pi} F(x(t)) \cdot x'(t)\, dt}$

$\boxed{15 \text{ pts}}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y^3 + \cos x & \sin y + z^2 & x \end{vmatrix} = (-2x, -1, -3y^2).$$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$$

$$= \iint_S -3y^2\, dS = \int_0^{2\pi} \int_0^1 -3r^3 \sin^2\theta\, dr\, d\theta$$

$$= \int_0^{2\pi} -3 \left[\frac{1}{4} r^4 \right]_0^1 \sin^2\theta\, d\theta = -\frac{3\pi}{4}.$$

15. (20 points) Use Gauss's theorem to find the volume of the solid region bounded by the paraboloids $z = 9 - x^2 - y^2$ and $z = 3x^3 + 3y^2 - 16$.