Linear space, basis, and linear transformation

1 Linear space

**Def.** (Defn 4.1.1 in our book, Linear spaces, some books call it as Vector spaces)

A linear space $V$ is a set endowed with a rule for addition (if $f$ and $g$ are in $V$, then so is $f + g$) and a rule for scalar multiplication (if $f$ is in $V$ and $k$ in $\mathbb{R}$, then $kf$ is in $V$) such that these operations satisfy the following eight rules (for all $f, g, h$ and all $c, k$ in $\mathbb{R}$):

1. $(f + g) + h = f + (g + h)$.
2. $f + g = g + f$.
3. There exists a neutral element $n$ in $V$ such that $f + n = f$, for all $f$ in $V$. This $n$ is unique and denoted by $0$.
4. For each $f$ in $V$ there exists a $g$ in $V$ such that $f + g = 0$. This $g$ is unique and denoted by $(-f)$.
5. $k(f + g) = kf + kg$.
6. $(c + k)f = cf + kf$.
7. $c(kf) = (ck)f$.
8. $1f = f$.

This is an abstract definition. To determine if a set $V$ is (or is not) a linear space, we could only verify $V$ and its $+$ and scalar multiplication satisfies (or does not satisfy) the definition.

**Example 1.**

a) (Example 3 in the textbook, page 154)

$$F(\mathbb{R}, \mathbb{R}) = \{ \text{all functions from } \mathbb{R} \text{ to } \mathbb{R} \}.$$  

b) 

$$C(\mathbb{R}) = \{ \text{all continuous functions from } \mathbb{R} \text{ to } \mathbb{R} \}.$$ 

c) 

$$P = \{ \text{all polynomials with one variable} \}.$$ 

d) 

$$P_n = \{ \text{all polynomials with degree } \leq n \}.$$ 

We can use the definition of linear space to verify that these are all linear spaces.

1.1 Subspaces

**Def.** (Defn 4.1.2 in our book, Subspaces)

A subset $W$ of a linear space $V$ is called a subspace of $V$ if

a. $W$ contains the neutral element $0$ of $V$.
b. $W$ is closed under addition (if $f$ and $g$ are in $W$), then so is $f + g$.
c. $W$ is closed under scalar multiplication (if $f$ is in $W$ and $k$ is a scalar, then $kf$ is in $W$).

**Example 1’.** In example 1, there is a sequence of linear spaces, each preceding one are subspaces of successive ones.

$$P_n \subset P \subset C(\mathbb{R}) \subset F(\mathbb{R}, \mathbb{R}).$$

**Example 2.** Determine is the following set are subspaces of $\mathbb{R}^{n \times m}$, where $\mathbb{R}^{n \times m}$ consists of all $n \times m$ matrices.

a) $V_0 = \{ \text{all rank 0 matrices} \}$. 

b) \( V_1 = \{ \text{all rank 1 matrices} \} \).

**Solution.** a) Rank 0 matrix has only 1 possibility, that is 0 matrix. Therefore, \( V_0 = \{ 0 \} \) is a subspace. (Note that \( \{ 0 \} \) is the simplest linear space.)

b) \( V_1 \) does not contain 0 matrix, i.e., neutral element. So \( V_1 \) doesn’t satisfy the a) in the definition of subspace. Therefore, \( V_1 \) is not a subspace of \( \mathbb{R}^{n \times m} \).

## 2 Basis

Basis is an important concept in linear algebra. Let’s introduce its definition first.

**Def.** (Defn 4.1.3 in our book, **Span**, linear independence, basis)
Consider the elements \( f_1, \ldots, f_n \) in a linear space \( V \).

a. We say that \( f_1, \ldots, f_n \) span \( V \) if every \( f \) in \( V \) can be expressed as a linear combination of \( f_1, \ldots, f_n \).

b. We say that \( f_1, \ldots, f_n \) are linearly independent if the equation

\[
\sum_{i=1}^{n} c_i f_i = 0
\]

has only the trivial solution

\[
\sum_{i=1}^{n} c_i = 0.
\]

(This definition is equivalent to say \( f_1, \ldots, f_n \) are redundant Defn 3.2.3. If you are interested, you can show it yourself.)

c. We say that elements \( f_1, \ldots, f_n \) are a basis of \( V \) if they span \( V \) and are linearly independent.

**Def.** (fact 4.1.5 in our book, **Dimension**, and Defn 4.1.8. **Finite dimensional linear spaces, Infinite dimensional spaces.**)

If a linear space \( V \) has a basis with \( n \) elements, then

\[
\dim(V) = n,
\]

and \( V \) is called finite dimensional. Otherwise, the space is called infinite dimensional.

**Example 3.** In example 1, \( P, C(\mathbb{R}), \) and \( F(\mathbb{R}, \mathbb{R}) \) are all infinite dimensional spaces since \( \{ 1, x, x^2, \ldots, x^n, \ldots \} \) are infinitely many linearly independent vectors in the linear spaces. We couldn’t find a basis with finitely many vectors.

**Example 4.** (Another generic infinite dimensional linear space) Example 5 in the textbook, page 154.

The set of all infinite sequences of real numbers is a linear space, where addition and scalar multiplication are defined term by term:

\[
(x_0, x_1, x_2, \ldots) + (y_0, y_1, y_2, \ldots) = (x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots)
\]

\[
k(x_0, x_1, x_2, \ldots) = (kx_0, kx_1, kx_2, \ldots).
\]

## 2.1 Generic finite dimensional linear space and their basis

Note that the basis of a linear space is **NOT** unique. For instance, if \( \{ \vec{v}_1, \ldots, \vec{v}_n \} \) is a basis, then \( \{ k_1 \vec{v}_1, \ldots, k_n \vec{v}_n \} \), where \( k_i \) are all nonzero scalars, and \( \{ \vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3, \ldots, \vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n \} \) are both bases as well.

**Example 5.**

a)

\[
\mathbb{R}^{n \times m} = \{ \text{all } n \times m \text{ matrices} \}.
\]
Its standard basis is \( \{E_{ij}\}, 1 \leq i \leq n, 1 \leq j \leq m \), where \( E_{ij} \) is the matrix with the entry \( a_{ij} = 1 \), i.e., the entry located in the \( i \)th row and the \( j \)th column is 1, else is 0.

Therefore, \( \dim(\mathbb{R}^{n \times m}) = mn \).

b) \( \mathbb{R}^{n \times n} = \{ \text{all} \ n \times n \text{ matrices} \} \).

Its standard basis is \( \{E_{ij}\}, 1 \leq i, j \leq n \). And \( \dim(\mathbb{R}^{n \times n}) = n^2 \).

c) \( U^{n \times n} = \{ \text{all upper triangular} \ n \times n \text{ matrices} \} \).

Its basis is \( \{E_{ij}\}, 1 \leq i \leq j \leq n \). And \( \dim(U^{n \times n}) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \).

d) \( D^{n \times n} = \{ \text{all diagonal} \ n \times n \text{ matrices} \} \).

Its basis is \( \{E_{ij}\}, 1 \leq i \leq n \). And \( \dim(D^{n \times n}) = n \).

e) \( P_n = \{ \text{all polynomials with degree} \ \leq n \} \) as defined in example 1.

It has a standard basis \( \{1, x, x^2, \ldots, x^n\} \). Generally, \( \{1, (x-a), (x-a)^2, \ldots, (x-a)^n\} \), where \( a \) is any real number, is also a basis. \( \dim P_n = n+1 \).

f) \( \mathbb{R}^n \). The most familiar finite dimensional linear space so far.

Its standard basis \( \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \). And \( \dim(\mathbb{R}^n) = n \).

### 2.2 Why basis is important

In this section, we only consider finite dimensional linear space.

1. You can view basis \( \mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\} \) as a coordinate system. Every vector \( \vec{v} \) can be uniquely written as a linear combination of \( \{\vec{v}_1, \ldots, \vec{v}_n\} \), i.e., \( \vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n \), and\
   \[
   \begin{bmatrix}
   c_1 \\
   c_2 \\
   \vdots \\
   c_n
   \end{bmatrix}
   \]
   is called the coordinates of \( \vec{v} \) with respect to \( \mathcal{B} \).

2. For a linear transformation \( T : V \to V \), \( T \) is uniquely determined by its image on a basis \( \mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\} \). And there is a matrix \( B = \begin{bmatrix} [T(\vec{v}_1)]_{\mathcal{B}} & \cdots & [T(\vec{v}_n)]_{\mathcal{B}} \end{bmatrix} \), such that \( T \) is given by formula
   \[
   [T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}, \forall \vec{x} \in V.
   \]

### 3 Linear Transformation

**Def.** (fact 4.2.1 in our book, Linear transformation, image, kernel, rank, nullity)

Consider two linear spaces \( V \) and \( W \). A function \( T \) from \( V \) to \( W \) is called a **linear transformation** if

\[
T(f + g) = T(f) + T(g) \quad \text{and} \quad T(kf) = kT(f)
\]

for all elements \( f \) and \( g \) of \( V \) and for all scalars \( k \).

For a linear transformation \( T \) from \( V \) to \( W \), we set

\[
\text{im} \ T = \{T(f) : f \in V\}
\]

and

\[
\ker(T) = \{f \in V : T(f) = 0\}.
\]

Note that \( \text{im}(T) \) is a subspace of codomain \( W \) and that \( \ker(T) \) is a subspace of domain \( V \).

If the image of \( T \) is finite dimensional, then \( \dim(\text{im} \ T) \) is called the **rank** of \( T \), and if the kernel of \( T \) is finite dimensional, then \( \dim(\ker T) \) is the **nullity** of \( T \).

If \( V \) is finite dimensional, then the rank-nullity theorem holds (see Fact 3.3.7):

\[
\dim V = \text{rank } T + \text{nullity } T = \dim(\text{im} \ T) + \dim(\ker T).
\]
Example 6. Show $T : P \to P$, where $P$ is defined in example 1, given by

$$T(f(x)) = \int_0^x f(t)dt,$$

is a linear transformation. And find $\ker T$ and $\text{im} T$.

**Solution.** By the definition of linear transformation, we need to show

$$T(f(x) + g(x)) = T(f(x)) + T(g(x)) \text{ and } T(kf(x)) = kT(f(x)).$$

And this is true because the linearity of integration.

Suppose $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial in $P$. If $f(x) \in \ker T$, i.e.,

$$T(f(x)) = \int_0^x f(t)dt = a_0x + \frac{1}{2}a_1x^2 + \cdots + \frac{a_n}{n+1}x^{n+1} = 0 \Rightarrow a_0 = a_1 = \ldots = a_n = 0 \Rightarrow f(x) = 0.$$

Therefore, $\ker T = \{0\}$.

From the formula,

$$T(f(x)) = \int_0^x f(t)dt = a_0x + \frac{1}{2}a_1x^2 + \cdots + \frac{a_n}{n+1}x^{n+1},$$

the image of a polynomial could be any polynomial except nonzero constant. Therefore,

$$\text{im} T = \{\text{all polynomials with degree } \geq 1 \text{ or } 0\}.$$

Thus $\text{im}(T) \neq P$, i.e., $T$ is not onto.

3.1 Isomorphisms

**Def.** (defn 4.2.2 in our book, *Isomorphisms and isomorphic spaces*)

An invertible linear transformation is called an **isomorphism**. We say that the linear space $V$ is **isomorphic** to the linear space $W$ if there exists an isomorphism from $V$ to $W$.

**Fact 4.2.4** in our book, *Properties of isomorphism*

a. A linear transformation $T$ from $V$ to $W$ is an isomorphism if (and only if) $\ker T = \{0\}$ and $\text{im}(T) = W$.

In parts (b) and (c), the linear spaces $V$ and $W$ are assumed to be finite dimensional.

b. If $V$ is isomorphic to $W$, then $\dim V = \dim W$.

c. Suppose $T$ is a linear transformation from $V$ to $W$ with $\ker T = \{0\}$. If $\dim V = \dim W$, then $T$ is an isomorphism.

**Example 7.** Determine if the linear transformation given in example 6 is an isomorphism.

Because $P$ is infinite dimensional, we couldn’t use property c). So we need to use property a). Since $\text{im} T \neq P$, we get $T$ is not an isomorphism.