Kernel and Image of a linear transformation

Kernel and image is not very hard! Here I gave some examples to illustrate these concepts.

1 Function case

The parallel concepts in function case of kernel and image is $f^{-1}(0)$ and range of $f$.

**Example 1.** Let $f(x) = x^2$ from $\mathbb{R} \to \mathbb{R}$ be a function.

Then, by definition,

$$f^{-1}(0) = \{ x \in \mathbb{R} | f(x) = 0 \}, \quad \text{range } f = \{ f(x) \text{ for all } x \in \mathbb{R} \}$$

From the definition, $f^{-1}(0)$ is a set consists of all $x \in \mathbb{R}$ such that $f(x) = 0$. So in our problem, $f(x) = x^2 = 0 \Rightarrow x = 0$. Therefore, $f^{-1}(0)$ consists of only one element 0, i.e., $f^{-1}(0) = \{0\}$.

range $f = \{ f(x) = x^2 \text{ for all } x \}$, so range $f = [0, \infty)$.

Therefore, in the problem, domain of $f$ is $\mathbb{R}$, codomain of $f$ is $\mathbb{R}$, $f^{-1}(0) = \{0\}$ and range $f = [0, \infty) \neq \text{codomain}$.

**Example 2.** Let $f(x) = \sin x$ from $\mathbb{R} \to \mathbb{R}$ be a function.

Then $f^{-1}(0)$ is the set consists of all $x$ such that $f(x) = 0$. Therefore,

$$f^{-1}(0) = \{ \cdots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \cdots \}.$$  

range $f$ is the set consists all $f(x) = \sin x$, so range $f = [-1, 1]$.

Therefore, in the problem, domain of $f$ is $\mathbb{R}$, codomain of $f$ is $\mathbb{R}$, $f^{-1}(0) = \{ \cdots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \cdots \}$ and range $f = [-1, 1] \neq \text{codomain}$.

2 Linear transformation case

**Example 3.** Let $T(\vec{v}) = \vec{0}$ from $V$ to $W$, where $V$ and $W$ are two linear spaces (V and W may be infinite dimensional). Show $T$ is a linear transformation and find ker $T$ and im $T$.

To show $T$ is a linear transformation, we need to verify two conditions

$$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) \quad \text{and} \quad T(k\vec{v}) = kT(\vec{v}).$$

I only show $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$, you can verify the other.

By definition of $T$, LHS=$T(\vec{v}_1 + \vec{v}_2) = \vec{0}$. RHS=$T(\vec{v}_1) + T(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0}$=LHS.

So $T$ is a linear transformation.

ker $T$ is the SET in $V$ consists of all vectors $\vec{v} \in V$ such that its image $T(\vec{v}) = \vec{0}$. By our definition of $T$, all vectors $\vec{v} \in V$ satisfy $T(\vec{v}) = \vec{0}$. Therefore, ker $T = V$.

im $T$ is the SET in $W$ consists of all vectors $T(\vec{v})$. By the definition of $T$, im $T$ consists of only one element $\vec{0}$. Therefore, im $T = \{0\}$.

Therefore, domain of $T$ is $V$, codomain of $T$ is $W$, ker $T = V$ and im $T = \{0\}$.

3 How to find ker $T$ and im $T$

To answer this question, I couldn’t find one method work for any case. By my experience, it depends on how do problems give linear transforms $T$. I try to illustrate this by using examples.
3.1 Using the matrix corresponding to linear transformation

Example 4 (Midterm 1.) $T$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ with the properties that $T(\vec{e}_1) = \vec{0}, T(\vec{e}_2) = \vec{0},$ and $T(\vec{v}) = \vec{v}$.

We can use column by column method to find the matrix $B$ with respect to the basis $\mathfrak{B} = \{\vec{e}_1, \vec{e}_2, \vec{v}\}$.

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $\ker T = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathfrak{B}}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathfrak{B}} \right\} = \text{span} \{\vec{e}_1, \vec{e}_2\}$.

(Note that there are two free variables in the matrix $B$ and the coordinate $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathfrak{B}}$ represents vector $\vec{e}_1$.)

$\text{im} T = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathfrak{B}} \right\} = \text{span} \{\vec{v}\}$. A basis of $\ker T$ is $\{\vec{e}_1, \vec{e}_2\}$. A basis of $\text{im} T$ is $\{\vec{v}\}$. Nullity of $T = 2$ and rank of $T = 1$.

Example 5. (Exer 4.3 prob 6.) $T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $M$ from $U^{2\times 2}$ to $U^{2\times 2}$.

First we need to find the matrix $B$ with respect to a basis $\mathfrak{B}$. Here we can choose the standard basis $\mathfrak{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ of $U^{2\times 2}$.

By computation, we get

$$T\left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

So

$$\left[ T\left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \right]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

Similarly, we can get the other two coordinates. Therefore, we can get the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$ 

There is no free variables in $B$. So $\ker T = \{0\}$.

$\text{im} T = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathfrak{B}}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}_{\mathfrak{B}} \right\} = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \right\} = U^{2\times 2}$.

Therefore, $T$ is an isomorphism.

Can you find the general way from the matrix corresponding to the linear transformation to find $\ker$ and $\text{im}$?
3.2 Using definition of ker and im

**Example 5**. The problem is the same as example 5, and we try to use definition of ker and im to find them.

By the definition, $\ker T$ is the set consists of all matrices $M \in U^{2 \times 2}$ such that $T(M) = 0$.

$$T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \Rightarrow M = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^{-1} \cdot 0 = 0,$$

notice that $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is invertible. Therefore, $\ker T = \{0\}$.

And by the formula, nullity+rank of $T=\dim U^{2 \times 2}$, we get rank of $T = \dim \ker T = \dim U^{2 \times 2} = 3$ since nullity=dim ker $T = 0$. So $\text{im } T = U^{2 \times 2}$. (Here we use a general result, if $W$ is a subspace of $V$ and $\dim W = \dim V$, then $W = V$. And $\text{im } T$ is a subspace of $U^{2 \times 2}$.)

If we want to find $\ker T$ and $\im T$ of a linear transformation from $V$ to $W$, where $V$ or $W$ is infinite dimensional linear space, then we **have to** use definition way to find them. We couldn’t use matrix way to find them.

$V$ denotes the space of infinite sequences of real numbers. Notice that $V$ is a infinite dimensional linear space.

**Example 6.** (Exer 4.2 prob 34.) $T(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots)$ from $V$ to $V$.

This is our homework problem, and we know that $\ker T = \{0\}$ and

$$\text{im } T = \{(0, x_0, x_1, x_2, \ldots) | x_i \text{ can be arbitrary real numbers}\}.$$  

So $\text{im } T \neq V$.

**Example 7.** (Exer 4.2 prob 33.) $T(x_0, x_1, x_2, x_3, x_4, \ldots) = (x_0, x_2, x_4, \ldots)$ from $V$ to $V$ (we are dropping every other term).

By definition, $\ker T$ consists of all sequences $(x_0, x_1, x_2, x_3, x_4, \ldots)$ such that

$$T(x_0, x_1, x_2, x_3, x_4, \ldots) = (x_0, x_2, x_4, \ldots) = (0, 0, 0, \ldots).$$

Therefore, $\ker T = \{(0, x_1, 0, x_3, 0, x_5, 0, x_7, \ldots) | x_i \text{ can be arbitrary, } i \text{ is odd}\}$. So $\ker T \neq \{0\}$, i.e., $T$ is not an isomorphism.

$\text{im } T$ is the set consists of all $T(x_0, x_1, x_2, x_3, x_4, \ldots) = (x_0, x_2, x_4, \ldots)$. Therefore, $\text{im } T = V$, i.e., $T$ is onto.

You can choose an easier way to find $\ker$ or $\im$.

4 Isomorphisms and invertible linear transformations

An invertible linear transformation is called an **isomorphism**. To make clear of isomorphisms we need to make clear of invertible linear transformations and how to determine if a linear transformation is invertible.

4.1 Finding $T^{-1}$

If we could find $T^{-1}$ of $T$, then $T$ is invertible.

**Example 5’** The problem is the same as example 5. And we’ve already known $T$ is invertible (or an isomorphism). We try to use another way to show the same answer, i.e., trying to find $T^{-1}$.

Notice that $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is an invertible matrix, so we can define $T^{-1}(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^{-1} M$. And we can verify $T^{-1}T = TT^{-1} = \text{identity map from } U^{2 \times 2} \text{ to } U^{2 \times 2}$.

In general, if a linear transformation $T$ is given by $T(M) = AM$ or $T(M) = MA$ where $A$ is an invertible matrix, then $T$ is an invertible linear transformation.

(Actually, some of you use this way to solve homework problems.)
4.2 Using \( \ker T \) and \( \im T \)

Basically, this way is to find \( \ker \) and \( \im \), and use some properties to determine if \( T \) is invertible or isomorphism.

There are three properties.

**G.** A linear transformation \( T \) from \( V \) to \( W \) is an isomorphism if and only if \( \ker T = \{0\} \) and \( \im T = W \). (This property works for any case, finite dimensional linear space or infinite dimensional linear space)

**F1.** If we’ve known that \( V \) is a finite dimensional linear spaces and \( T \) is a linear transformation from \( V \) to \( V \), then \( \ker T = \{0\} \) **OR** \( \im T = W \) implies \( T \) is invertible or an isomorphism. (Do you notice the difference between this and the above one?)

**F2.** Suppose \( T \) is a linear transformation from \( V \) to \( W \) with \( \ker T = \{0\} \), where \( V \) and \( W \) are finite dimensional linear spaces. If \( \dim V = \dim W \), then \( T \) is an isomorphism.

You can see example 5,6,7.

4.3 using matrix corresponding to \( T \)

The above two ways can determine if \( T \) is an isomorphism in general. *In general* means \( T \) can be a linear transformation from arbitrary \( V \) to \( W \), where \( V \) and \( W \) can be infinite dimensional or finite dimensional.

However this way, using matrix, only works for finite dimensional linear spaces case, i.e., \( V \) and \( W \) are finite dimensional linear space.

**Example 5** The problem is the same as example 5. In example 5, we’ve already known the matrix \( B \) corresponding to \( T \) wrt \( \mathcal{B} \) is

\[
B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 3
\end{bmatrix}.
\]

\( B \) is an invertible matrix. Therefore, \( T \) is an isomorphism. (Notice that if \( B \) is not invertible, then \( T \) is not invertible or an isomorphism.)

So this way is to find the matrix \( B \) corresponding to \( T \) first. And the invertibility of \( B \) determines the invertibility or isomorphism of \( T \). Actually, some of you use this way to solve homework exercises.