VARIATIONAL METHODS

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Contents

1. Functionals

1.1. **Introduction.** In the course of Calculus, we study derivatives of functions with variables of real numbers. For example, the following function

$$f(x) = x^2, \quad x \in \mathbf{R}$$

is continuous, $\min_{x\in\mathbf{R}} f(x) = f(0) = 0$, and the first derivative of f at $x = 0$ vanishes.

In this course, we shall study "derivatives" of "functions" with variables of functions. We will explain the mentioned two terminologies which are usually called as *variations* and *functionals*, respectively. Let $\mathscr{X}$ denote the set of all functions defined on $\mathbf{R}$ and consider a map

$$\mathcal{F}(f) := f(0)^2, \quad f \in \mathscr{X}.$$

Clearly that $\min_{f\in\mathscr{X}} \mathcal{F}(f) = 0 = \mathcal{F}(0)$ and (at least) the zero function minimizes the map $\mathcal{F}$. Then we may ask a natural question on how to define the first "derivative" of the map $\mathcal{F}$ so that the first derivative of $\mathcal{F}$ vanishes at the zero function.

1.2. **Vector spaces.** By a vector space (over the set of real numbers $\mathbf{R}$) we mean a set $\mathscr{X}$ of elements $x, y, z, \cdots$, referred to as **vectors**, together with two operations of addition and multiplication, satisfies the following rules:

(1) $x + y \in \mathscr{X}$ for any $x, y \in \mathscr{X}$.
(2) $ax \in \mathscr{X}$ for any $x \in \mathscr{X}$ and $a \in \mathbf{R}$.
(3) $x + y = y + x$ for any $x, y \in \mathscr{X}$.
(4) $(x + y) + z = x + (y + z)$ for any $x, y, z \in \mathscr{X}$.
(5) $\mathscr{X}$ contains an element 0, called the **zero vector**, such that $x + 0 = x$ for every $x \in \mathscr{X}$.
(6) For every $x \in \mathscr{X}$, there is a vector $-x \in \mathscr{X}$ such that $x + (-x) = 0$.
(7) $a(bx) = (ab)x$ for any $a, b \in \mathbf{R}$ and any $x \in \mathscr{X}$.
(8) $a(x + y) = ax + ay$ for any $a \in \mathbf{R}$ and any $x, y \in \mathscr{X}$.
(9) $(a + b)x = ax + bx$ for any $a, b \in \mathbf{R}$ and any $x \in \mathscr{X}$.
(10) $1x = x$ for all $x \in \mathscr{X}$.

Example 1.1. (1) $\mathbf{R}$ is a vector space. More generally, the n-dimensional Euclidean space $\mathbf{R}^n = \{x = (x^1, \cdots, x^n) : x^i \in \mathbf{R},\ 1 \le i \le n\}$ is a vector space.

(2) For a fixed interval $I \subset \mathbf{R}$, let $\mathscr{X}$ be the set of all real-valued functions defined on I. If we define

$$(\phi + \psi)(x) \doteqdot \phi(x) + \psi(x), \tag{1.2.1}$$

$$(a\phi)(x) \doteqdot a\phi(x), \tag{1.2.2}$$

where $a \in \mathbf{R}$ and $\phi, \psi \in \mathscr{X}$, then $\mathscr{X}$ is a vector space.

Example 1.2. Let

$$\mathscr{X}' = \{f \in \mathscr{X} : f(0) - f(1) = 1\}$$

where the vector space $\mathscr{X}$ is given by Example 1.1 with $I = [0, 1]$. However, the set $\mathscr{X}'$ is not a vector space. If we choose $f(x) = 1 - x$ and $g(x) = 1 - x^2$, then $f, g \in \mathscr{X}'$ but $f + g \notin \mathscr{X}'$.

If $\mathscr{Y}$ is a subset of a vector space $\mathscr{X}$, then it is a **subspace** of $\mathscr{X}$ if

(a) $x + y \in \mathscr{Y}$ for any $x, y \in \mathscr{Y}$, and
(b) $ax \in \mathscr{Y}$ for any $a \in \mathbf{R}$ and $x \in \mathscr{Y}$.

Note that $\mathscr{Y}$ is itself a vector space.

Example 1.3. For a fixed interval $I \subset \mathbf{R}$, let $\mathscr{C}^k(I)$ be the set of all real-valued functions on I which have continuous derivative of all orders up to and including k-th order. When $I = [a, b]$, we write $\mathscr{C}^k(I)$ as $\mathscr{C}^k[a, b]$.

1.3. **Functionals.** A **functional** is a map $\mathcal{J}$ from the subset $D(\mathcal{J})$ of some vector space $\mathscr{X}$, the domain of $\mathcal{J}$, to $\mathbf{R}$.

(1) **Brachistochrone functional.** Consider a smooth curve γ in the (x, y)-plane joining two fixed points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ (assume that $y_0 > y_1$). The time T required for a bead to move from P_0 down to P_1 along γ is given by

$$T = \int_0^T dt = \int_\gamma \frac{ds}{v}, \tag{1.3.1}$$

where s measures the arc length along γ, ds/dt is the rate of change of arc length with respect to time t, and the speed of motion is $v = ds/dt$.

We assume that the earth's gravitational acts down along the negative y-direction. Then a bead located at the position (x, y) and sliding along γ under the force of gravity will have kinetic energy of motion given as $\frac{1}{2}mv^2$ and potential energy given as mgy, where m is the mass of the bead. By the conservation of energy, we have

$$\frac{1}{2}mv^2 + mgy = mgy_0, \tag{1.3.2}$$

provided that the bead starts from rest at P_0 with zero initial kinetic energy and initial potential energy equal to mgy_0.

When

$$\gamma : y = Y(x), \quad x_0 \le x \le x_1,$$

for some suitable function $Y(x)$, we have

$$v = \sqrt{2g[y_0 - Y(x)]}, \quad ds = \sqrt{1 + Y'(x)^2}dx.$$

Hence

$$T(Y) = \int_{x_0}^{x_1} \sqrt{\frac{1 + Y'(x)^2}{2g[y_0 - Y(x)]}}dx \tag{1.3.3}$$

for any Y in $D(T) = \{Y \in C^1[x_0, x_1] : Y(x_0) = y_0 \text{ and } Y(x_1) = y_1\}$ that is a subset of $C^1[x_0, x_1]$.

(2) **Area functional.** Consider **Chaplygin's problem** on the greatest area that can be encircled in a given time T by varying the closed path γ flown by an airplane at constant natural speed v_0 while a constant wind blows. The airplane is assumed to be flying in a fixed horizonal plane surface above the surface of the earth which we take to be the (x, y)-coordinate plane. Suppose that a fixed closed path in the xy-plane parametrically as

$$\gamma : \begin{cases} x = X(t), \\ y = Y(t), \end{cases}$$

for $t \in [0, T]$, where we require that

$$X(0) = X(T), \quad Y(0) = Y(T). \tag{1.3.4}$$

The area enclosed by γ is given by

$$A = \frac{1}{2}\int_0^T [X(t)Y'(t) - Y(t)X'(t)]dt. \tag{1.3.5}$$

We let w_0 be the constant wind speed with $0 \le w_0 \le v_0$ and assume that the x-direction coincides with w_0. Let $\alpha(t)$ be the steering angle between the positive x-direction and the direction of the axis of the airplane. The absolute velocity of the airplane relative to the ground is

$$\begin{cases} X'(t) = v_0 \cdot \cos[\alpha(t)] + w_0, \\ Y'(t) = v_0 \cdot \sin[\alpha(t)]. \end{cases} \tag{1.3.6}$$

Hence

$$\begin{cases} X(t) = x_0 + v_0 \int_0^t \cos[\alpha(\tau)]d\tau + w_0 t, \\ Y(t) = Y_0 + v_0 \int_0^t \sin[\alpha(\tau)]d\tau, \end{cases} \tag{1.3.7}$$

where we set

$$X(0) = x_0, \quad Y(0) = y_0. \tag{1.3.8}$$

By (1.3.4) we must have

$$\begin{cases} \int_0^T \cos[\alpha(t)]dt = -\frac{w_0}{v_0} T, \\ \int_0^T \sin[\alpha(t)]dt = 0. \end{cases} \tag{1.3.9}$$

We might also impose the additional constraint that the initial steering angle $\alpha(0)$ be specified as

$$\alpha(0) = \alpha_0 \tag{1.3.10}$$

for some given initial angle α_0. Therefore

$$\begin{aligned} A(\alpha) \quad = \quad & \frac{1}{2} \int_0^T \left\{ v_0 \cdot \sin[\alpha(t)] \left[x_0 + w_0 t + v_0 \int_0^t \cos[\alpha(\tau)]d\tau \right] \right. \\ & \left. - [v_0 \cdot \cos[\alpha(t)] + w_0] \left[y_0 + v_0 \int_0^t \sin[\alpha(\tau)]d\tau \right] \right\} dt, \end{aligned} \tag{1.3.11}$$

and the domain $D(A)$, a subset of $\mathscr{C}^0[0, T]$, is the set of all continuous functions $\alpha(t)$ on $[0, T]$ satisfying (1.3.9) and (1.3.10).

(3) **Transit time functional.** Consider the transit time of a boat crossing a river from a fixed initial point on one bank to a specified terminal point on the other bank. For simplicity, we assume that the river has parallel banks, where we let the y-axis coincide with the left bank. The river is ℓ units wide so that the right bank coincides with the line $x = \ell$.

We consider a river without cross currents so that the current velocity is everywhere directed downstream along the y-direction. Furthermore, we assume that this downstream current speed w depends only on x as

$$w = w(x), \quad x \in [0, \ell],$$

and the boat travels at a constant natural speed v_0 relative to the surrounding water. If the path of the boat is represented by

$$\gamma : \begin{cases} x = \xi(t), \\ y = \eta(t), \end{cases}$$

for $t \in [0, T]$. Then

$$\begin{cases} \frac{dx}{dt} = \xi'(t) = v_0 \cdot \cos[\alpha(t)], \\ \frac{dy}{dt} = \eta'(t) = v_0 \sin[\alpha(t)] + w(\xi(t)), \end{cases} \tag{1.3.12}$$

where $\alpha(t)$ is the steering angle of the boat measured between the positive x-direction and the direction of the axis of the boat.

The time of transit T of the boat is

$$T = \int_0^T dt = \int_0^\ell \frac{dt}{dx} dx = \int_0^\ell \frac{dx}{v_0 \cdot \cos\alpha}, \tag{1.3.13}$$

where $\alpha = \alpha(t) = \alpha(x)$. If

$$\gamma : y = Y(x), \quad x \in [0, \ell], \tag{1.3.14}$$

where the function $Y(x) = \eta(\xi^{-1}(x))$. Then

$$Y'(x) = \frac{\eta'(t)}{\xi'(t)}$$

so that

$$Y'(x) = \frac{\sin\alpha + e(x)}{\cos\alpha}, \quad e(x) \doteqdot \frac{w(x)}{v_0}. \tag{1.3.15}$$

From (1.3.15), we have

$$\cos\alpha = \frac{1 - e(x)^2}{\sqrt{1 - e(x)^2 + Y'(x)^2 - e(x)Y'(x)}} \tag{1.3.16}$$

and

$$T(Y) = \frac{1}{v_0} \int_0^\ell \frac{\sqrt{1 - e(x)^2 + Y'(x)^2} - e(x)Y'(x)}{1 - e(x)^2} dx. \tag{1.3.17}$$

The domain of the functional defined by (1.3.17) is the vector space $\mathscr{C}^1[0, \ell]$.

(4) **Cost functional.** We consider a company that manufactures and sells some particular product. We assume that the company has on hand sufficient long-term orders for its product so that it can predict its future sales rate

$$S = S(t) \tag{1.3.18}$$

with certainty. If the product is completely durable, it is natural to assume that the production rate $P(t)$ and the finished product inventory level $I(t)$ are related by

$$\dot{I} = P - S.$$

More generally, we allow for some spoilage of the inventory by considering instead the equation

$$\dot{I} = P - (S + \alpha I), \tag{1.3.19}$$

where the spoilage proportionality α is a given constant.

Finally, we assume that on the basis of the known sales forecast S of (1.3.18) the company has decided on a desired inventory level $\mathfrak{i}(t)$, resulting in a corresponding desired production rate $\mathfrak{p}(t)$, obtained from (1.3.19) as

$$\dot{\mathfrak{i}} = \mathfrak{p} - (S + \alpha\mathfrak{i}). \tag{1.3.20}$$

Consider the cost functional $\mathcal{C}$ defined as

$$\mathcal{C} \doteqdot \int_0^T \left\{\beta^2[I(t) - \mathfrak{i}(t)]^2 + [P(t) - \mathfrak{p}(t)]^2\right\} dt, \tag{1.3.21}$$

where β is a fixed constant which the company might want to specify so as to give different relative weights to the unwanted derivations of the inventory level and production rate away from their known desired levels.

Suppose that

$$I(0) = I_0 \tag{1.3.22}$$

holds for some given nonnegative constant I_0 with $I_0 \neq \mathfrak{i}_0 = \mathfrak{i}(0)$. The difference $I_0 - \mathfrak{i}_0$ furnishes a measure of the initial disturbance away from the desired state. From (1.3.19) we have

$$I_P(t) \doteqdot I(t) = e^{-\alpha t}\left(I_0 + \int_0^t e^{\alpha t}[P(\tau) - S(\tau)]d\tau\right). \tag{1.3.23}$$

Now the cost functional C given by (1.3.21) can be written as

$$\mathcal{C}(P) = \int_0^T \left\{\beta^2[I_P(t) - \mathfrak{i}(t)]^2 + [P(t) - \mathfrak{p}(t)]^2\right\} dt. \tag{1.3.24}$$

The domain of the functional $\mathcal{C}$ can be taken to be the vector space $\mathscr{C}^0[0, T]$.

1.4. **Normed vector spaces.** A vector space $\mathscr{X}$ is said to be a **normed vector space** if there is real-valued norm function $||\cdot||$ defined on $\mathscr{X}$ which assigns the number $||x||$ (called the **norm** of x) to the vector $x \in \mathscr{X}$ such that

(1) $||x|| \geq 0$ whenever $x \in \mathscr{X}$, and $||x|| = 0$ if and only if $x = 0$;
(2) $||ax|| = |a|||x||$ for every $x \in \mathscr{X}$ and every $\alpha \in \mathbf{R}$;
(3) (Triangle inequality) $||x + y|| \leq ||x|| + ||y||$ for any $x, y \in \mathscr{X}$.

Example 1.4. For $x = (x^1, \cdots, x^n) \in \mathbf{R}^n$, we define

$$||x|| \doteqdot \sqrt{(x^1)^2 + \cdots + (x^n)^2} \tag{1.4.1}$$

that is a norm on $\mathbf{R}^n$. The triangle inequality follows from Cauchy's inequality

$$\left(\sum_{i=1}^n x^i y^i\right)^2 \leq \left(\sum_{i=1}^n (x^i)^2\right)\left(\sum_{i=1}^n (y^i)^2\right). \tag{1.4.2}$$

Example 1.5. For $\phi \in \mathscr{C}^0[a, b]$, we define

$$||\phi||_{\mathscr{C}^0[a,b]} \doteqdot \left(\int_a^b |\phi(x)|^2 dx\right)^{1/2} \tag{1.4.3}$$

that is a norm on $\mathscr{C}^0[a, b]$. The triangle inequality follows from Schwarz's inequality

$$\left(\int_a^b \phi(x)\psi(x)dx\right)^2 \leq \left(\int_a^b \phi(x)^2 dx\right)\left(\int_a^b \psi(x)^2 dx\right). \tag{1.4.4}$$

Another norm is given by

$$||\phi||'_{\mathscr{C}^0[a,b]} \doteqdot \max_{a \leq x \leq b} |\phi(x)| \tag{1.4.5}$$

for any $\phi \in \mathscr{C}^0[a, b]$. We call (1.4.3) and (1.4.5) the L^2**-norm** and the **uniform norm** on $\mathscr{C}^0[a, b]$, respectively.

For $\phi \in \mathscr{C}^k[a, b]$, we define

$$||\phi||'_{\mathscr{C}^k[a,b]} \doteqdot \sum_{i=0}^k \max_{a \leq x \leq b} \left|\phi^{(i)}(x)\right|. \tag{1.4.6}$$

Note that any subspace $\mathscr{Y}$ of any given normed vector space $\mathscr{X}$ is itself a normed vector space with the same norm as used on $\mathscr{X}$.

If $(\mathscr{X}, ||\cdot||)$ is a normed vector space, we define the ball of radius ρ centered at x to be the set

$$B_\rho(x) \doteqdot \{y \in \mathscr{X} : ||y - x|| < \rho\}. \tag{1.4.7}$$

A subset D of a normed vector space $(\mathscr{X}, ||\cdot||)$ is said to be **open** in $\mathscr{X}$ whenever D contains along with each of its elements x some ball $B_\rho(x)$ in $\mathscr{X}$ centered at x for some positive ρ which may depend on x.

(a) A subspace $\mathscr{Y}$ of a normed vector space $(\mathscr{X}, ||\cdot||)$ is not open in $\mathscr{X}$ unless $\mathscr{Y}$ is all of $\mathscr{X}$. For example, $\mathbf{R}$ is a subspace of $\mathbf{R}^2$, but $\mathbf{R}$ is not open in $\mathbf{R}^2$ under the usual norm.

(b) Any open subspace $\mathscr{Y}$ is always considered to be a normed vector space itself with the same norm as used in $\mathscr{X}$, and then $\mathscr{Y}$ is an open subset of itself.

1.5. **Continuous functionals.** Let D be a fixed open set in a given normed vector space $(\mathscr{X}, ||\cdot||)$, and let $\mathcal{J}$ be a functional defined on D. $\mathcal{J}$ is said to have the number L as its **limit at** x in D if $\mathcal{J}(y)$ is close to L whenever y is close to x (but distinct from x). That is, $\mathcal{J}$ has the limit L at x if for every $\epsilon > 0$ there is a ball $B_\rho(x) \subset D$ such that

$$|L - \mathcal{J}(y)| < \epsilon$$

for all $y \in B_\rho(x) \setminus \{x\}$. Symbolically we write

$$\lim_{y \to x \text{ in } \mathscr{X}} \mathcal{J}(y) = L \tag{1.5.1}$$

whenever $\mathcal{J}$ has limit L at x.

$\mathcal{J}$ is said to be **continuous at** x **in** D if $\mathcal{J}$ has the limit $\mathcal{J}(x)$ at x,

$$\lim_{y \to x \text{ in } \mathscr{X}} \mathcal{J}(y) = \mathcal{J}(x). \tag{1.5.2}$$

$\mathcal{J}$ is said to be **continuous on** D if $\mathcal{J}$ is continuous at each point of D.

Example 1.6. The cost functional (1.3.24) is continuous on $\mathscr{C}^0[0,T]$ with the uniform norm (1.4.5). It suffices to show that for any such fixed $P \in \mathscr{C}^0[0,T]$ and for any given $\epsilon > 0$ we can find a positive number ρ such that

$$|\mathcal{C}(P) - \mathcal{C}(Q)| < \epsilon \tag{1.5.3}$$

whenever

$$0 \leq ||P - Q||'_{\mathscr{C}^0[0,T]} < \rho, \tag{1.5.4}$$

where

$$||P - Q||'_{\mathscr{C}^0[0,T]} = \max_{t \in [0,T]} |P(t) - Q(t)|. \tag{1.5.5}$$

Since

$$\mathcal{C}(Q) = \int_0^T \left\{\beta^2 [I_Q(t) - \mathfrak{i}(t)]^2 + [Q(t) - \mathfrak{p}(t)]^2\right\} dt, \tag{1.5.6}$$

where

$$I_Q(t) = e^{-\alpha t}\left(I_0 + \int_0^t e^{\alpha\tau}[Q(\tau) - S(\tau)]d\tau\right), \tag{1.5.7}$$

it follows that

$$\begin{aligned}\mathcal{C}(P)-\mathcal{C}(Q) &= \beta^2\int_0^T [I_P(t)-I_Q(t)][I_P(t)+I_Q(t)-2\mathfrak{i}(t)]dt\\ &+ \int_0^T [P(t)-Q(t)][P(t)+Q(t)-2\mathfrak{p}(t)]dt\end{aligned}$$

and hence

$$\begin{aligned}|\mathcal{C}(P)-\mathcal{C}(Q)| &\leq \beta^2\int_0^T |I_P(t)-I_Q(t)||I_P(t)+I_Q(t)-2\mathfrak{i}(t)|dt\\ &+ \int_0^T |P(t)-Q(t)||P(t)+Q(t)-2\mathfrak{p}(t)|dt.\end{aligned} \tag{1.5.8}$$

From

$$\begin{aligned}|I_P(t)+I_Q(t)-2\mathfrak{i}(t)| &= |2(I_P(t)-\mathfrak{i}(t))+(I_Q(t)-I_P(t)|\\ &\leq 2|I_P(t)-\mathfrak{i}(t)|+|I_P(t)-I_Q(t)|\end{aligned} \tag{1.5.9}$$

and similarly

$$|P(t)+Q(t)-2\mathfrak{p}(t)|\leq 2|P(t)-\mathfrak{p}(t)|+|P(t)-Q(t)|, \tag{1.5.10}$$

we have

$$\begin{aligned}|\mathcal{C}(P)-\mathcal{C}(Q)| &\leq \beta^2\int_0^T |I_P(t)-I_Q(t)|\left(2|I_P(t)-\mathfrak{i}(t)|+|I_P(t)-I_Q(t)|\right)dt\\ &+ \int_0^T |P(t)-Q(t)|\left(2|P(t)-\mathfrak{p}(t)|+|P(t)-Q(t)|\right)dt.\end{aligned} \tag{1.5.11}$$

By

$$|P(t)-Q(t)|\leq ||P-Q||'_{\mathscr{C}^0[0,T]}, \quad |P(t)-\mathfrak{p}(t)|\leq ||P-\mathfrak{p}||'_{\mathscr{C}^0[0,T]}, \tag{1.5.12}$$

we get

$$\begin{aligned}&\int_0^T |P(t)-Q(t)|\left(2|P(t)-\mathfrak{p}(t)|+|P(t)-Q(t)|\right)dt\\ \leq& \int_0^T ||P-Q||'_{\mathscr{C}^0[0,T]}\left(2||P-\mathfrak{p}||'_{\mathscr{C}^0[0,T]}+||P-Q||'_{\mathscr{C}^0[0,T]}\right)dt\\ =& ||P-Q||'_{\mathscr{C}^0[0,T]}\left(2||P-\mathfrak{p}||'_{\mathscr{C}^0[0,T]}+||P-Q||'_{\mathscr{C}^0[0,T]}\right)T.\end{aligned} \tag{1.5.13}$$

On the other hand, we have

$$|I_P(t)-I_Q(t)|\leq e^{-\alpha t}\int_0^t r^{\alpha t}|P(\tau)-Q(\tau)|d\tau\leq ||P-Q||'_{\mathscr{C}^0[0,T]}T \tag{1.5.14}$$

and

$$\begin{aligned}&\beta^2\int_0^T |I_P(t)-I_Q(t)|\left(2|I_P(t)-\mathfrak{i}(t)|+|I_P(t)-I_Q(t)|\right)dt\\ \leq& \beta^2||P-Q||'_{\mathscr{C}^0[0,T]}T\left(2||I_P-\mathfrak{i}||'_{\mathscr{C}^0[0,T]}+||P-Q||'_{\mathscr{C}^0[0,T]}T\right)T.\end{aligned} \tag{1.5.15}$$

It follows from (1.5.11), (1.5.13), and (1.5.15) that

$$\begin{aligned} |\mathcal{C}(P)-\mathcal{C}(Q)| \quad \leq \quad & ||P-Q||'_{\mathscr{C}^0[0,T]}T\left\{\beta^2 T\left(2||I_P-\mathfrak{i}||'_{\mathscr{C}^0[0,T]}\right.\right. \\ & \left.\left.+||P-Q||'_{\mathscr{C}^0[0,T]}T\right)+2||P-\mathfrak{p}||'_{\mathscr{C}^0[0,T]}+||P-Q||'_{\mathscr{C}^0[0,T]}\right\}. \end{aligned} \tag{1.5.16}$$

Letting $Q\to P$ in $\mathscr{C}^0[0,T]$, we have

$$\lim_{Q\to P \text{ in } \mathscr{C}^0[0,T]} \mathcal{C}(Q)=\mathcal{C}(P).$$

Remark 1.7. We give examples showing that the continuity or lack of continuity of a given functional defined on a vector space may depend on the particular norm used. This can happen only in infinite dimensional vector spaces, since all norms are equivalent to each other on any finite dimensional vector space.

(i) Let $\mathscr{X}=\mathscr{C}^0[0,1]$ equipped with the L^2-norm given as

$$||\phi||_{L^2[0,1]} \doteqdot \left(\int_0^1 |\phi(t)|^2 dt\right)^{1/2}$$

for any $\phi\in\mathscr{X}$. Consider the functional $\mathcal{J}$ defined by

$$\mathcal{J}(\phi)\doteqdot\phi(0), \quad \phi\in\mathscr{X}.$$

We show that $\mathcal{J}$ fails to be continuous at ϕ for each vector $\phi\in\mathscr{X}$. Let $\psi\doteqdot\phi+\chi$. Then

$$\mathcal{J}(\psi)-\mathcal{J}(\phi)=\psi(0)-\phi(0)=\chi(0).$$

If we choose

$$\chi_n(t)\doteqdot\begin{cases} 1, & 0\le t\le \frac{1}{n}, \\ \sqrt{2-nt}, & \frac{1}{n}\le t\le\frac{2}{n}, \\ 0, & \frac{2}{n}\le t\le 1, \end{cases}$$

then

$$\begin{aligned} ||\chi_n||^2_{L^2[0,1]} \quad &= \quad \int_0^1 |\chi_n(t)|^2 dt \quad = \quad \int_0^{\frac{1}{n}} dt+\int_{\frac{1}{n}}^{\frac{2}{n}}(2-nt)dt \\ &= \quad \frac{1}{n}+\left(2t-\frac{n}{2}t^2\right)\Big|_{\frac{1}{n}}^{\frac{2}{n}} \quad = \quad \frac{3}{2n}. \end{aligned}$$

Consequently, $\psi_n:=\phi+\chi_n\to\phi$ as $n\to\infty$ in L^2-norm, but $\mathcal{J}(\psi_n)-\mathcal{J}(\phi)=1$.

(ii) Let $\mathscr{Y}=\mathscr{C}^0[0,1]$ equipped with the uniform norm as

$$||\phi||_{\mathscr{C}^0[0,1]}\doteqdot\max_{t\in[0,1]}|\phi(t)|$$

for any $\phi\in\mathscr{Y}$. Consider the same functional $\mathcal{J}$ defined by

$$\mathcal{J}(\phi)\doteqdot\phi(0), \quad \phi\in\mathscr{Y}.$$

We show that $\mathcal{J}$ is continuous at each vector $\phi\in\mathscr{Y}$. Indeed,

$$|\mathcal{J}(\psi)-\mathcal{J}(\phi)|=|\psi(0)-\phi(0)|\le||\psi-\phi||_{\mathscr{C}^0[0,1]}\to 0$$

as $||\psi-\phi||_{\mathcal{C}^0[0,1]}\to 0$.

1.6. **Linear functionals.** A functional $\mathscr{J}$ is said to be **linear** if the domain of $\mathscr{J}$ consists of some entire vector space $\mathscr{X}$ and if $\mathscr{J}$ satisfies the linearity relation

$$\mathscr{J}(ax+by) = a\mathscr{J}(x) + b\mathscr{J}(y) \tag{1.6.1}$$

for all $a, b \in \mathbf{R}$ and all vectors $x, y \in \mathscr{X}$.

Remark 1.8. (a) The cost functional $\mathcal{C}$ is not linear.

(b) If $\mathscr{J}$ is a linear functional, then $\mathscr{J}(0) = 0$. In fact, by (1.6.1), we have

$$\mathscr{J}(0) = \mathscr{J}(0+0) = \mathscr{J}(0) + \mathscr{J}(0) = 2\mathscr{J}(0).$$

(c) A linear functional is continuous on its domain $\mathscr{X}$ if and only if it is continuous at the zero vector in $\mathscr{X}$. The proof is very easy. Suppose first that a linear functional $\mathscr{J}$ is continuous at $0 \in \mathscr{X}$, then for any given $\epsilon > 0$ there is a number $\rho > 0$ such that

$$|\mathscr{J}(y)| = |\mathscr{J}(y) - \mathscr{J}(0)| < \epsilon$$

whenever $||y|| = ||y - 0|| < \rho$. If x is any vector in $\mathscr{X}$, then

$$\mathscr{J}(z) - \mathscr{J}(x) = \mathscr{J}(z - x)$$

holds for all vectors $z \in \mathscr{X}$ and so

$$|\mathscr{J}(z) - \mathscr{J}(x)| = |\mathscr{J}(z - x)|.$$

If $y := z - x$ and $||y|| < \rho$, then $|\mathscr{J}(z) - \mathscr{J}(x)| = |\mathscr{J}(y)| < \epsilon$. Thus $\mathscr{J}$ is continuous at $x \in \mathscr{X}$.

(d) A linear functional $\mathscr{J}$ is continuous at the zero vector in its domain $\mathscr{X}$ if

$$|\mathscr{J}(x)| \le C||x|| \tag{1.6.2}$$

for all $x \in \mathscr{X}$ and some fixed constant C depending only on the functional $\mathscr{J}$.

(e) Let $e_1, \cdots, e_n$ be the unit vectors in $\mathbf{R}^n$ given as

$$e_1 = (1, 0, \cdots, 0), \quad e_n = (0, 0, \cdots, 1),$$

where e_i has its ith component equal to 1 and all other components equal to 0. If $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$ is any vector, then x can be written as a linear combination of

$$x = \sum_{i=1}^{n} x_i e_i.$$

Suppose that $\mathscr{J}$ is a linear functional on $\mathbf{R}^n$. Then for any $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$, we have

$$\mathscr{J}(x) = \mathscr{J}\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i \mathscr{J}(e_i).$$

Since

$$\begin{aligned} |\mathscr{J}(x)| &\le \sum_{i=1}^{n} |x_i| \cdot |\mathscr{J}(e_i)| \le \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |\mathscr{J}(e_i)|^2\right)^{1/2} \\ &= \left(\sum_{i=1}^{n} |\mathscr{J}(e_i)|^2\right)^{1/2} ||x|| \end{aligned}$$

by Cauchy's inequality, it follows from (1.6.2) that $\mathscr{J}$ is continuous at the zero vector in $\mathbf{R}^n$ and then is continuous on $\mathbf{R}^n$ by (c).

Note that every linear functional on a *finite* dimensional normed vector space $\mathscr{X}$ is continuous.

(e) However the above result is not true for every normed vector space. For example, consider the functional

$$\mathcal{J}(\phi) := \phi'(0) = \frac{d}{dx}\Big|_{x=0}\phi(x)$$

on the normed vector space $\mathscr{C}^1[-1,1]$ equipped with the uniform norm defined as $||\phi|| := \max_{x\in[-1,1]}|\phi(x)|$ for any vector $\phi \in \mathscr{C}^1[-1,1]$. Consider the vectors

$$\phi_k(x) := \frac{1}{\sqrt{k}}\sin(kx), \quad x \in [-1,1], \quad k \in \mathbf{N},$$

in $\mathscr{C}^1[-1,1]$. For any $a, b \in \mathbf{R}$ and $\phi, \psi \in \mathscr{C}^1[-1,1]$, we have

$$\mathcal{J}(a\phi + b\psi) = \frac{d}{dx}\Big|_{x=0}[a\phi(x) + b\psi(x)] = a\phi'(0) + b\psi'(0) = a\mathcal{J}(\phi) + b\mathcal{J}(\psi)$$

and then $\mathcal{J}$ is linear. On the other hand,

$$\begin{aligned} ||\phi_k|| &= \max_{x\in[-1,1]}\frac{|\sin(kx)|}{\sqrt{k}} &= \frac{1}{\sqrt{k}}, \\ |\mathcal{J}(\phi_k)| &= |\sqrt{k}\cdot\cos(kx)|_{x=0}| &= \sqrt{k}, \end{aligned}$$

which imply that $\lim_{\phi_k\to 0 \text{ in } \mathscr{C}^1[-1,1]}\mathcal{J}(\phi_k) = \lim_{k\to\infty}\mathcal{J}(\phi_k) = +\infty$. Hence, $\mathcal{J}$ is not continuous at the zero vector and then is not continuous on $\mathscr{C}^1[-1,1]$.

2. A fundamental necessary condition for an extremum

In this section we introduce the Gâteaux[1] variation and the Fréchet differential of a functional.

2.1. Gâteaux variation.

Let $\mathscr{D}$ be a fixed nonempty subset of a normed vector space $\mathscr{X}$, and let $\mathcal{J}$ be a functional defined on $\mathscr{D}$.

Definition 2.1. A vector $x^* \in \mathscr{D}$ is said to be a **maximum vector in $\mathscr{D}$ for $\mathcal{J}$** if $\mathcal{J}(x) \le \mathcal{J}(x^*)$ for all vectors $x \in \mathscr{D}$. The vector x^* in $\mathscr{D}$ is a **local maximum vector in $\mathscr{D}$ for $\mathcal{J}$** if there is some ball $\mathscr{B}_\rho(x^*)$ in $\mathscr{X}$ centered at x^* such that $\mathcal{J}(x) \le \mathcal{J}(x^*)$ for all vectors $x \in \mathscr{D} \cap \mathscr{B}_\rho(x^*)$. If $\mathscr{D}$ is an open subset of $\mathscr{X}$, we require that the ball $\mathscr{B}_\rho(x^*)$ to be contained in $\mathscr{D}$. A **local minimum vector in $\mathscr{D}$ for $\mathcal{J}$** is defined similarly.

We say that x^* is a **local extremum vector in $\mathscr{D}$ for $\mathcal{J}$** if x^* is either a local maximum vector or a local minimum vector, and in this case we say that $\mathcal{J}$ has a **local extremum at x^***. The functional value $\mathcal{J}(x^*)$ is said to be a **local extreme value for $\mathcal{J}$ in $\mathscr{D}$**.

Definition 2.2. A functional $\mathcal{J}$ defined on an open subset $\mathscr{D}$ of a normed vector space $\mathscr{X}$ is said to have a **Gâteaux variation at a vector x in $\mathscr{D}$** whenever there is a function $\delta\mathcal{J}(x)$ with values $\delta\mathcal{J}(x;h)$ defined for all vectors h in $\mathscr{X}$ and such that

$$\lim_{\epsilon\to 0}\frac{\mathcal{J}(x+\epsilon h) - \mathcal{J}(x)}{\epsilon} = \delta\mathcal{J}(x;h) \tag{2.1.1}$$

[1] René Eugène Gâteaux (1889–1914): A French mathematician who is known for the Gâteaux derivative. Part of his work has been posthumously published by Paul Lévy. Gâteaux was killed during World War I. The word "Gâteaux" is pronounced as gah/toh.

holds for every vector h in $\mathscr{X}$. The functional $\delta\mathcal{J}(x)$ is called the **Gâteaux variation of $\mathcal{J}$ at** x. Note that $\delta\mathcal{J}(x) : \mathscr{X} \to \mathbf{R}$.

Theorem 2.3. *If a functional $\mathcal{J}$ defined on an open set $\mathscr{D}$ contained in a normed vector space $\mathscr{X}$ has a local extremum at a vector x^* in $\mathscr{D}$, and if $\mathcal{J}$ has a Gâteaux variation at x^*, then the Gáteaux variation of $\mathcal{J}$ at x^* must vanish; that is,*

$$\delta\mathcal{J}(x^*; h) = 0 \tag{2.1.2}$$

for all vectors h in $\mathscr{X}$.

Proof. Without loss of generality, we may assume that x^* is a local minimum vector in $\mathscr{D}$ for $\mathcal{J}$. If $h \in \mathscr{X}$, then

$$\mathcal{J}(x^* + \epsilon h) - \mathcal{J}(x^*) \geq 0$$

holds for all sufficiently small number $\epsilon > 0$, since $x^* + \epsilon h \in \mathscr{D}$ by the openness of $\mathscr{D}$. Hence

$$\frac{\mathcal{J}(x^* + \epsilon h) - \mathcal{J}(x^*)}{\epsilon} \geq 0$$

for all sufficiently small number $\epsilon > 0$. Hence

$$\liminf_{\epsilon \to 0+} \frac{\mathcal{J}(x^* + \epsilon h) - \mathcal{J}(x^*)}{\epsilon} \geq 0.$$

Similarly,

$$\limsup_{\epsilon \to 0-} \frac{\mathcal{J}(x^* + \epsilon h) - \mathcal{J}(x^*)}{\epsilon} \leq 0.$$

Since $\mathcal{J}$ has a Gâteaux variation at x^*, it follows that (2.1.2) holds. □

Remark 2.4. (1) Let $\mathcal{J}$ be a real-valued function defined in some open interval $\mathscr{D} = (a, b) \subset \mathbf{R}$. If $\mathcal{J}$ is differentiable at x, then $\mathcal{J}$ has a variation at x given as $\delta\mathcal{J}(x; h) = \mathcal{J}'(x)h$ for any $h \in \mathbf{R}$. Hence (2.1.2) is equivalent to $\mathcal{J}'(x^*) = 0$.

(2) Let $\mathcal{J}$ be a real-valued function defined in some open region $\mathscr{D} \subset \mathbf{R}^n$. If $\mathcal{J}$ has continuous first-order partial derivatives at x, denoted as $\mathcal{J}_{x_i}(x) \doteqdot \partial\mathcal{J}(x)/\partial x_i$ for $i = 1, \cdots, n$, then $\mathcal{J}$ has a variation at x given as

$$\delta\mathcal{J}(x; h) = \sum_{i=1}^{n} \mathcal{J}_{x_i}(x) h_i \tag{2.1.3}$$

for any vector $h = (h_1, \cdots, h_n) \in \mathbf{R}^n$. Hence (2.1.2) is equivalent to $\mathcal{J}_{x_i}(x^*) = 0$ for $i = 1, \cdots, n$.

(3) (2.1.2) is a necessary condition but may not be a sufficient condition for $\mathcal{J}(x^*)$ being a local extreme value for $\mathcal{J}$ in $\mathscr{D}$. Indeed, (2.1.2) may hold also at certain non-extremum vectors x^* such as **saddle points** or certain **inflection points** of $\mathcal{J}$ in $\mathscr{D}$.

- (Inflection points) Let $\mathcal{J}(x) = x^3$, $x \in \mathbf{R}$. The variation of $\mathcal{J}$ vanishes at $x^* = 0$, but, the point $x^* = 0$ is not a local extremum vector in $\mathbf{R}$ for $\mathcal{J}$.
- (Saddle points) Let $\mathcal{J}(x) = x_2^2 - x_1^2$, where $x = (x_1, x_2) \in \mathbf{R}^2$. The variation of $\mathcal{J}$ vanishes at $x^* = (0, 0)$, but, the point $x^* = (0, 0)$ is not a local extremum vector in $\mathbf{R}^2$ for $\mathcal{J}$.

(4) The limit (2.1.1) is unique if it exists, hence a functional can have at most one variation at x.

(5) The value of the variation is the ordinary derivative of the function $\mathscr{J}(x+\epsilon h)$ considered as a function of the real number ϵ and evaluated at $\epsilon = 0$; i.e.,

$$\delta\mathscr{J}(x;h) = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathscr{J}(x+\epsilon h). \tag{2.1.4}$$

(6) The variation satisfy the homogeneity relation

$$\delta\mathscr{J}(x;ah) = a\cdot\delta\mathscr{J}(x;h) \tag{2.1.5}$$

for any $a \in \mathbf{R}$.

(7) We usually use the symbol "Δx" rather than h to denote the second argument in the variation $\delta\mathscr{J}(x;\Delta x)$:

$$\delta\mathscr{J}(x;\Delta x) = \lim_{\epsilon\to 0}\frac{\mathscr{J}(x+\epsilon\Delta x)-\mathscr{J}(x)}{\epsilon} = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathscr{J}(x+\epsilon\Delta x). \tag{2.1.6}$$

(8) Let the functional $\mathscr{J}$ be defined on an open set $\mathscr{D}$ in a normed vector space $\mathscr{X}$ by

$$\mathscr{J}(x) = \mathcal{K}(x)\mathcal{L}(x), \quad x \in \mathscr{D},$$

where $\mathcal{K}$ and $\mathcal{L}$ are given functionals on $\mathscr{D}$ which are known to have variations at a vector $x_0 \in \mathscr{D}$. Then

$$\delta\mathscr{J}(x_0;h) = \mathcal{K}(x_0)\delta\mathcal{L}(x_0;h) + \delta\mathcal{K}(x_0;h)\mathcal{L}(x_0) \tag{2.1.7}$$

for any vector $h \in \mathscr{X}$.

(9) Let $\mathcal{K}$ and $\mathcal{L}$ be as in (8) and define

$$\mathscr{J}(x) = \frac{\mathcal{K}(x)}{\mathcal{L}(x)}$$

for any $x \in \mathscr{D}$ for which $\mathcal{L}(x) \neq 0$. If $\mathcal{K}$ and $\mathcal{L}$ are known to have variations at a vector $x_0 \in \mathscr{D}$ at which $\mathcal{L}(x_0) \neq 0$, then

$$\delta\mathscr{J}(x_0;h) = \frac{\mathcal{L}(x_0)\delta\mathcal{K}(x_0;h) - \mathcal{K}(x_0)\delta\mathcal{L}(x_0;h)}{\mathcal{L}(x_0)^2} \tag{2.1.8}$$

for any vector $h \in \mathscr{X}$.

Example 2.5. (1) Consider the functional

$$\mathcal{L}(\phi) := \int_0^{\pi/2}[2\phi(x)^3 + 9(\sin(x))\phi(x)^2 + 12(\sin^2(x))\phi(x) - \cos(x)]dx$$

for any function $\phi \in \mathscr{C}^0[0,\pi/2]$. Since

$$\begin{aligned}\mathcal{L}(\phi+\epsilon\psi) &= \int_0^{\pi/2}[2(\phi(x)+\epsilon\psi(x))^3 + 9\sin(x)(\phi(x)+\epsilon\psi(x))^2 \\ &\quad +12\sin^2(x)(\phi(x)+\epsilon\psi(x)) - \cos(x)]dx \\ &= \mathcal{L}(\phi) + \int_0^{\pi/2}[6\epsilon\phi(x)^2\psi(x) + 6\epsilon\phi(x)\psi(x)^2 + 2\epsilon^3\psi(x)^3 \\ &\quad +18\epsilon\sin(x)\phi(x)\psi(x) + 9\epsilon^2\sin(x)\psi(x)^2 + 12\epsilon\sin^2(x)\psi(x)]dx,\end{aligned}$$

we obtain

$$\begin{aligned}\frac{\mathcal{L}(\phi+\epsilon\psi)-\mathcal{L}(\phi)}{\epsilon} &= \int_0^{\pi/2}[6\phi(x)^2\psi(x) + 18\sin(x)\phi(x)\psi(x) + 12\sin^2(x)\psi(x)]dx \\ &\quad + \epsilon\int_0^{\pi/2}[6\phi(x)\psi(x)^2 + 2\epsilon\psi(x)^3 +_9 \sin(x)\psi(x)^2]dx\end{aligned}$$

and hence

$$\delta\mathcal{L}(\phi;\psi)=\int_0^{\pi/2}[6\phi(x)^2\psi(x)+18\sin(x)\phi(x)\psi(x)+12\sin^2(x)\psi(x)]dx.$$

If $\phi^*\in\mathscr{C}^0[0,\pi/2]$ is a local extremum vector, then by Theorem 2.3 we conclude that

$$0=6\phi^*(x)^2+18\sin(x)\phi^*(x)+12\sin^2(x)=6[\phi^*(x)+2\sin(x)][\phi^*(x)+\sin(x)].$$

(2) Consider the functional $\mathcal{J}$ defined in Remark 1.7. We have shown that $\mathcal{J}$ is not continuous on $\mathscr{C}^0[0,1]$. However, $\mathcal{J}$ has a Gâteaux variation at each vector $\phi\in\mathscr{C}^0[0,1]$ as

$$\delta\mathcal{J}(\phi;\Delta\phi)=\lim_{\epsilon\to0}\frac{\mathcal{J}(\phi+\epsilon\Delta\phi)-\mathcal{J}(\phi)}{\epsilon}=\lim_{\epsilon\to0}\frac{\epsilon\Delta\phi(0)}{\epsilon}=\Delta\phi(0).$$

(3) Let $\mathcal{J}$ be defined for any vector $x=(x_1,x_2)\in\mathbf{R}^2$ by

$$\mathcal{J}(x)=\begin{cases}\frac{x_1x_2^2}{x_1^2+x_2^4}, & x_1\neq0,\\ 0, & x_1=0.\end{cases}$$

From $\mathcal{J}(0)=0$ we have

$$\begin{aligned}\delta\mathcal{J}(0;h) &= \lim_{\epsilon\to0}\frac{\mathcal{J}(0+\epsilon h)-\mathcal{J}(0)}{\epsilon} = \begin{cases}\lim_{\epsilon\to0}\frac{\epsilon^2h_1h_2^2}{\epsilon^2h_1^2+\epsilon^4h_2^4}, & h_1\neq0,\\ 0, & h_1=0.\end{cases}\\ &= \begin{cases}\frac{h_2^2}{h_1}, & h_1\neq0,\\ 0, & h_1=0.\end{cases}\end{aligned}$$

However,

$$\lim_{(x_2^2,x_2)\to0}\mathcal{J}(x)=\lim_{x_2\to0}\frac{x_2^4}{x_2^4+x_2^4}=\frac{1}{2}\neq0.$$

Thus $\mathcal{J}$ has a Gâteaux variation at the origin $0=(0,0)$ but is not continuous at $0=(0,0)$.

Remark 2.6. A functional $\mathcal{J}$ is **continuous along each fixed direction at** x if

$$\lim_{\epsilon\to0}\mathcal{J}(x+\epsilon h)=\mathcal{J}(x) \tag{2.1.9}$$

for each fixed vector $h\in\mathscr{X}$.

(a) If $\mathcal{J}$ has a Gâteaux variation at $x\in\mathscr{X}$, then $\mathcal{J}$ is continuous along each fixed direction at x. Indeed,

$$\lim_{\epsilon\to0}[\mathcal{J}(x+\epsilon h)-\mathcal{J}(x)]=\lim_{\epsilon\to0}\frac{\mathcal{J}(x+\epsilon h)-\mathcal{J}(x)}{\epsilon}\cdot\epsilon=\delta\mathcal{J}(x;h)\cdot\lim_{\epsilon\to0}\epsilon=0.$$

(b) The functional defined in Example 2.5 (3) is continuous along each fixed direction at the origin $0=(0,0)$. Since the functional $\mathcal{J}$ has a Gâteaux variation at 0, it follows from (a) that $\mathcal{J}$ is continuous along each fixed direction at 0. Directly computation shows that

$$\lim_{\epsilon\to0}\mathcal{J}(\epsilon h)=\begin{cases}\lim_{\epsilon\to0}\frac{\epsilon^3h_1h_2^2}{\epsilon^2h_1^2+\epsilon^4h_2^4}=\lim_{\epsilon}\frac{\epsilon h_1h_2^2}{h_1^2+\epsilon^2h_2^4}, & h_1\neq0,\\ 0, & h_1=0.\end{cases}=0.$$

(c) The functional defined in Example 2.5 (2) is continuous along each fixed direction at the each vector $\phi \in \mathscr{C}^0[0,1]$. By Example 2.5 (2), we have shown that

$$\mathcal{J}(\phi + \epsilon\Delta\phi) - \mathcal{J}(\phi) = \epsilon\Delta\phi(0)$$

which tends to 0 as $\epsilon \to 0$.

2.2. **Examples.** In this subsection we will calculate the variations of the functionals considered in Subsection 1.3.

(1) **Cost functional.** Consider the cost functional $\mathcal{C}$ defined by (1.3.23) and (1.3.24) on the vector space $\mathscr{C}^0[0,T]$:

$$\begin{aligned} C(P) &= \int_0^T \left\{\beta^2[I_P(t) - \mathfrak{i}(t)]^2 + [P(t) - \mathfrak{p}(t)]^2\right\} dt, \\ I_P(t) &= e^{-\alpha t}\left(I_0 + \int_0^t e^{\alpha\tau}[P(\tau) - S(\tau)]d\tau\right). \end{aligned}$$

Since

$$\begin{aligned} I_{P+\epsilon\Delta P}(t) &= e^{-\alpha t}\left(I_0 + \int_0^t e^{\alpha\tau}[P(\tau) + \epsilon\Delta P(\tau) - S(\tau)]d\tau\right) \\ &= I_P(t) + \epsilon e^{-\alpha t}\int_0^t e^{\alpha\tau}\Delta P(\tau)d\tau, \end{aligned}$$

it follows that

$$\begin{aligned} C(P+\epsilon\Delta P) &= \int_0^T \left\{\beta^2[I_{P+\epsilon\Delta P}(t) - \mathfrak{i}(t)]^2 + [P(t) + \epsilon\Delta P(t) - \mathfrak{p}(t)]^2\right\} dt \\ &= C(P) + 2\epsilon\int_0^T \Bigg\{\beta^2[I_P(t) - \mathfrak{i}(t)]e^{-\alpha t}\int_0^t e^{\alpha\tau}\Delta P(\tau)d\tau \\ &\quad +[P(t) - \mathfrak{p}(t)]\Delta P(t)\Bigg\}dt \\ &\quad + \epsilon^2\int_0^T \left\{\beta^2\left[e^{-\alpha t}\int_0^t e^{\alpha\tau}\Delta P(\tau)d\tau\right]^2 + [\Delta P(t)]^2\right\} dt \end{aligned}$$

and therefore

$$\begin{aligned} \delta C(P;\Delta P) &= 2\int_0^T \Bigg\{\beta^2[I_P(t) - \mathfrak{i}(t)]e^{-\alpha t}\int_0^t e^{\alpha\tau}\Delta P(\tau)d\tau \\ &\quad +[P(t) - \mathfrak{p}(t)]\Delta P(t)\Bigg\}dt \end{aligned} \tag{2.2.1}$$

for any vector $\Delta P \in \mathscr{C}^0[0,T]$.

(2) **Area functional.** Recall the area functional defined in (1.3.11):

$$\begin{aligned} A(\alpha) &= \frac{1}{2}\int_0^T \Bigg\{v_0 \cdot \sin[\alpha(t)]\left[x_0 + w_0 t + v_0\int_0^t \cos[\alpha(\tau)]d\tau\right] \\ &\quad -[v_0 \cdot \cos[\alpha(t)] + w_0]\left[y_0 + v_0\int_0^t \sin[\alpha(\tau)]d\tau\right]\Bigg\} dt, \end{aligned}$$

where $\alpha \in \mathscr{C}^0[0,T]$. Then

$$
\begin{aligned}
& A(\alpha+\epsilon\Delta\alpha) \\
= \; & \frac{1}{2}\int_0^T \left\{ v_0 \cdot \sin[\alpha(t)+\epsilon\Delta\alpha(t)] \left[x_0 + w_0 t + v_0 \int_0^t \cos[\alpha(\tau)+\epsilon\Delta\alpha(\tau)]d\tau \right] \right. \\
& \left. -[w_0 + v_0 \cdot \cos[\alpha(t)+\epsilon\Delta\alpha(t)]] \left[y_0 + v_0 \int_0^t \sin[\alpha(\tau)+\epsilon\Delta\alpha(\tau)]d\tau \right] \right\} dt \\
\doteqdot \; & \int_0^T f(t,\epsilon)dt
\end{aligned}
$$

for any $\epsilon > 0$ and any vectors $\alpha, \Delta\alpha \in \mathscr{C}^0[0,T]$. Since

$$
\frac{d}{d\epsilon}\int_0^T f(t,\epsilon)dt = \int_0^T \frac{\partial f(t,\epsilon)}{\partial \epsilon}dt, \tag{2.2.2}
$$

it follows that

$$
\frac{d}{d\epsilon}A(\alpha+\epsilon\Delta\alpha) = \int_0^T \frac{\partial f(t,\epsilon)}{\partial \epsilon}dt. \tag{2.2.3}
$$

Now

$$
\begin{aligned}
\frac{\partial f(t,\epsilon)}{\partial\epsilon}\Big|_{\epsilon=0} = \; & \frac{v_0\Delta\alpha(t)}{2}\left\{ \cos[\alpha(t)] \left[x_0 + w_0 t + v_0\int_0^t \cos[\alpha(\tau)]d\tau \right] \right. \\
& \left. + \sin[\alpha(t)] \left[y_0 + v_0 \int_0^t \sin[\alpha(\tau)]d\tau \right] \right\} \\
& + \frac{v_0}{2}\sin[\alpha(t)]\left[-v_0 \int_0^t \Delta\alpha(\tau)\cdot\sin[\alpha(\tau)]d\tau \right] \\
& - \left[\frac{w_0}{2} + \frac{v_0}{2}\cdot\cos[\alpha(t)] \right] \left[v_0 \int_0^t \Delta\alpha(\tau)\cdot\cos[\alpha(\tau)]d\tau \right] \\
= \; & \frac{v_0^2}{2}\int_0^t \cos[\alpha(t)-\alpha(\tau)][\Delta\alpha(t)-\Delta\alpha(\tau)]d\tau \\
& - \frac{v_0 w_0}{2}\int_0^t \Delta\alpha(\tau)\cdot\cos[\alpha(\tau)]d\tau \\
& + \frac{v_0}{2}[(x_0+w_0t)\cos[\alpha(t)] + y_0\sin[\alpha(t)]]\Delta\alpha(t),
\end{aligned}
$$

so that

$$
\begin{aligned}
\delta A(\alpha;\Delta\alpha) = \; & \frac{v_0^2}{2}\int_0^T\int_0^t \cos[\alpha(t)-\alpha(\tau)][\Delta\alpha(t)-\Delta\alpha(\tau)]d\tau dt \\
& - \frac{v_0 w_0}{2}\int_0^T\int_0^t \Delta\alpha(\tau)\cdot\cos[\alpha(\tau)]d\tau \\
& + \frac{v_0}{2}\int_0^T [(x_0+w_0t)\cos[\alpha(t)] + y_0\sin[\alpha(t)]]dt.
\end{aligned} \tag{2.2.4}
$$

(3) **Brachistochrone functional and transit time functional.** Recall the Brachistochrone functional defined in (1.3.3)

$$
T(Y) = \int_{x_0}^{x_1} \sqrt{\frac{1+Y'(x)^2}{2g[y_0 - Y(x)]}}dx
$$

for any $Y \in \mathscr{D}(T) = \{Y \in \mathscr{C}^1[x_0, x_1] : Y(x_0) = y_0 \text{ and } Y(x_1) = y_1\}$, and the transit time functional defined in (1.3.17)

$$T(Y) = \int_0^\ell \frac{\sqrt{1 - e(x)^2 + Y'(x)^2} - e(x)Y'(x)}{v_0[1 - e(x)^2]} dx$$

for any $Y \in \mathscr{C}^1[0, \ell]$.

Those two functionals are special examples of a wider class of functionals

$$\mathscr{J}(Y) = \int_{x_0}^{x_1} F(x, Y(x), Y'(x))dx, \tag{2.2.5}$$

where the function $F = F(x, y, z)$ is a specified given function defined in some open set of $\mathbf{R}^3$. For instance, in the case of Brachistochrone functional, we have

$$F(x, y, z) = \sqrt{\frac{1 + z^2}{2g(y_0 - y)}}, \tag{2.2.6}$$

while in the case of transit time functional, we have

$$F(x, y, z) = \frac{\sqrt{1 - e(x)^2 + z^2} - e(x)z}{v_0[1 - e(x)^2]}. \tag{2.2.7}$$

We now assume that the functional $\mathscr{J}$ is defined by (2.2.5) for all vectors Y in some open subset $\mathscr{D}$ of the normed vector space $\mathscr{C}^1[x_0, x_1]$ with a suitable norm. It is easily to see that

$$\delta\mathscr{J}(Y; \Delta Y) = \int_{x_0}^{x_1} \left[F_y(x, Y(x), Y'(x))\Delta Y(x) + F_z(x, Y(x), Y'(x))\Delta Y'(x)\right] dx \tag{2.2.8}$$

for any vector Y in the domain $\mathscr{D}$ of $\mathscr{J}$ and any vector $\Delta Y \in \mathscr{C}^1[x_0, x_1]$. Hence, for the Brachistochrone functional, we have

$$\begin{aligned} \delta T(Y; \Delta Y) &= \int_{x_0}^{x_1} \left\{ \frac{1}{2[y_0 - Y(x)]} \sqrt{\frac{1 + Y'(x)^2}{2g[y_0 - Y(x)]}} \Delta Y(x) \right. \\ &\quad \left. + \frac{Y'(x)\Delta Y'(x)}{\sqrt{2g[y_0 - Y(x)][1 + Y'(x)^2]}} \right\} dx; \end{aligned} \tag{2.2.9}$$

for the transit time functional, we have

$$\delta T(Y; \Delta Y) = \int_0^\ell \frac{Y'(x) - e(x)\sqrt{1 - e(x)^2 + Y'(x)^2}}{v_0[1 - e(x)^2]\sqrt{1 - e(x)^2 + Y'(x)^2}} \Delta Y'(x) dx. \tag{2.2.10}$$

Example 2.7. The functional

$$A(Y) = 2\pi \int_{x_0}^{x_1} Y(x)\sqrt{1 + Y'(x)^2} dx$$

gives the area of the surface of revolution obtained by rotating the curve γ about the x-axis, where γ is given as $\gamma : y = Y(x)$ for $x \in [x_0, x_1]$. Then

$$\begin{aligned} \delta A(Y; \Delta Y) &= 2\pi \int_{x_0}^{x_1} \left(\Delta Y \sqrt{1 + Y'(x)^2} + Y(x) \frac{Y'(x)\Delta Y(x)}{\sqrt{1 + Y'(x)^2}} \right) dx \\ &= 2\pi \int_{x_0}^{x_1} \frac{1 + Y'(x)^2 + Y(x)Y'(x)}{\sqrt{1 + Y'(x)^2}} \Delta Y(x) dx \end{aligned}$$

from which we get that a local extremum Y^* satisfies $1+Y^{*\prime}(x)^2+Y^*(x)Y^{*\prime}(x)=0$.

2.3. **An optimization problem in production planning.** If P^* is local extremum vector in $\mathscr{C}^0[0,T]$ for C, then by Theorem 2.3 and (2.2.1), we have

$$\begin{aligned} 0 &= \int_0^T \left\{\beta^2[I_{P^*}(t)-\mathfrak{i}(t)]e^{-\alpha t}\int_0^t e^{\alpha\tau}\Delta P(\tau)d\tau+[P^*(t)-\mathfrak{p}(t)]\Delta P(t)\right\}dt \\ &= \int_0^T \left\{P^*(t)-\mathfrak{p}(t)+\beta^2 e^{\alpha t}\int_t^T e^{-\alpha\tau}[I_{P^*}(\tau)-\mathfrak{i}(\tau)]d\tau\right\}\Delta P(t)dt \end{aligned} \tag{2.3.1}$$

for any $\Delta P\in\mathscr{C}^0[0,T]$, where

$$I_{P^*}(t)=e^{-\alpha t}\left(I_0+\int_0^t e^{\alpha\tau}[P^*(\tau)-S(\tau)]d\tau\right).$$

Hence P^* satisfies

$$P^*(t)-\mathfrak{p}(t)+\beta^2 e^{\alpha t}\int_t^T e^{-\alpha\tau}[I_{P^*}(\tau)-\mathfrak{i}(\tau)]d\tau=0,\quad t\in[0,T]. \tag{2.3.2}$$

Using the following fact that

$$\frac{d}{dt}\int_t^T h(\tau)d\tau=-h(t) \tag{2.3.3}$$

for any $h\in\mathscr{C}[0,T]$, we conclude that

$$\frac{d}{dt}[P^*(t)-\mathfrak{p}(t)]=\beta^2[I_{P^*}(t)-\mathfrak{i}(t)]+\alpha[P^*(t)-\mathfrak{p}(t)] \tag{2.3.4}$$

and

$$\begin{aligned} \frac{d^2}{dt^2}[P^*(t)-\mathfrak{p}(t)] &= \alpha^2[P^*(t)-\mathfrak{p}(t)] \\ &\quad+\beta^2\left\{\frac{d}{dt}[I_{P^*}(t)-\mathfrak{i}(t)]+\alpha[I_{P^*}(t)-\mathfrak{i}(t)]\right\}. \end{aligned} \tag{2.3.5}$$

On the other hand, (1.3.19) and (1.3.20) imply that

$$\frac{d}{dt}[I_{P^*}(t)-\mathfrak{i}(t)]+\alpha[I_{P^*}(t)-\mathfrak{i}(t)]=P^*(t)-\mathfrak{p}(t), \tag{2.3.6}$$

so that (2.3.6) becomes

$$\frac{d^2}{dt^2}[P^*(t)-\mathfrak{p}(t)]=(\alpha^2+\beta^2)[P^*(t)-\mathfrak{p}(t)],\quad t\in[0,T]. \tag{2.3.7}$$

The general solution of (2.3.7) is

$$P^*(t)-\mathfrak{p}(t)=Ae^{\gamma t}+Be^{-\gamma t},\quad \gamma=\sqrt{\alpha^2+\beta^2}, \tag{2.3.8}$$

for any $A,B\in\mathbf{R}$.

However, letting $t=T$ in (2.3.3) yields

$$P^*(T)-\mathfrak{p}(T)=0, \tag{2.3.9}$$

and, letting $t=0$ in (2.3.5) yields

$$\left.\frac{d}{dt}\right|_{t=0}[P^*(t)-\mathfrak{p}(t)]-\alpha[P^*(0)-\mathfrak{p}(0)]=\beta^2(I_0-\mathfrak{i}(0)) \tag{2.3.10}$$

where $I_0 \doteqdot I_{P^*}(0)$ and $\mathfrak{i}_0 \doteqdot \mathfrak{i}(0)$ are known constants. The necessary conditions (2.3.9) and (2.3.10) let us determine the constants A and B as

$$A = \frac{\beta^2(I_0 - \mathfrak{i}_0)e^{-\gamma T}}{(\gamma+\alpha)e^{\gamma T} + (\gamma-\alpha)e^{-\gamma T}}, \quad B = \frac{-\beta^2(I_0 - \mathfrak{i}_0)e^{\gamma T}}{(\gamma+\alpha)e^{\gamma T} + (\gamma-\alpha)e^{-\gamma T}}. \tag{2.3.11}$$

Plugging (2.3.11) into (2.3.8) we arrive at

$$P^*(t) = \mathfrak{p}(t) + \beta^2(\mathfrak{i}_0 - I_0)\frac{e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{(\gamma+\alpha)e^{\gamma T} + (\gamma-\alpha)e^{-\gamma T}}. \tag{2.3.12}$$

Furthermore,

$$I_{P^*}(t) = \mathfrak{i}(t) + (I_0 - \mathfrak{i}_0)\frac{(\gamma+\alpha)e^{\gamma(T-t)} + (\gamma-\alpha)e^{-\gamma(T-t)}}{(\gamma+\alpha)e^{\gamma T} + (\gamma-\alpha)e^{-\gamma T}}, \tag{2.3.13}$$

$$C(P^*) = \beta^2(I_0 - \mathfrak{i}_0)^2\frac{e^{\gamma T} - e^{-\gamma T}}{(\gamma+\alpha)e^{\gamma T} + (\gamma-\alpha)e^{-\gamma T}}. \tag{2.3.14}$$

Finally we check that P^* gives a minimum value to the cost functional $\mathcal{C}$. The computation in Subsection 2.2, we have

$$\begin{aligned} & C(P^*+Q) - C(P^*) \\ & + 2\int_0^T \left\{\beta^2[I_{P^*}(t) - \mathfrak{i}(t)]e^{-\alpha t}\int_0^t e^{\alpha\tau}Q(\tau)d\tau + [P(t) - \mathfrak{p}(t)]Q(t)\right\} dt \\ & + \int_0^T \left\{\beta^2\left[e^{-\alpha t}\int_0^t e^{\alpha\tau}Q(\tau)d\tau\right]^2 + Q(t)^2\right\} dt \end{aligned}$$

for any $Q \in \mathscr{C}^0[0,T]$; using (2.3.1) we conclude that

$$C(P^*+Q) - C(P^*) = \int_0^T \left\{\beta^2\left[e^{-\alpha t}\int_0^t e^{\alpha\tau}Q(\tau)d\tau\right]^2 + Q(t)^2\right\} dt \geq 0 \tag{2.3.15}$$

for any $Q \in \mathscr{C}^0[0,T]$. Thus the cost functional C has a minimum in $\mathscr{C}^0[0,T]$ at the vector P^*.

2.4. **Fréchet differential.** We say that a functional $\mathcal{J}$ defined on an open subset $\mathscr{D}$ of a normed vector space $\mathscr{X}$ is **Fréchet differential**[2] or **differential** at a vector $x \in \mathscr{D}$ whenever there is a continuous linear functional $d\mathcal{J}(x)$ with values $d\mathcal{J}(x;h)$ defined for all vectors $h \in \mathscr{X}$ and for which

$$\lim_{h\to 0 \text{ in } \mathscr{X}} \frac{\mathcal{J}(x+h) - \mathcal{J}(x) - d\mathcal{J}(x;h)}{||h||} = 0 \tag{2.4.1}$$

holds. The continuous linear functional $d\mathcal{J}(x)$ is called the **(Fréchet) differential of $\mathcal{J}$ at** x. If $\mathcal{J}$ is differentiable at each vector $x \in \mathscr{D}$, we say that $\mathcal{J}$ is differentiable on $\mathscr{D}$.

Remark 2.8. (1) If we let

$$\mathcal{E}(x;h) := \frac{\mathcal{J}(x+h) - \mathcal{J}(x) - d\mathcal{J}(x;h)}{||h||} \tag{2.4.2}$$

for any nonzero vector $h \in \mathscr{X}$, then (2.4.1) is equivalent to

$$\lim_{h\to 0 \text{ in } \mathscr{X}} \mathcal{E}(x;h) = 0. \tag{2.4.3}$$

[2]Maurice Fréchet (1878–1973): a French mathematician. He made major contributions to the topology of point sets and introduced the entire concept of metric spaces.

(2) If a functional $\mathscr{J}$ is differentiable at x, then the variation of $\mathscr{J}$ at x exists and is equal to the differentiable at x,

$$\delta\mathscr{J}(x;h) = d\mathscr{J}(x;h) \tag{2.4.4}$$

for all $h \in \mathscr{X}$. In fact,

$$\frac{\mathscr{J}(x+\epsilon h) - \mathscr{J}(x)}{\epsilon} = \frac{d\mathscr{J}(x;\epsilon h) + \mathcal{E}(x;\epsilon h)||\epsilon h||}{\epsilon} = d\mathscr{J}(x;h) + \mathcal{E}(x;\epsilon h)||h||\frac{|\epsilon|}{\epsilon}.$$

Letting $\epsilon \to 0$ yields (2.4.4).

(3) A functional $\mathscr{J}$ may have a variation at a vector x even if $\mathscr{J}$ is not differentiable at x. For example, the functional $\mathscr{J}$ defined in Remark 1.7 (i) fails to be differentiable at each fixed vector $\phi \in \mathscr{C}^0[0,1]$ equipped with the L^2-norm.

3. Euler-Lagrange necessary condition for an extremum with constraints

3.1. **Extremum problems.** Let $\mathscr{X}$ be a normed vector space and $\mathscr{D}$ an open subset of $\mathscr{X}$. For two functionals $\mathscr{J}$ and $\mathcal{K}$ which are defined and have variations on $\mathscr{D}$, consider the problem:

$$\text{find extremum vectors for } \mathscr{J} \text{ in } \mathscr{D}[\mathcal{K} = k_0] := \{x \in \mathscr{D} : \mathcal{K}(x) = k_0\} \neq \emptyset, \tag{3.1.1}$$

where k_0 is some specified fixed number.

Remark 3.1. (1) Note that the set $\mathscr{D}[\mathcal{K} = k_0]$ may not open so that we can not use Theorem 2.3.

(2) The variation of $\mathscr{J}$ need not vanish at a local extremum vector $x^* \in \mathscr{D}[\mathcal{K} = k_0]$ if this set is not open in $\mathscr{X}$.

Example 3.2. Consider

$$\mathscr{J}(x) = x^2, \quad \mathcal{K}(x) = x^2 - 1, \quad x \in \mathscr{D} = \mathbf{R}.$$

Since $\mathscr{D}[\mathcal{K} = 0] = \{-1, 1\}$, it follows that $\mathscr{J}$ attains its maximum and minimum at -1 and 1. However, the variation of $\mathscr{J}$ fails to vanish at each point in $\mathscr{D}[\mathcal{K} = 0]$.

3.2. **Weak continuity.** If $\mathscr{J}$ is a functional which has a variation on an open set $\mathscr{D}$ contained in a normed vector space $\mathscr{X}$, and if for some vector $x \in \mathscr{D}$,

$$\lim_{y \to x \text{ in } \mathscr{X}} \delta\mathscr{J}(y;\Delta x) = \delta\mathscr{J}(x;\Delta x) \tag{3.2.1}$$

holds for every vector $\Delta x \in \mathscr{X}$, then we say that the variation of $\mathscr{J}$ is **weakly continuous at** x.

Remark 3.3. (1) The variation of $\mathscr{J}$ is weakly continuous at x is equivalent to saying that, for each fixed vector $\Delta x \in \mathscr{X}$, the variation $\delta\mathscr{J}(y;\Delta x)$ is continuous at $y = x$.

(2) Let $\mathscr{J}$ be defined in Remark 2.4 (1), then the variation of $\mathscr{J}$ is weakly continuous at x if and only if the function $\mathscr{J}$ is continuously differentiable at x.

(3) Let $\mathscr{J}$ be defined in Remark 2.4 (2), then the variation of $\mathscr{J}$ is weakly continuous at x if and only if the function $\mathscr{J}$ has continuous first-order partial derivatives at x.

(4) There is a functional $\mathscr{J}$ such that it has a weakly continuous variation even though $\mathscr{J}$ is not itself continuous. Consider the functional $\mathscr{J}$ defined in Remark 1.7 (i). Since $\delta\mathscr{J}(\phi;\Delta\phi) = \Delta\phi$, it follows that the variation of $\mathscr{J}$ is weakly continuous, but, we have showed that $\mathscr{J}$ is not continuous.

Example 3.4. The variation of the cost functional C is weakly continuous on $\mathscr{C}^0[0,T]$. From (2.2.1), we have

$$\begin{aligned}\delta C(Q;\Delta P)-\delta C(P;\Delta P) &= 2\int_0^T [I_Q(t)-I_P(t)]e^{-\alpha t}\int_0^t e^{\alpha\tau}\Delta P(\tau)d\tau \\ &\quad + 2\int_0^T [Q(t)-P(t)]\Delta P(t)dt\end{aligned}$$

for any vectors $Q, P, \Delta P \in \mathscr{C}^0[0,T]$. Hence

$$\begin{aligned}|\delta C(Q;\Delta P)-\delta C(P;\Delta P)| &\le 2\beta^2\int_0^T |I_Q(t)-I_P(t)|e^{-\alpha t}\int_0^t e^{\alpha\tau}|\Delta P(\tau)|d\tau dt \\ &\quad + 2\int_0^T |Q(t)-P(t)||\Delta P(t)|dt;\end{aligned}$$

however,

$$|I_Q(t)-I_P(t)| \le ||Q-P||e^{-\alpha t}\int_0^t e^{\alpha\tau}d\tau = ||Q-P||_{\mathscr{C}^0[0,T]}\frac{1-e^{-\alpha t}}{\alpha},$$

where we used the uniform norm on $\mathscr{C}^0[0,T]$:

$$||Q-P||_{\mathscr{C}^0[0,T]} = \max_{t\in[0,T]} |Q(t)-P(t)|.$$

Consequently,

$$\begin{aligned}&|\delta C(Q;\Delta P)-\delta C(P;\Delta P)| \\ \le\ & 2\int_0^T \left[\beta^2\left(\frac{e^{-\alpha t}-e^{-2\alpha t}}{\alpha}\right)\int_0^t e^{\alpha\tau}|\Delta P(\tau)|d\tau + |\Delta P(t)|\right]dt||Q-P||_{\mathscr{C}^0[0,T]}.\end{aligned}$$

Thus

$$\lim_{Q\to P \text{ in } \mathscr{C}^0[0,T]} \delta C(Q;\Delta P) = \delta C(P;\Delta P).$$

3.3. **Euler-Lagrange multiplier theorem for a single constraint.** The extremum problem (3.1.1) can be solved by the following

Theorem 3.5. *Let $\mathcal{J}$ and $\mathcal{K}$ be functionals which are defined and have variations on an open subset $\mathscr{D}$ of a normed vector space $\mathscr{X}$, and let x^* be a local extremum vector in $\mathscr{D}[\mathcal{K}=k_0]$ for $\mathcal{J}$, where k_0 is any given fixed number for which the set $\mathscr{D}[\mathcal{K}=k_0]$ is nonempty. Assume that both the variation of $\mathcal{J}$ and the variation of $\mathcal{K}$ are weakly continuous near x^*. Then at least one of the following two possibilities must hold:*

(i) *The variation of $\mathcal{K}$ at x^* vanishes identically, i.e.,*

(3.3.1) $$\delta\mathcal{K}(x^*;\Delta x)=0$$

for every vector $\Delta x \in \mathscr{X}$; or

(ii) *The variation of $\mathcal{J}$ is a constant multiple of the variation of $\mathcal{K}$ at x^*, i.e., there is a constant λ such that*

(3.3.2) $$\delta\mathcal{J}(x^*;\Delta x)=\lambda\delta\mathcal{K}(x^*;\Delta x)$$

for every vector $\Delta x \in \mathscr{X}$.

Example 3.6. Let

$$\mathcal{J}(x) = x^2, \quad \mathcal{K}(x) = x^2 + 2x + \frac{3}{4}, \quad x \in \mathscr{D} = \mathbf{R}.$$

We shall find extremum vectors in $\mathscr{D}[\mathcal{K} = 0]$ for $\mathcal{J}$. Note that

$$\delta\mathcal{J}(x; \Delta x) = 2x\Delta x, \quad \delta\mathcal{K}(x; \Delta x) = 2(x+1)\Delta x, \quad \Delta x \in \mathbf{R}. \tag{3.3.3}$$

From (3.3.3) it is easily to see that the variations of $\mathcal{J}$ and $\mathcal{K}$ are weakly continuous on $\mathbf{R}$. Letting $\delta\mathcal{K}(x^*; \Delta x) = 0$, we find that $x^* = -1 \notin \mathscr{D}[\mathcal{K} = 0]$. Hence we need only consider the second possibility of Theorem 3.5. By (3.3.2), we have

$$2x^*\Delta x = 2\lambda(x^* + 1)\Delta x$$

or

$$x^* - \lambda(x^* + 1) = 0.$$

Hence

$$x^* = \frac{\lambda}{1-\lambda}, \quad \lambda \neq 1.$$

Since $\mathcal{K}(x^*) = 0$, it follows that

$$\lambda^2 - 2\lambda - 3 = 0,$$

from which $\lambda = -1$ or $\lambda = 3$ and $x^* = -1/2$ or $x^* = -3/2$.

Example 3.7. We consider the problem of finding the dimensions of the rectangle having the smallest perimeter among all rectangles with given fixed are A. Let x_1 be the length and x_2 the width of any such rectangle. Define

$$\mathcal{J}(x) = 2(x_1 + x_2), \quad \mathcal{K}(x) = x_1 x_2 \tag{3.3.4}$$

for any vector $x = (x_1, x_2) \in \mathbf{R}^2$. The problem is then to find a minimum point for the function $\mathcal{J}$ in the open set

$$\mathscr{D} = \{x = (x_1, x_2) \in \mathbf{R}^2 : x_1, x_2 > 0\} \tag{3.3.5}$$

subject to the constraint

$$\mathcal{K}(x) = A. \tag{3.3.6}$$

From

$$\delta\mathcal{J}(x) = 2\Delta x_1 + 2\Delta x_2, \quad \delta\mathcal{K}(x; \Delta x) = x_2\Delta x_1 + x_1\Delta x_2,$$

we find that $\delta\mathcal{K}(x^*; \Delta x) = 0$ implies that $x^* = 0 \notin \mathscr{D}[\mathcal{K} = 0]$ and hence

$$2\Delta x_1 + 2\Delta x_2 = \lambda(x_2^*\Delta x_1 + x_1^*\Delta x_2)$$

which gives us $x^* = (2/\lambda, 2/\lambda)$. On the other hand, $\mathcal{K}(x^* = 0$, we conclude that $\lambda = 2/\sqrt{A}$ and $x^* = (\sqrt{A}, \sqrt{A})$.

Finally, we shall check that x^* is a minimum vector in $\mathscr{D}[\mathcal{K} = 0]$ for $\mathcal{J}$. Choosing any vector $\Delta x \in \mathbf{R}^2$ so that $x^* + \Delta x \in \mathscr{D}[\mathcal{K} = 0]$, we get

$$A = (x_1^* + \Delta x_1)(x_2^* + \Delta x_2) = A + \sqrt{A}(\Delta x_1 + \Delta x_2) + \Delta x_1 \Delta x_2,$$

and

$$\begin{aligned}\mathcal{J}(x^* + \Delta x) - \mathcal{J}(x^*) &= 2(\Delta x_1 + \Delta x_2) \\ &= 2\left(\Delta x_1 - \frac{\sqrt{A}\Delta x_1}{\sqrt{A} + \Delta x_1}\right) \\ &= \frac{2(\Delta x_1)^2}{\sqrt{A} + \Delta x_1}.\end{aligned}$$

Thus $\mathcal{J}(x^* + \Delta x) \geq \mathcal{J}(x^*)$ for any vector $x^* + \Delta x \in \mathscr{D}[\mathcal{K} = 0]$.

Example 3.8. Consider the problem of minimizing the value of the functional

$$\mathcal{J}(\phi) = \int_1^2 x\phi(x)^2 dx$$

on the vector space $\mathscr{C}^0[1,2]$ subject to the constraint $\mathcal{K}(\phi) = \log 2$, where the functional $\mathcal{K}$ is defined by

$$\mathcal{K}(\phi) = \int_1^2 \phi(x) dx$$

for any $\phi \in \mathscr{C}^0[1,2]$. Since

$$\delta\mathcal{J}(\phi; \Delta\phi) = 2\int_1^2 x\phi(x)\Delta\phi(x)dx, \quad \delta\mathcal{K}(\phi; \Delta\phi) = \int_1^2 \Delta\phi(x)dx, \quad \Delta\phi \in \mathscr{C}^0[1,2],$$

it follows that $\delta\mathcal{K}(\phi; \Delta\phi)] = 0$ has empty solution so that we need only consider the second possibility in Theorem 3.5; that is,

$$2\int_1^2 x\phi^*(x)\Delta\phi(x)dx = \lambda\int_1^2 \Delta\phi(x)dx$$

from which we get $\phi^*(x) = \lambda/2x$ for $x \in [1,2]$. However, $\mathcal{K}(\phi^*) = \log 2$, we conclude that $\lambda = 2$ and

$$\phi^*(x) = \frac{1}{x}, \quad x \in [1,2].$$

Finally, we shall check that ϕ^* is a minimum vector in $\mathscr{C}^0[1,2][\mathcal{K} = \log 2]$ for $\mathcal{J}$. If $\phi^* + \psi \in \mathscr{C}^0[1,2][\mathcal{K} = \log 2]$, then

$$\int_1^2 (\phi^*(x) + \psi(x))dx = \log 2$$

and hence

$$\int_1^2 \psi(x)dx = 0.$$

Using this and $x\phi^*(x) \equiv 1$ yields

$$\begin{aligned}\mathcal{J}(\phi^* + \psi) - \mathcal{J}(\phi^*) &= \int_1^2 x[\phi^*(x) + \psi(x)]^2 dx - \int_1^2 x\phi^*(x)^2 dx \\ &= \int_1^2 x\psi(x)^2 dx + 2\int_1^2 \psi(x)dx \\ &= \int_1^2 x\psi(x)^2 dx.\end{aligned}$$

Thus $\mathcal{J}(\phi^* + \psi) \geq \mathcal{J}(\phi^*)$ whenever $\phi^* + \psi \in \mathscr{C}^0[1,2][\mathcal{K} = \log 2]$.

The proof of Theorem 3.5. We assume that equation (3.3.1) fails to hold in general, so that we may choose a fixed vector $\Delta\bar{x} \in \mathscr{X}$ such that

$$\delta\mathcal{K}(x^*, \Delta\bar{x}) \neq 0. \tag{3.3.7}$$

Letting $\Delta x = \Delta\bar{x}$ in (3.3.2 yields

$$\delta\mathcal{J}(x^*; \Delta\bar{x}) = \lambda\delta\mathcal{K}(x^*; \Delta\bar{x})$$

and then

$$\delta\mathcal{J}(x^*; \Delta x)\delta\mathcal{K}(x^*; \Delta\bar{x}) = \delta\mathcal{J}(x^*; \Delta\bar{x})\delta\mathcal{K}(x^*; \Delta x).$$

By the above motivation, we claim that

$$\det\begin{pmatrix} \delta\mathcal{J}(x^*; \Delta x) & \delta\mathcal{J}(x^*; \Delta y) \\ \delta\mathcal{K}(x^*; \Delta x) & \delta\mathcal{K}(x^*; \Delta y) \end{pmatrix} = 0, \tag{3.3.8}$$

for any vectors $\Delta x, \Delta y \in \mathscr{X}$. Letting $\Delta y = \Delta\bar{x}$ in (3.3.8) we have

$$\delta\mathcal{J}(x^*; \Delta x) = \lambda\delta\mathcal{K}(x^*; \Delta x) \tag{3.3.9}$$

for all $\Delta x \in \mathscr{X}$, where $\lambda \doteqdot \delta\mathcal{J}(x^*; \Delta\bar{x})/\delta\mathcal{K}(x^*; \Delta\bar{x})$.

Now we give a proof of (3.3.8). Without loss of generality, we may assume that $\Delta x, \Delta y \neq 0$. Since x^* lies in the open set $\mathscr{D}$, we can find sufficiently small numbers α and β so that

$$x^* + \alpha\Delta x + \beta\Delta y \in \mathscr{D}.$$

Set

$$j \doteqdot \mathfrak{J}(\alpha, \beta) = \mathcal{J}(x^* + \alpha\Delta x + \beta\Delta y), \tag{3.3.10}$$

$$k \doteqdot \mathfrak{K}(\alpha, \beta) = \mathcal{K}(x^* + \alpha\Delta x + \beta\Delta y). \tag{3.3.11}$$

Since $\mathcal{J}$ and $\mathcal{K}$ are weakly continuous near x^*, we can find a number $\rho > 0$ such that both the variations of $\mathcal{J}$ and $\mathcal{K}$ are weakly continuous at each vector in

$$\mathscr{U} = \{x^* + \alpha\Delta x + \beta\Delta y : (\alpha, \beta) \in U\} \tag{3.3.12}$$

where $U = \{(\alpha, \beta) \in \mathbf{R}^2 : \alpha^2 + \beta^2 < \rho^2\}$. Hence the mapping (j, k) maps the disc U of the (α, β)-plane into the (j, k)-plane. The origin in the (α, β)-plane maps onto the point $(j_0, k_0) \doteqdot (\mathcal{J}(x^*), k_0)$

(i) The functions $\mathfrak{J}$ and $\mathfrak{K}$ are continuously differentiable in U and

$$\det\begin{pmatrix} \frac{\partial\mathfrak{J}}{\partial\alpha} & \frac{\partial\mathfrak{J}}{\partial\beta} \\ \frac{\partial\mathfrak{K}}{\partial\alpha} & \frac{\partial\mathfrak{K}}{\partial\beta} \end{pmatrix}\Big|_{(\alpha,\beta)=(0,0)} = \det\begin{pmatrix} \delta\mathcal{J}(x^*; \Delta x) & \delta\mathcal{J}(x^*; \Delta y) \\ \delta\mathcal{K}(x^*; \Delta x) & \delta\mathcal{K}(x^*; \Delta y) \end{pmatrix}. \tag{3.3.13}$$

For example,

$$\begin{aligned} \frac{\partial\mathfrak{J}(\alpha,\beta)}{\partial\alpha} &= \lim_{\epsilon\to 0}\frac{\mathfrak{J}(\alpha+\epsilon,\beta) - \mathfrak{J}(\alpha,\beta)}{\epsilon} \\ &= \lim_{\epsilon\to 0}\frac{\mathcal{J}(x^* + \alpha\Delta x + \beta\Delta y + \epsilon\Delta x) - \mathcal{J}(x^* + \alpha\Delta x + \beta\Delta y)}{\epsilon} \\ &= \delta\mathcal{J}(x^* + \alpha\Delta x + \beta\Delta y; \Delta x); \end{aligned}$$

since $\mathfrak{J}$ is weakly continuous in $\mathscr{U}$, it follows that $\partial\mathfrak{J}/\partial\alpha$ is continuous.

(ii) If (3.3.8) does not hold, then we can find nonzero vectors $\Delta x, \Delta y \in \mathscr{X}$ so that

$$\det\begin{pmatrix} \frac{\partial\mathfrak{J}}{\partial\alpha} & \frac{\partial\mathfrak{J}}{\partial\beta} \\ \frac{\partial\mathfrak{K}}{\partial\alpha} & \frac{\partial\mathfrak{K}}{\partial\beta} \end{pmatrix}\Big|_{(\alpha,\beta)=(0,0)} \neq 0. \tag{3.3.14}$$

According to the inverse function theorem, we have

$$\alpha = \mathfrak{A}(j,k), \quad \beta = \mathfrak{B}(j,k) \tag{3.3.15}$$

in some disk V centered at the point (j_0, k_0) in the (j,k)-plane and satisfy

$$\mathfrak{J}(\mathfrak{A}(j,k), \mathfrak{B}(j,k)) = j, \quad \mathfrak{K}(\mathfrak{A}(j,k), \mathfrak{B}(j,k)) = k \tag{3.3.16}$$

for all $(j,k) \in V$. Furthermore,

$$\mathfrak{A}(j_0, k_0) = \mathfrak{B}(j_0, k_0) = 0. \tag{3.3.17}$$

(iii) Let $(j^*, k_0) \in V$ and consider

$$\alpha^* \doteqdot \mathfrak{A}(j^*, k_0), \quad \beta^* = \mathfrak{B}(j^*, k_0). \tag{3.3.18}$$

From (3.3.15), (3.3.16), and (3.3.18), we conclude that

$$\begin{aligned} \mathcal{J}(x^* + \alpha^*\Delta x + \beta^*\Delta y) &= \mathfrak{J}(\alpha^*, \beta^*) = \mathfrak{J}(\mathfrak{A}(j^*, k_0), \mathfrak{B}(j^*, k_0)) = j^*, \\ \mathcal{K}(x^* + \alpha^*\Delta x + \beta^*\Delta y) &= \mathfrak{K}(\alpha^*, \beta^*) = \mathfrak{K}(\mathfrak{A}(j^*, k_0), \mathfrak{B}(j^*, k_0)) = k_0. \end{aligned}$$

Since $\Delta x, \Delta y$ are nonzero, if we choose j^* sufficiently small to j_0 but not equal to j_0, we can make $x^* + \alpha^*\Delta x + \beta^*\Delta y \in \mathscr{D}[\mathcal{K} = k_0]$. This contradicts the fact that x^* is a local extremum vector in $\mathscr{D}[\mathcal{K} = k_0]$ for $\mathcal{J}$ and therefore proves that (3.3.8) hold for any vectors $\Delta x, \Delta y \in \mathscr{X}$.

Example 3.9. We consider a problem in investment planning for a person who has a certain known annual income and some accumulated savings which he has invested and which earn him a known annual return. We assume that the total available annual resources for consumption consist of his current annual income, his previous savings, and his current annual return on those savings which were invested.

Let $S = S(t)$ denote the savings which are accumulated and invested at time t. We assume first that

$$\dot{S} = I + R - C, \tag{3.3.19}$$

where I is the annual income, R is the annual return generated by savings, and C is the annual consumption. Set

$$S(0) = S_0 \tag{3.3.20}$$

for a given nonnegative constant S_0. We then assume that

$$R = \alpha S \tag{3.3.21}$$

for a given positive constant α. From (3.3.20) and (3.3.21) we get

$$\dot{S} - \alpha S = I - C, \tag{3.3.22}$$

which gives us

$$S(t) = e^{\alpha t} S_0 + e^{\alpha t} \int_0^t e^{-\alpha\tau}[I(\tau) - C(\tau)]d\tau. \tag{3.3.23}$$

Here we assume that the income function $I = I(t)$ is known and the optimization problem will involve making a suitable choice for the unknown consumption function $C = C(t)$.

Finally, we assume that

$$S(T) = S_T \tag{3.3.24}$$

for a given nonnegative constant S_T. Evaluating (3.3.23) at $t = T$ yields

$$\int_0^T e^{-\alpha t} C(t)dt = S_0 - e^{-\alpha T} S_T + \int_0^T e^{-\alpha t} I(t)dt. \tag{3.3.25}$$

If we define a functional $\mathcal{K}$ on the vector space $\mathscr{C}^0[0,T]$ by

$$\mathcal{K}(C) \doteqdot \int_0^T e^{-\alpha t} C(t)dt, \tag{3.3.26}$$

then (3.3.25) can be rewritten as

$$\mathcal{K}(C) = S_0 - e^{-\alpha T} S_T + \int_0^T e^{-\alpha t} I(t)dt \tag{3.3.27}$$

for any function $C \in \mathscr{C}^0[0,T]$.

The optimization problem we shall consider now is to maximize the satisfaction derived from consumption subject to the constraint (3.3.27). In general, we may consider some suitable measure of the satisfaction of the form

$$\int_0^T F(t, C(t))dt, \tag{3.3.28}$$

where $F = F(t, C)$ would be some suitable given function of t and C. For simplicity, we consider the form

$$F(t, C) \doteqdot e^{-\beta t} \log(1 + C) \tag{3.3.29}$$

for any $t \geq 0$ and for any $C > 0$. We define a satisfaction functional $\mathcal{S}$ by

$$\mathcal{S}(C) = \int_0^T e^{-\beta t} \log[1 + C(t)]dt \tag{3.3.30}$$

for any $C \in \mathscr{D} = \{C \in \mathscr{C}^0[0,T] : C(t) > 0\}$. If we define a constant k_0 by

$$k_0 \doteqdot S_0 - e^{-\alpha T} S_T + \int_0^T e^{-\alpha t} I(t)dt, \tag{3.3.31}$$

then the problem is to find a maximum vector C^* in the set $\mathscr{D}[\mathcal{K} = k_0]$ for $\mathcal{S}$. Note that

$$\delta\mathcal{K}(C; \Delta C) = \int_0^T e^{-\alpha t} \Delta C(t)dt, \tag{3.3.32}$$

$$\delta\mathcal{S}(C; \Delta C) = \int_0^T \frac{e^{-\beta t}}{1 + C(t)} \Delta C(t)dt \tag{3.3.33}$$

for any vector $\Delta C \in \mathscr{C}^0[0,T]$ and for any $C \in \mathscr{D}$. By Theorem 3.5, we must have

$$\delta\mathcal{S}(C^*; \Delta C) = \lambda \delta\mathcal{K}(C^*; \Delta C)$$

for all vectors $\Delta C \in \mathscr{C}^0[0,T]$. Thus,

$$\frac{e^{-\beta t}}{1 + C^*(t)} = \lambda e^{-\alpha t} \tag{3.3.34}$$

for all $t \in [0,T]$. Therefore

$$C^*(t) = -1 + \frac{1}{\lambda} e^{(\alpha - \beta)t}, \quad t \in [0,T]. \tag{3.3.35}$$

By the constraint (3.3.27), we get

$$\frac{1}{\lambda} = \left(S_0 - e^{-\alpha T}S_T + \int_0^T e^{-\alpha t}I(t)dt + \frac{1-e^{-\alpha T}}{\alpha}\right)\frac{\beta}{1-e^{-\beta T}} \tag{3.3.36}$$

and

$$C^*(t) = -1 + \left(S_0 - e^{-\alpha T}S_T + \int_0^T e^{-\alpha t}I(t)dt + \frac{1-e^{-\alpha T}}{\alpha}\right)\frac{\beta}{1-e^{-\beta T}}e^{(\alpha-\beta)t}. \tag{3.3.37}$$

However, we have assumed that $C^* \in \mathscr{D}$ so that we should furthermore impose the conditions

$$\alpha > \beta, \tag{3.3.38}$$

$$S_0 + \int_0^T e^{-\alpha t}I(t)dt + \frac{1-e^{-\alpha T}}{\alpha} \geq e^{-\alpha T}S_T + \frac{1-e^{-\beta T}}{\beta}. \tag{3.3.39}$$

Now we check that the satisfaction functional $\mathcal{S}$ has a maximum in $\mathscr{D}[\mathcal{K} = k_0]$ at the vector C^* provided that (3.3.38) and (3.3.39) hold. Since $\mathscr{D}$ is open, we can choose any function $\Delta C \in \mathscr{C}^0[0,T]$ so that $C^* + \Delta C \in \mathscr{D}[\mathcal{K} = k_0]$. Since

$$\begin{aligned} \mathcal{S}(C^*+\Delta C) - \mathcal{S}(C^*) &= \int_0^T e^{-\beta t}\log\left[1 + \frac{\Delta C(t)}{1+C^*(t)}\right]dt \\ &= \int_0^T e^{-\beta t}\log\left[1 + \lambda e^{(\beta-\alpha)t}\Delta C(t)\right]dt, \\ \int_0^T e^{-\alpha t}\Delta C(t)dt &= k_0 - \int_0^T e^{-\alpha t}C^*(t)dt = 0, \end{aligned}$$

it follows that

$$\mathcal{S}(C^*+\Delta C) - \mathcal{S}(C^*) \leq \int_0^T e^{-\beta t}\lambda e^{(\beta-\alpha)t}\Delta C(t)dt = 0$$

for any function $\Delta C \in \mathscr{C}^0[0,T]$ such that $C^* + \Delta C \in \mathscr{D}[\mathcal{K} = k_0]$.

From (3.3.30) and (3.3.37) we conclude that

$$\begin{aligned} \mathcal{S}(C^*) &= \frac{T(\beta-\alpha)e^{-\beta T}}{\beta} \\ &+ \left(\frac{1-e^{-\beta T}}{\beta}\right)\left[\frac{\alpha-\beta}{\beta} + \log\frac{\beta}{1-e^{-\beta T}} + \log\left(k_0 + \frac{1-e^{-\alpha T}}{\alpha}\right)\right]. \end{aligned} \tag{3.3.40}$$

Letting $k \doteqdot k_0$ we find

$$\frac{\partial \mathcal{S}(C^*)}{\partial k} = \frac{1}{k + [(1-e^{-\alpha T})/\alpha]}\frac{1-e^{-\beta T}}{\beta} = \lambda. \tag{3.3.41}$$

Hence in this case the Euler-Lagrange multiplier λ gives the rate of change of the extreme value $\mathcal{S}(C^*)$ with respect to the constraint value k.

3.4. **The Euler-Lagrange multiplier theorem for many constraints.** Let $\mathcal{K}_1, \cdots, \mathcal{K}_m$ be any collection of functionals which are defined and have variations on an open subset $\mathscr{D}$ of a normed vector space $\mathscr{X}$, and let $\mathscr{D}[\mathcal{K}_i = k_i \text{ for } i = 1, \cdots, m]$ denote the subset of $\mathscr{D}$ which consists of all vectors $x \in \mathscr{D}$ which simultaneously satisfy all the following constraints:

$$\mathcal{K}_1(x) = k_1, \cdots, \mathcal{K}_m(x) = k_m. \tag{3.4.1}$$

Here $k_1, \cdots, k_m$ may be any given numbers, and we assume that there is at least one vector in $\mathscr{D}$ which satisfies all the constraints of (3.4.1) so that the set $\mathscr{D}[\mathcal{K}_i = k_i \text{ for } i = 1, \cdots, m]$ is not empty.

Theorem 3.10. *Let $\mathcal{J}, \mathcal{K}_1, \cdots, \mathcal{K}_m$ be functionals which are defined and have variations on an open subset $\mathscr{D}$ of a normed vector space $\mathscr{X}$, and let x^* be a local extremum vector in $\mathscr{D}[\mathcal{K}_i = k_i$ for $i = 1, \cdots, m]$ for $\mathcal{J}$, where $k_1, \cdots, k_m$ are any given fixed numbers for which the set $\mathscr{D}[\mathcal{K}_i = k_i$ for $i = 1, \cdots, m]$ is nonempty. Assume that the variation of $\mathcal{J}$ and the variation of each $\mathcal{K}_i$ (for $i = 1, \cdots, m$) are weakly continuous near x^*. Then at least one of the following two possibilities must hold:*

(i) *The following determinant vanishes identically,*

$$\det \begin{pmatrix} \delta\mathcal{K}_1(x^*;\Delta x_1) & \delta\mathcal{K}_1(x^*;\Delta x_2) & \cdots & \delta\mathcal{K}_1(x^*;\Delta x_m) \\ \delta\mathcal{K}_2(x^*;\Delta x_1) & \delta\mathcal{K}_2(x^*;\Delta x_2) & \cdots & \delta\mathcal{K}_2(x^*;\Delta x_m) \\ \vdots & \vdots & & \vdots \\ \delta\mathcal{K}_m(x^*;\Delta x_1) & \delta\mathcal{K}_m(x^*;\Delta x_2) & \cdots & \delta\mathcal{K}_m(x^*;\Delta x_m) \end{pmatrix} = 0 \tag{3.4.2}$$

for all vectors $\Delta x_1, \cdots, \Delta x_m \in \mathscr{X}$; or

(ii) *The variation of $\mathcal{J}$ at x^* is a linear combination of the variations of $\mathcal{K}_1, \cdots, \mathcal{K}_m$ at x^*, i.e., there are constants $\lambda_1, \cdots, \lambda_m$ such that*

$$\delta\mathcal{J}(x^*;\Delta x) = \sum_{i=1}^{m} \lambda_i \delta\mathcal{K}_i(x^*;\Delta x) \tag{3.4.3}$$

holds for every vector $\Delta x \in \mathscr{X}$.

Theorem 3.11. *Let $\mathcal{J}, \mathcal{K}_1, \cdots, \mathcal{K}_m$ be functionals which are defined on an open set $\mathscr{D}$ contained in the normed vector space $\mathscr{X} = \mathbf{R}^n$. We assume that*

(i) *all those functionals are differentiable and weakly continuous on $\mathscr{D}$,*

(ii) *the set $\mathscr{D}[\mathcal{K}_i = k_i$ for $i = 1, \cdots, m]$ is nonempty for all choices of $k_1, \cdots, k_m$ considered,*

(iii) *$\mathcal{J}$ has a local maximum or minimum in $\mathscr{D}[\mathcal{K}_i = k_i$ for $i = 1, \cdots, m]$ at some vector x^* and the determinant (3.4.2) is not identically zero. Then Theorem 3.10 and (2.4.4) imply that there are numbers $\lambda_1, \cdots, \lambda_m$ such that*

$$d\mathcal{J}(x^*;\Delta x) = \sum_{i=1}^{m} \lambda_i d\mathcal{K}_i(x^*;\Delta x) \tag{3.4.4}$$

for every vector $\Delta x \in \mathbf{R}^n$. Any such extremum vector x^* is written as

$$x^* = x^*(k_1, \cdots, k_m) \tag{3.4.5}$$

and assumed that $x_i^* = x_i^*(k) = x_i^*(k_1, \cdots, k_m)$, $i = 1, \cdots, n$, are continuously differentiable,

(iv) *the functional $\mathcal{J}(x^*(k_1, \cdots, k_m))$ has continuous first-order partial derivatives with respect to $k_1, \cdots, k_m$.*

Then

$$\frac{\partial}{\partial k_i} \mathcal{J}(x^*(k_1, \cdots, k_m)) = \lambda_i \tag{3.4.6}$$

for $i = 1, \cdots, m$.

Proof. By (iii), we have

$$x_j^*(k+\Delta k) = x_j^*(k) + \sum_{i=1}^{m} \frac{\partial x_j^*(k)}{\partial k_i}\Delta k_i + R_j(k;\Delta k)|\Delta k| \tag{3.4.7}$$

for $j = 1, \cdots, n$, where

$$\lim_{\Delta k\to 0 \text{ in } \mathbf{R}^n} R_j(k;\Delta k) = 0. \tag{3.4.8}$$

Thus

$$x^*(k+\Delta k) = x^*(k) + \sum_{i=1}^{m} \frac{\partial x^*(k)}{\partial k_i}\Delta k_i + R(k;\Delta k)|\Delta k|. \tag{3.4.9}$$

If we denote by $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ the i-th unit vector in $\mathbf{R}^m$, then

$$x^*(k+\Delta k_i e_i) = x^*(k) + \frac{\partial x^*(k)}{\partial k_i}\Delta k_i + R(k;\Delta k_i e_i)|\Delta k_i| \tag{3.4.10}$$

for all small nonzero number Δk_i. Consequently,

$$\begin{aligned} & \frac{\mathcal{J}(x^*(k+\Delta k_i e_i)) - \mathcal{J}(x^*(k))}{\Delta k_i} \\ = \quad & \frac{d\mathcal{J}(x^*(k); \frac{\partial x^*(k)}{\partial k_i}\Delta k_i + R(k;\Delta k_i e_i)|\Delta k_i|) + \mathcal{E}(x^*(k);\Delta x)|\Delta x|}{\Delta k_i} \\ = \quad & d\mathcal{J}\left(x^*(k); \frac{\partial x^*(k)}{\partial k_i}\right) + \frac{|\Delta k_i|}{\Delta k_i} d\mathcal{J}(x^*(k); R(k;\Delta k_i e_i)) + \frac{|\Delta x|}{\Delta k_i}\mathcal{E}(x^*(k);\Delta x) \end{aligned}$$

where

$$\Delta x \doteqdot \frac{\partial x^*(k)}{\partial x_i}\Delta k_i + R(k;\Delta k_i e_i)|\Delta k_i|.$$

Letting now $\Delta k_i \to 0$ yields

$$\frac{\partial \mathcal{J}(x^*(k))}{\partial k_i} = \lim_{\Delta k_i\to 0} \frac{\mathcal{J}(x^*(k+\Delta k_i e_i)) - \mathcal{J}(x^*(k))}{\Delta k_i} = d\mathcal{J}\left(x^*(k); \frac{\partial x^*(k)}{\partial k_i}\right).$$

Hence

$$\frac{\partial \mathcal{J}(x^*(k))}{\partial k_i} = \sum_{j=1}^{m} \lambda_j d\mathcal{K}_j\left(x^*(k); \frac{\partial x^*(k)}{\partial k_i}\right). \tag{3.4.11}$$

On the other hand, since

$$\mathcal{K}_j(x^*(k+\Delta k_i e_i)) = \begin{cases} k_i + \Delta k_i, & j = i, \\ k_j, & j \neq i, \end{cases}$$

it follows that

$$\begin{aligned} d\mathcal{K}_j\left(x^*(k); \frac{\partial x^*(k)}{\partial k_i}\right) &= \lim_{\Delta k_i\to 0} \frac{\mathcal{K}_j(x^*(k+\Delta k_i e_i)) - \mathcal{K}_j(x^*(k))}{\Delta k_i} \\ &= \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases} \end{aligned}$$

Plugging it into (3.4.11), we prove (3.4.6). □

3.5. **Chaplygin's problem.** This problem is to find a steering control $\alpha^* \in \mathscr{C}^0[0,T]$ with the uniform norm that will alow an airplane to encircle a maximum area in time T while flying at constant natural speed v_0 (relative to the surrounding air) and while a constant wind is blowing.

For any $\alpha \in \mathscr{C}^0[0,T]$, we define

$$\mathcal{K}_1(\alpha) = \int_0^T \cos[\alpha(t)]dt, \tag{3.5.1}$$

$$\mathcal{K}_2(\alpha) = \int_0^T \sin[\alpha(t)]dt, \tag{3.5.2}$$

$$\mathcal{K}_3(\alpha) = \alpha(0). \tag{3.5.3}$$

Then the constraints (1.3.9) and (1.3.10) become

$$\mathcal{K}_1(\alpha) = -\frac{w_0}{v_0}T, \quad \mathcal{K}_2(\alpha) = 0, \quad \mathcal{K}(\alpha) = \alpha_0. \tag{3.5.4}$$

The optimization problem is then to fins an extremum vector $\alpha^* \in \mathscr{D}[\mathcal{K}_1 = -\frac{w_0}{v_0}T,\ \mathcal{K}_2 = 0,\ \mathcal{K}_3 = \alpha_0]$ for the functional $\mathcal{A}$ defined blew, where $\mathscr{D} = \mathscr{C}^0[0,T]$. Recall that

$$\begin{aligned} A(\alpha) &= \frac{1}{2}\int_0^T \Big\{ v_0 \cdot \sin[\alpha(t)] \left[x_0 + w_0 t + v_0 \int_0^t \cos[\alpha(\tau)]d\tau \right] \\ &\quad -[v_0 \cdot \cos[\alpha(t)] + w_0] \left[y_0 + v_0 \int_0^t \sin[\alpha(\tau)]d\tau \right] \Big\} dt. \end{aligned}$$

It is easily to see that

$$\delta\mathcal{K}_1(\alpha;\Delta\alpha) = -\int_0^T \sin[\alpha(t)]\Delta\alpha(t)dt, \tag{3.5.5}$$

$$\delta\mathcal{K}_2(\alpha;\Delta\alpha) = \int_0^T \cos[\alpha(t)]\Delta\alpha(t)dt, \tag{3.5.6}$$

$$\delta\mathcal{K}_3(\alpha;\Delta\alpha) = \Delta\alpha(0) \tag{3.5.7}$$

for any vector $\Delta\alpha \in \mathscr{C}^0[0,T]$.

(a) The determinant

$$\det \doteqdot \det \begin{pmatrix} \delta\mathcal{K}_1(\alpha;\Delta\alpha_1) & \delta\mathcal{K}_1(\alpha;\Delta\alpha_2) & \delta\mathcal{K}_1(\alpha;\Delta\alpha_3) \\ \delta\mathcal{K}_2(\alpha;\Delta\alpha_1) & \delta\mathcal{K}_2(\alpha;\Delta\alpha_2) & \delta\mathcal{K}_2(\alpha;\Delta\alpha_3) \\ \delta\mathcal{K}_3(\alpha;\Delta\alpha_1) & \delta\mathcal{K}_3(\alpha;\Delta\alpha_2) & \delta\mathcal{K}_3(\alpha;\Delta\alpha_3) \end{pmatrix}$$

does not vanish identically for any vectors $\Delta\alpha_1, \Delta\alpha_2, \Delta\alpha_3 \in \mathscr{C}^0[0,T]$. Since any continuous function on $[0,T]$ can be approximated by smooth functions, it suffices to check when α is continuously differentiable with $d\alpha(t)/dt \neq 0$ for $t \in [0,T]$. Without loss generality, we may furthermore assume that $\alpha(T) - \alpha(0) = 2\pi n$ for some nonzero integer n. Taking

$$\Delta\alpha_1(t) = \sin[\alpha(t)]\frac{d\alpha(t)}{dt}, \quad \Delta\alpha_2(t) = \cos[\alpha(t)]\frac{d\alpha(t)}{dt},$$

we have

$$\begin{aligned}\delta\mathcal{K}_1(\alpha;\Delta\alpha_1) &= -\int_0^T \sin^2[\alpha(t)]d\alpha(t) = -\int_{\alpha(0)}^{\alpha(T)} \sin^2 x dx \\ &= -\left(\frac{x}{2} - \frac{\sin(2x)}{4}\right)\Big|_{\alpha(0)}^{\alpha(T)} = -\pi n, \\ \delta\mathcal{K}_1(\alpha;\Delta\alpha_2) &= -\int_0^T \sin[\alpha(t)]\cos[\alpha(t)]d\alpha(t) = -\int_{\alpha(0)}^{\alpha(T)} \sin x\cdot\cos x dx \\ &= -\frac{1}{2}\sin^2 x\Big|_{\alpha(0)}^{\alpha(T)} = 0;\end{aligned}$$

similarly, we have

$$\delta\mathcal{K}_2(\alpha;\Delta\alpha_1) = 0, \quad \delta\mathcal{K}_2(\alpha;\Delta\alpha_2) = \pi n.$$

Hence

$$\det = -\pi^2 n^2 \Delta\alpha_3(0) + \pi n\dot\alpha(0)\int_0^T \cos[\alpha(t)-\alpha(0)]\Delta\alpha_3(t)dt.$$

Choosing $\Delta\alpha_3 = d\alpha/dt$ yields

$$\begin{aligned}\det &= -\pi^2 n^2\dot\alpha(0) + \pi n\dot\alpha(0)\int_0^{2\pi n}\cos x dx \\ &= -\pi^2 n^2 \dot\alpha(0)\end{aligned}$$

which is nonzero.

(b) By Theorem 3.10, there are constants $\lambda_1, \lambda_2, \lambda_3$ such that

(3.5.8) $$\delta A(\alpha^*;\Delta\alpha) = \lambda_1\delta\mathcal{K}_1(\alpha^*;\Delta\alpha) + \lambda_2\delta\mathcal{K}_2(\alpha^*;\Delta\alpha) + \lambda_3\delta\mathcal{K}_3(\alpha^*;\Delta\alpha)$$

for any vector $\Delta\alpha \in \mathscr{C}^0[0,T]$. From (2.2.4), we get

$$\begin{aligned}&\frac{v_0^2}{2}\int_0^T \Delta\alpha(t)\int_0^t \cos[\alpha^*(t)-\alpha^*(\tau)]d\tau dt \\ &-\frac{v_0^2}{2}\int_0^T\int_0^t \left\{\cos[\alpha^*(t)-\alpha^*(\tau)] + \frac{w_0}{v_0}\cos[\alpha^*(\tau)]\right\}\Delta\alpha(\tau)d\tau dt \\ = &\int_0^T \left\{-\sin[\alpha^*(t)]\left(\lambda_1 + \frac{v_0 y_0}{2}\right) + \cos[\alpha^*(t)]\left(\lambda_2 - \frac{v_2}{2}(w_0 t + x_0)\right)\right\}\Delta\alpha(t)dt \\ &+ \lambda_3\Delta\alpha(0);\end{aligned}$$

a simple calculation shows that

$$\begin{aligned}\lambda_3\Delta\alpha(0) &= \int_0^T\left\{v_0^2\int_0^t \cos[\alpha^*(t)-\alpha^*(\tau)]d\tau + \left(\lambda_1+\frac{v_0y_0}{2}\right)\sin[\alpha^*(t)]\right. \\ &\quad \left. + \left(-\lambda_2 + \frac{v_0x_0}{2} + v_0w_0t\right)\cos[\alpha^*(t)]\right\}\Delta\alpha(t)dt \qquad (3.5.9)\end{aligned}$$

for any function $\Delta\alpha \in \mathscr{C}^0[0,T]$. In particular,

$$\begin{aligned}0 &= \int_0^T\left\{v_0^2\int_0^t \cos[\alpha^*(t)-\alpha^*(\tau)]d\tau + \left(\lambda_1+\frac{v_0y_0}{2}\right)\sin[\alpha^*(t)]\right. \\ &\quad \left. + \left(-\lambda_2 + \frac{v_0x_0}{2} + v_0w_0t\right)\cos[\alpha^*(t)]\right\}\Delta\alpha(t)dt \qquad (3.5.10)\end{aligned}$$

for any function $\Delta\alpha \in \mathscr{C}^0[0,T]$ with $\Delta\alpha(0) = \Delta\alpha(T) = 0$. Using part (c) below, we conclude that

$$\begin{aligned} 0 &= v_0^2 \int_0^t \cos[\alpha^*(t) - \alpha^*(\tau)]d\tau + \Big(\lambda_1 + \frac{v_0 y_0}{2}\Big)\sin[\alpha^*(t)] \\ &\quad + \Big(-\lambda_2 + \frac{v_0 x_0}{2} + v_0 w_0 t\Big)\cos[\alpha^*(t)] \end{aligned} \tag{3.5.11}$$

and hence

$$\lambda_3 = 0. \tag{3.5.12}$$

If we write $\alpha^* = \alpha^*(v_0, w_0, T, \alpha_0)$, then Theorem 3.11 implies that

$$\frac{\partial}{\partial \alpha_0}\mathcal{A}(\alpha^*) = 0. \tag{3.5.13}$$

(c) **Du Bois-Reymond's lemma.** Let $f(x)$ be any given continuous real-valued function on $[a,b]$ and suppose that for some nonnegative integer $n \in \mathbf{N} \cup \{0\}$

$$\int_a^b f(x)h(x)dx = 0$$

holds for all functions $h \in \mathscr{C}^n[a,b]$ which vanish at the end points along with their derivatives of order up to and including order n,

$$h^{(k)}(a) = h^{(k)}(b) = 0, \quad k = 0, 1, \cdots, n.$$

Then $f \equiv 0$ on $[a,b]$.

Proof. Without loss of generality, we may show that f vanishes in the open interval (a,b). Suppose that there is some interior point $x^* \in (a,b)$ for which $f(x^*) \neq 0$. We may assume that $f(x^*) > 0$, otherwise, we consider the function $-f$.

Since f is continuous, it follows that $f(x) > 0$ for all $x \in (\alpha, \beta)$ anf for some sufficiently small open interval (α,β) of x^*. Define

$$h^*(x) = \begin{cases} (x-\alpha)^{n+1}(\beta - x)^{n+1}, & x \in [\alpha, \beta], \\ 0, & x \in [a, \alpha) \cup (\beta, b]. \end{cases}$$

Then $h^* \in \mathscr{C}^n[a,b]$ and vanishes at the end points $x = a$ and $x = b$ along with all its derivatives. The assumption implies that

$$\int_\alpha^\beta f(x)(x-\alpha)^{n+1}(\beta - x)^{n+1}dx = 0.$$

However, the function $f(x)(x-\alpha)^{n+1}(\beta-x)^{n+1} > 0$ for all $x \in (\alpha,\beta)$, we obtain a contradiction! Hence, the original assumption $f(x^*) \neq 0$ is impossible. □

(d) Letting $t = 0$ and $t = T$ in (3.5.11) respectively and using the constraints (3.5.4), we have

$$\Big(\lambda_1 + \frac{v_0 y_0}{2}\Big)\sin\alpha_0 + \Big(-\lambda_2 + \frac{v_0 x_0}{2}\Big)\cos\alpha_0 = 0, \tag{3.5.14}$$

$$\Big(\lambda_1 + \frac{v_0 y_0}{2}\Big)\sin[\alpha^*(T)] + \Big(-\lambda_2 + \frac{v_0 x_0}{2}\Big)\cos[\alpha^*(T)] = 0. \tag{3.5.15}$$

The determine of (3.5.14) and (3.5.15) is

$$\det = \cos\alpha_0 \cdot \sin[\alpha^*(T)] - \sin\alpha_0 \cdot \cos[\alpha^*(T)].$$

If det $\neq 0$, then

$$\lambda_1 = -\frac{v_0 y_0}{2}, \quad \lambda_2 = \frac{v_0 x_0}{2};$$

if det $= 0$, that is, $\tan\alpha_0 = \tan[\alpha^*(T)]$, or $\alpha^*(T) = \alpha_0 + 2\pi n$ for some $n \in \mathbf{N}$, we have

$$\lambda_1 = -\frac{v_0 y_0}{2} - \mu v_0^2 \cos\alpha_0, \quad \lambda_2 = \frac{v_0 x_0}{2} - \mu v_0^2 \sin\alpha_0 \tag{3.5.16}$$

for any constant μ. Thus (3.5.16) is a general solution for (3.5.14) and (3.5.15). Plugging (3.5.16) into (3.5.11) and putting

$$e_0 \doteqdot \frac{w_0}{v_0} \in [0,1), \tag{3.5.17}$$

we arrive at

$$\int_0^t \cos[\alpha^*(t) - \alpha^*(\tau)]d\tau = \mu \sin[\alpha^*(t) - \alpha_0] - e_0 t \cos[\alpha^*(t)]. \tag{3.5.18}$$

On the other hand, from (1.3.7) and the identity

$$\cos[\alpha^*(t) - \alpha^*(\tau)] = \cos[\alpha^*(t)]\cos[\alpha^*(\tau)] + \sin[\alpha^*(t)]\sin[\alpha^*(\tau)],$$

we have

$$\int_0^t \cos[\alpha^*(t) - \alpha^*(\tau)]d\tau = \left(\frac{X(t) - x_0}{v_0} - e_0 t\right)\cos[\alpha^*(t)] + \left(\frac{Y(t) - y_0}{v_0}\right)\sin[\alpha^*(t)] \tag{3.5.19}$$

(where $(X(t), Y(t))$ corresponds to α^*) and then

$$(X(t) - x_0 + \mu v_0 \sin\alpha_0)\cos[\alpha^*(t)] + (Y(t) - y_0 - \mu v_0 \cos\alpha_0)\sin[\alpha^*(t)] = 0. \tag{3.5.20}$$

If we introduce

$$R(t) \doteqdot \left([X(t) - x_0 + \mu v_0 \sin\alpha_0]^2 + [Y(t) - y_0 - \mu v_0 \cos\alpha_0]^2\right)^{1/2}, \tag{3.5.21}$$

then

$$X(t) - x_0 + \mu v_0 \sin\alpha_0 = R(t)\sin[\alpha^*(t)], \tag{3.5.22}$$
$$Y(t) - y_0 - \mu v_0 \cos\alpha_0 = -R(t)\cos[\alpha^*(t)]. \tag{3.5.23}$$

Since

$$\begin{aligned} R'(t) &= \sin[\alpha^*(t)]X'(t) - \cos[\alpha^*(t)]Y'(t) \\ &= \frac{Y'(t)}{v_0}X'(t) - \frac{X'(t) - w_0}{v_0}Y'(t) \\ &= e_0 Y'(t) \end{aligned}$$

and $R(0) = \mu v_0$, it follows that

$$R(t) = e_0 Y(t) + (\mu v_0 - e_0 y_0). \tag{3.5.24}$$

Consequently,

$$(X(t) - x_0 + \mu v_0 \sin\alpha_0)^2 + (Y(t) - y_0 - \mu v_0 \cos\alpha_0)^2 = [e_0(y - y_0) + \mu v_0]^2. \tag{3.5.25}$$

The above equation can be rewritten as

$$\frac{(x - x_1)^2}{a^2} + \frac{(y - y_1)^2}{b^2} = 1, \tag{3.5.26}$$

with

$$\begin{aligned} (x,y) &= (X(t),Y(t), \\ (x_1,y_1) &= \left(x_0-\mu v_0\sin\alpha_0, y_0+\mu v_0\frac{e_0+\cos\alpha_0}{1-e_0^2}\right), \\ a &= \mu n_0\frac{1+e_0\cos\alpha_0}{\sqrt{1-e_0^2}}, \\ b &= \mu v_0\frac{1+e_0\cos\alpha_0}{1-e_0^2}. \end{aligned}$$

The area now is given by

$$\text{area} = \pi ab = \pi\mu^2 v_0^2\frac{(1+e\cos\alpha_0)^2}{(1-e_0^2)^{3/2}}. \tag{3.5.27}$$

(e) The equation (3.5.26) together with (1.3.4) implies that

$$X'(T) = X'(0), \quad Y'(T) = Y(0). \tag{3.5.28}$$

Thus

$$\alpha^*(T) = \alpha_0 + 2\pi n \tag{3.5.29}$$

for some integer n. We claim that

$$\ddot{\alpha}^*(1+e_0\cos\alpha^*) + 2(\dot{\alpha}^*)^2 e_0\sin\alpha^* = 0. \tag{3.5.30}$$

Differentiating (3.5.18) twice we have

$$\begin{aligned} & -\int_0^t \sin[\alpha^*(t)-\alpha^*(\tau)]d\tau\dot{\alpha}^*(t) + 1 \\ = \; & \mu\cos[\alpha^*(t)-\alpha_0]\dot{\alpha}^*(t) - e_0\cos[\alpha^*(t)] + e_0 t\sin[\alpha^*(t)]\dot{\alpha}^*(t), \\ & -\int_0^t \sin[\alpha^*(t)-\alpha^*(\tau)]d\tau\ddot{\alpha}^*(t) - \int_0^t \cos[\alpha^*(t)-\alpha^*(\tau)]d\tau\,(\dot{\alpha}^*(t))^2 \\ = \; & -\mu\sin[\alpha^*(t)-\alpha_0]\,(\dot{\alpha}^*(t))^2 + \mu\cos[\alpha^*(t)-\alpha_0]\ddot{\alpha}(t) + e_0\sin[\alpha^*(t)]\dot{\alpha}^*(t) \\ & + e_0 t\sin[\alpha^*(t)]\ddot{\alpha}^*(t) + e_0\sin[\alpha^*(t)]\dot{\alpha}^*(t) + e_0 t\cos[\alpha^*(t)]\,(\dot{\alpha}^*(t))^2. \end{aligned}$$

Resolving the above two integrals and using (3.5.18) again we arrive at (3.5.30). We may rewrite (3.5.30) as

$$\frac{\ddot{\alpha}^*}{\dot{\alpha}^*} = \frac{-2\dot{\alpha}^* e_0\sin\alpha^*}{1+e_0\cos\alpha^*} \tag{3.5.31}$$

or

$$\frac{d}{dt}\log\dot{\alpha}^* = 2\frac{d}{dt}\log(1+e_0\cos\alpha^*). \tag{3.5.32}$$

Letting

$$\dot{\alpha}_0 \doteqdot \dot{\alpha}(0) \tag{3.5.33}$$

yields

$$\dot{\alpha}^* = \dot{\alpha}_0\left(\frac{1+e_0\cos\alpha^*}{1+e_0\cos\alpha_0}\right)^2. \tag{3.5.34}$$

Therefore

$$\int_{\alpha_0}^{\alpha^*}\frac{d\alpha}{(1+e_0\cos\alpha)^2} = \frac{\dot{\alpha}_0 t}{(1+e_0\cos\alpha_0)^2}. \tag{3.5.35}$$

On the other hand, differentiating (3.5.18) and setting $t = 0$ yields

$$\dot{\alpha}_0 = \frac{1 + e_0 \cos \alpha_0}{\mu} \tag{3.5.36}$$

which gives us

$$\int_{\alpha_0}^{\alpha^*} \frac{d\alpha}{(1 + e_0 \cos \alpha)^2} = \frac{t}{\mu(1 + e_0 \cos \alpha_0)}. \tag{3.5.37}$$

(f) We claim that

$$I \doteqdot \int \frac{d\alpha}{(1 + e_0 \cos \alpha)^2} = \frac{\left(\frac{-e_0 \sin \alpha}{1+e_0 \cos \alpha} + \frac{2}{\sqrt{1-e_0^2}} \arctan \left(\sqrt{\frac{1-e_0}{1+e_0}} \tan \frac{\alpha}{2} \right) \right)}{1 - e_0^2}. \tag{3.5.38}$$

We first review the basic identities

$$\sin \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}, \quad \cos \alpha = \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}, \quad \tan \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}}.$$

We first show that

$$J \doteqdot \int \frac{d\alpha}{1 + e_0 \cos \alpha} = \frac{2}{\sqrt{1 - e_0^2}} \arctan \left(\sqrt{\frac{1 - e_0}{1 + e_0}} \tan \frac{\alpha}{2} \right). \tag{3.5.39}$$

To prove (3.5.39) we use the above basic identities to conclude that

$$\begin{aligned} J &= \int \frac{d\alpha}{1 + e_0 \frac{1-\tan^2 \frac{\alpha}{2}}{1+\tan^2 \frac{\alpha}{2}}} = \int \frac{(1 + \tan^2 \frac{\alpha}{2}) d\alpha}{(1 + e_0) + (1 - e_0) \tan^2 \frac{\alpha}{2}} \\ &= \frac{2}{1 - e_0} \int \frac{d \tan \frac{\alpha}{2}}{\left(\sqrt{\frac{1+e_0}{1-e_0}} \right)^2 + \tan^2 \frac{\alpha}{2}} \\ &= \frac{2}{1 - e_0} \sqrt{\frac{1 - e_0}{1 + e_0}} \arctan \left(\sqrt{\frac{1 - e_0}{1 + e_0}} \tan \frac{\alpha}{2} \right) \\ &= \frac{2}{\sqrt{1 - e_0^2}} \arctan \left(\sqrt{\frac{1 - e_0}{1 + e_0}} \tan \frac{\alpha}{2} \right). \end{aligned}$$

Now a basic idea to prove (3.5.38) is to lower the power for $1 + e_0 \cos \alpha$. From

$$\begin{aligned} \left(\frac{\sin \alpha}{1 + e_0 \cos \alpha} \right)' &= \frac{\cos \alpha (1 + e_0 \cos \alpha) + e_0 \sin^2 \alpha}{(1 + e_0 \cos \alpha)^2} \\ &= \frac{e_0 + \cos \alpha}{(1 + e_0 \cos \alpha)^2} = \frac{\frac{1}{e_0}}{1 + e_0 \cos \alpha} + \frac{\frac{e_0^2 - 1}{e_0}}{(1 + e_0 \cos \alpha)^2} \end{aligned}$$

so that

$$\frac{1}{(1 + e_0 \cos \alpha)^2} = \frac{e_0}{e_0^2 - 1} \left(\frac{\sin \alpha}{1 + e_0 \cos \alpha} \right)' - \frac{1}{e_0^2 - 1} \frac{1}{1 + e_0 \cos \alpha}$$

and

$$J = \frac{e_0}{e_0^2 - 1} \frac{\sin \alpha}{1 + e_0 \cos \alpha} - \frac{1}{e_0^2 - 1} J.$$

(g) From (3.5.37), (3.5.38), and (3.5.29), we have

$$\mu_n = \frac{(1-e_0^2)^{1/2}T}{2\pi n(1+e_0\cos\alpha_0)} \tag{3.5.40}$$

for any nonzero integer n. For convenience, we take n to be a positive integer $n \in \mathbf{N}$ and write

$$\mu_{-n} = -\mu_n, \quad n \in \mathbf{N}.$$

Then

$$a_n = \frac{(1-e_0^2)v_0T}{2\pi n}, \tag{3.5.41}$$

$$b_n = \frac{\sqrt{1-e_0^2}v_0T}{2\pi n} = \frac{a_n}{\sqrt{1-e_0^2}}, \tag{3.5.42}$$

$$A(\alpha_n^*) = \frac{(1-e_0^2)^{3/2}(v_0T)^2}{4\pi n^2}. \tag{3.5.43}$$

Hence the maximum area among all admissible functions when $n=1$ and $A(\alpha_1^*) = A(\alpha_{-1}^*) = (1-e_0^2)^{3/2}(v_0T)/4\pi$.

3.6. **John multiplier theorem.** In Theorem 3.10 we considered the set $\mathscr{D}[\mathcal{K}_i = k_i$ for $i=1,\cdots,m]$, where $\mathscr{D}$ is a given open set in a normed vector space $\mathscr{X}$. Now, we may wish to find an extremum vector for a functional $\mathcal{J}$ in a constraint set of the type $\mathscr{D}[\mathcal{K}_i \le k_i$ for $i=1,\cdots,m]$.

Theorem 3.12. (John multiplier theorem) *Let $\mathcal{J},\mathcal{K}_1,\cdots,\mathcal{K}_m$ be differentiable functions on an open subset $\mathscr{D}$ of a normed vector space $\mathscr{X}$, and let x^* be a local minimum vector in $\mathscr{D}[\mathcal{K}_i \le k_i$ for $i=1,\cdots,m]$ for $\mathcal{J}$. Assume also that the differential of $\mathcal{J}$ and the differential of each $\mathcal{K}_i$ are weakly continuous near x^*. Then there are nonnegative constants $\mu_0,\mu_1,\cdots,\mu_m$, which do not all vanish, such that*

$$\mu_0 d\mathcal{J}(x^*;\Delta x) + \sum_{i=1}^{m}\mu_i d\mathcal{K}_i(x^*;\Delta x) = 0 \tag{3.6.1}$$

for all vectors $\Delta x \in \mathscr{X}$ and such that

$$[\mathcal{K}_i(x^*) - k_i]\mu_i = 0 \tag{3.6.2}$$

for each $i=1,\cdots,m$.

Proof. We give a proof when $m=1$. Suppose that the vector x^* is a local minimum vector in $\mathscr{D}[\mathcal{K}_1 \le k_1]$ for $\mathcal{J}$, i.e., $\mathcal{K}_1(x^*) \le k_1$, and there is some ball $\mathscr{B}_\rho(x^*)$ in $\mathscr{X}$ centered at x^* such that $\mathcal{J}(x^*) \le \mathcal{J}(x)$ for all vectors x which simultaneously are in $\mathscr{B}_\rho(x^*)$ and satisfy $\mathcal{K}_1(x) \le k_1$.

We now consider two possible cases: (1) $\mathcal{K}_1(x^*) = k_1$ and (2) $\mathcal{K}_1(x^*) < k_1$.

(1) $\mathcal{K}_1(x^*) = k_1$. By Theorem 3.10 we have

$$\mu_0\delta\mathcal{J}(x^*;\Delta x) + \mu_1\delta\mathcal{K}_1(x^*;\Delta x) = 0$$

for all vectors $\Delta x \in \mathscr{X}$, and $(\mu_0,\mu_1) = (0,1)$ or $(1,-\lambda)$. We now claim that $\lambda \le 0$. According to (3.4.6) we get

$$\lambda = \frac{\partial}{\partial k_1}\mathcal{J}(x^*(k_1))$$

and hence it suffices to show that $\partial\mathcal{J}(x^*(k_1))/\partial k_1 \leq 0$. Since x^* is a local minimum vector in $\mathscr{D}[\mathcal{K}_1 \leq k_1]$ for $\mathcal{J}$, it follows that $\mathcal{J}(x^*(k_1))$ is increasing if k_1 decreases.

(2) $\mathcal{K}_1(x^*) < k_1$. By continuity, $\mathcal{K}_1(x) < k_1$ hold for all vectors x in some ball $\mathscr{B}_\rho(x^*)$ centered at x^* and it follows that x^* is a local minimum vector in $\mathscr{D}$ for $\mathcal{J}$. By Theorem 2.3, we have

$$\delta\mathcal{J}(x^*; \Delta x) = 0$$

for all vectors $\Delta x \in \mathscr{X}$. Hence this case can be included also in the previous case. □

4. Application I: Calculus of variations

4.1. **Problems with fixed end points.** We consider the problem of maximizing or minimizing the value of a functional $\mathcal{J}$ defined by

$$\mathcal{J}(Y) \doteqdot \int_{x_0}^{x_1} F(x, Y(x), Y'(x))dx \tag{4.1.1}$$

in terms of a given known function F on $\mathbf{R}^3$. Here $Y \in \mathscr{C}^1[x_0, x_1]$ and

$$Y(x_0) = y_0, \quad Y(x_1) = y_1 \tag{4.1.2}$$

for given constants y_0 and y_1.

Defining functionals $\mathcal{K}_0$ and $\mathcal{K}_1$ by

$$\mathcal{K}_0(Y) = Y(x_0), \quad \mathcal{K}_1(Y) = Y(x_1) \tag{4.1.3}$$

for any function $Y \in \mathscr{C}^1[x_0, x_1]$, we see that the problem is equivalent to find extremum vectors in $\mathscr{D}[\mathcal{K}_0 = y_0, \mathcal{K}_1 = y_1]$ for $\mathcal{J}$ (Here $\mathscr{D} = \mathscr{C}^1[x_0, x_1]$ with any suitable norm so that functionals have weakly continuous variations).

Note that

$$\delta\mathcal{K}_0(Y; \Delta Y) = \Delta Y(x_0), \tag{4.1.4}$$

$$\delta\mathcal{K}_1(Y; \Delta Y) = \Delta Y(x_1), \tag{4.1.5}$$

$$\begin{aligned}\delta\mathcal{J}(Y; \Delta Y) = & \int_{x_0}^{x_1} [F_y(x, Y(x), Y'(x))\Delta Y(x) \\ & + F_z(x, Y(x), Y'(x))\Delta Y'(x)]\, dx\end{aligned} \tag{4.1.6}$$

for any vectors $Y, \Delta Y \in \mathscr{C}^1[x_0, x_1]$.

(a) The determinant defined in (3.4.2) does not vanish identically for all functions $\Delta Y_0, \Delta Y_1 \in \mathscr{C}^1[x_0, x_1]$. In fact,

$$\det = \det\begin{pmatrix} \delta\mathcal{K}_0(Y; \Delta Y_0) & \delta\mathcal{K}_0(Y; \Delta Y_1) \\ \delta\mathcal{K}_1(Y; \Delta Y_0) & \delta\mathcal{K}_1(Y; \Delta Y_1) \end{pmatrix} = \Delta Y_0(x_0)\Delta Y_1(x_1) - \Delta Y_0(x_1)\Delta Y_1(x_0);$$

taking $\Delta Y_0(x) = 1$ and $\Delta Y_1(x) = \frac{x - x_0}{x_1 - x_0}$ for $x \in [x_0, x_1]$, we find $\det = 1 \neq 0$.

(b) By Theorem 3.10 we can find constants λ_0, λ_1 such that

$$\delta\mathcal{J}(Y; \Delta Y) = \lambda_0 \delta\mathcal{K}_0(Y; \Delta Y) + \lambda_1 \delta\mathcal{K}_1(Y; \Delta Y) \tag{4.1.7}$$

for any function $\Delta Y \in \mathscr{C}^1[x_0, x_1]$, and then

$$\begin{aligned}& \int_{x_0}^{x_1} [F_y(x, Y(x), Y'(x))\Delta Y(x) + F_z(x, Y(x), Y'(x))\Delta Y'(x)]dx \\ = \; & \lambda_0 \Delta Y(x_0) + \lambda_1 \Delta Y(x_1)\end{aligned} \tag{4.1.8}$$

for any function $\Delta Y \in \mathscr{C}^1[x_0, x_1]$.

(c) (Euler-Lagrange equation) We now assume that

$$F_z(x, Y(x), Y'(x)) \in \mathscr{C}^1[x_0, x_1] \tag{4.1.9}$$

as a function of x. From

$$\begin{aligned} \frac{d}{dx}\left[F_z(x, Y(x), Y'(x))\Delta Y(x)\right] &= F_z(x, Y(x), Y'(x))\Delta Y'(x) \\ &+ \left[\frac{d}{dx}F_z(x, Y(x), Y'(x))\right]\Delta Y(x), \end{aligned}$$

we have

$$\begin{aligned} &\int_{x_0}^{x_1} F_z(x, Y(x), Y'(x))\Delta Y'(x)dx \\ = &\ F_z(x_1, Y(x_1), Y'(x_1))\Delta Y(x_1) - F_z(x_0, Y(x_0), Y'(x_0))\Delta Y(x_0) \\ - &\int_{x_0}^{x_1}\left[\frac{d}{dx}F_z(x, Y(x), Y'(x))\right]\Delta Y(x)dx. \end{aligned}$$

The above formula, together with (4.1.8), gives

$$\begin{aligned} &\int_{x_0}^{x_1}\left[F_y(x, Y(x), Y'(x)) - \frac{d}{dx}F_z(x, Y(x), Y'(x))\right]\Delta Y(x)dx \\ = &\ \left[\lambda_0 + F_z(x_0, y_0, Y'(x_0))\right]\Delta Y(x_0) + \left[\lambda_1 - F_z(x_1, y_1, Y'(x_1))\right]\Delta Y(x_1) \end{aligned} \tag{4.1.10}$$

for all vectors $\Delta Y \in \mathscr{C}^1[x_0, x_1]$. In particular,

$$\int_{x_0}^{x_1}\left[F_y(x, Y(x), Y'(x)) - \frac{d}{dx}F_z(x, Y(x), Y'(x))\right]\Delta Y(x)dx = 0 \tag{4.1.11}$$

for all vectors $\Delta Y \in \mathscr{C}^1[x_0, x_1]$ satisfying $\Delta Y(x_0) = \Delta Y(x_1) = 0$. By Du Bois-Reymond's lemma, we get

$$F_y(x, Y(x), Y'(x)) - \frac{d}{dx}F_z(x, Y(x), Y'(x)) = 0 \tag{4.1.12}$$

for all $x \in [x_0, x_1]$, provided that (4.1.9) holds. The above equation (4.1.12) is called the **Euler-Lagrange equation**.

(d) (Du Bois-Reymond, 1879) Du Bois-Reymond derived (4.1.12) without assuming that the function $F_z(x, Y(x), Y'(x))$ is differentiable. Define

$$g(x) \doteqdot \int_{x_0}^{x} F_y(\xi, Y(\xi), Y'(\xi))d\xi. \tag{4.1.13}$$

Then g is continuous differentiable and

$$g'(x) = F_y(x, Y(x), Y'(x)) \tag{4.1.14}$$

so that

$$\int_{x_0}^{x_1} F_y(x, Y(x), Y'(x))\Delta Y(x)dx = -\int_{x_0}^{x_1} g(x)\Delta Y'(x)dx \tag{4.1.15}$$

for any function $\Delta Y \in \mathscr{C}^1[x_0, x_1]$ vanishing at the end points x_0 and x_1. Hence, equation (4.1.8) can be rewritten as

$$\int_{x_0}^{x_1}\left[-g(x) + F_z(x, Y(x), Y'(x))\right]\Delta Y'(x)dx = 0 \tag{4.1.16}$$

for any function $\Delta Y \in \mathscr{C}^1[x_0, x_1]$ vanishing at the end points x_0 and x_1.

We claim that the function $-g(x) + F_z(x, Y(x), Y'(x))$ is everywhere constant. Define

$$c \doteqdot \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} [-g(x) + F_z(x, Y(x), Y'(x))]\, dx \tag{4.1.17}$$

so that

$$\int_{x_0}^{x_1} [-g(x) + F_z(x, Y(x), Y'(x)) - c]\, \Delta Y'(x) dx = 0$$

which must hold for all functions $\Delta Y \in \mathscr{C}^1[x_0, x_1]$ vanishing at the end points $x = x_0$ and $x = x_1$. By Du Bois-Reymond's lemma, it follows that

$$-g(x) + F_z(x, Y(x), Y'(x)) - c \equiv 0 \tag{4.1.18}$$

for all $x \in [x_0, x_1]$. From (4.1.18), we see that $F_z(x, Y(x), Y'(x))$ is continuous differentiable and then we get (4.1.12).

(e) From (4.1.10) and (4.1.12), we have

$$\lambda_0 = -F_z(x_0, y_0, Y'(x_0)), \quad \lambda_1 = F_z(x_1, y_1, Y'(x_1)). \tag{4.1.19}$$

Consequently, Theorem 3.11 shows that

$$\frac{\partial}{\partial y_0} \mathcal{J}(Y) = -F_z(x_0, y_0, Y'(x_0)), \quad \frac{\partial}{\partial y_1} \mathcal{J}(Y) = F_z(x_1, y_1, Y'(x_1)). \tag{4.1.20}$$

Example 4.1. (Shortest distance between two points/geodesics) This problem is to find a curve $y = Y(x)$, $x \in [x_0, x_1]$, that minimizes the distance between two given points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ in $\mathbf{R}^2$. Let

$$\mathcal{J}(Y) \doteqdot \int_{x_0}^{x_1} \sqrt{1 + Y'(x)^2} dx$$

be the length of any such curve $y = Y(x)$. By (4.1.12), we have

$$\frac{d}{dx} \left[\frac{Y'(x)}{\sqrt{1 + Y'(x)^2}} \right] = 0.$$

This equation can be integrated to give

$$\frac{Y'(x)}{\sqrt{1 + Y'(x)^2}} = \text{constant},$$

and then $Y'(x) \equiv A$ for some constant A. Finally, we find for the extremum function the result $Y(x) = Ax + B$, where the constants A and B are determined by

$$A = \frac{y_2 - y_1}{x_2 - x_1}, \quad B = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}.$$

Example 4.2. (1) If $F = F(y, z)$ is independent on the first argument x, then

$$\begin{aligned}
& \frac{d}{dx} [F(Y(x), Y'(x)) - Y'(x) F_z(Y(x), Y'(x))] \\
= \;& F_y(Y(x), Y'(x)) Y'(x) + F_z(Y(x), Y'(x)) Y''(x) \\
& - Y''(x) F_z(Y(x), Y'(x)) - Y'(x) \frac{d}{dx} F_z(Y(x), Y'(x)) \\
= \;& Y'(x) \left[F_y(Y(x), Y'(x)) - \frac{d}{dx} F_z(Y(x), Y'(x)) \right].
\end{aligned}$$

Hence, if $Y(x)$ is any solution of the Euler-Lagrange equation (4.1.12), we find that

$$\frac{d}{dx}\left[F(Y(x),Y'(x)) - Y'(x)F_z(Y(x),Y'(x))\right] = 0, \tag{4.1.21}$$

which is equivalent to

$$F(Y(x),Y'(x)) - Y'(x)F_z(Y(x),Y'(x)) = C \tag{4.1.22}$$

for some constant C.

(2) If $F = F(x,z)$ is independent on the second argument y, then the Euler-Langrange equation (4.1.12) gives us

$$\frac{d}{dx}F_z(x,Y'(x)) = 0, \tag{4.1.23}$$

which is equivalent to

$$F_z(x,Y'(x)) = C \tag{4.1.24}$$

for some constant C.

Example 4.3. (Mimimum transit time of a boat) We take $P_0 = (x_0, y_0) = (0,0)$ and $P_1 = (x_1, y_1) = (\ell, y_1)$, and we then seek a curve γ connecting P_0 and P_1 given as

$$\gamma : y = Y(x), \quad x \in [0,\ell]$$

along with the boat can travel from P_0 to P_1 in minimum time. Recall (1.3.17) that

$$\mathcal{T}(Y) = \int_0^\ell F(x,Y'(x))dx$$

where

$$F(x,Y'(x)) = \frac{\sqrt{1-e(x)^2+Y'(x)^2} - e(x)Y'(x)}{v_0[1-e(x)^2]}.$$

Since

$$F_z(x,Y'(x)) = \frac{Y'(x) - e(x)\sqrt{1-e(x)^2+Y'(x)^2}}{v_0[1-e(x)^2]\sqrt{1-e(x)^2+Y'(x)^2}},$$

it follows from (4.1.24) that

$$\frac{Y'(x) - e(x)\sqrt{1-e(x)^2+Y'(x)^2}}{v_0[1-e(x)^2]\sqrt{1-e(x)^2+Y'(x)^2}} = C$$

which can be simplified as

$$Y'(x)^2 = \frac{[e(x) + A(1-e(x)^2)]^2}{1 - 2Ae(x) - A^2[1-e(x)^2]}, \quad A = v_0 C. \tag{4.1.25}$$

Hence

$$Y(x) = \int_0^x \frac{e(\xi) + A[1-e(\xi)^2]}{\sqrt{1-2Ae(\xi) - A^2[1-e(\xi)^2]}}d\xi \tag{4.1.26}$$

by imposing the constraint $Y(0) = 0$. Therefore

$$T_{\min} = \frac{1}{v_0}\int_0^\ell \frac{1 - Ae(x)}{\sqrt{1-2Ae(x) - A^2[1-e(x)^2]}}dx. \tag{4.1.27}$$

The constant A can be determined by $Y(\ell) = y_1$.

Example 4.4. (1)(Brachistochrone problem) This problem is to find a minimum vector in $\mathscr{D}[\mathcal{K}_0 = y_0, \mathcal{K}_1 = y_1]$ for the functional $\mathcal{T}$ of (1.3.3), where $\mathscr{D} = \mathscr{C}^1[x_0, x_1]$, and where

$$\mathcal{K}_0(Y) = Y(x_0) = y_0, \quad \mathcal{K}_1(Y) = Y(x_1) = y_1.$$

Now

$$F(Y(x), Y'(x)) = \sqrt{\frac{1 + Y'(x)^2}{2g[y_0 - Y(x)]}}.$$

From

$$F_z(Y(x), Y'(x)) = \frac{Y'(x)}{\sqrt{2g[y_0 - Y(x)][1 + Y'(x)^2]}},$$

we have from (4.1.22) that

$$\sqrt{\frac{1 + Y'(x)^2}{y_0 - Y(x)}} - \frac{Y'(x)^2}{\sqrt{[y_0 - Y(x)][1 + Y'(x)^2]}} = \sqrt{2g}C,$$

or

$$[y_0 - Y(x)][1 + Y'(x)^2] = A \tag{4.1.28}$$

with $A^{-1} = 2gC^2$. Hence

$$Y'(x) = -\sqrt{\frac{A - [y_0 - Y(x)]}{y_0 - Y(x)}}. \tag{4.1.29}$$

Letting

$$y_0 - Y(x) = A\left[\sin\left(\frac{\theta(x)}{2}\right)\right]^2, \tag{4.1.30}$$

we get

$$A\left[\sin\left(\frac{\theta(x)}{2}\right)\right]^2 \frac{d\theta(x)}{dx} = 1 \tag{4.1.31}$$

and then

$$x = x_0 + \frac{A}{2}(\theta - \sin\theta). \tag{4.1.32}$$

Thus, we find for any such extremum curve γ that

$$\gamma : \begin{cases} x = x_0 + \frac{A}{2}(\theta - \sin\theta), \\ y = y_0 - \frac{A}{2}(1 - \cos\theta), \end{cases} \tag{4.1.33}$$

for $\theta \in [\theta_0, \theta_1]$. Consequently,

$$\begin{aligned} T_{\min} &= \int_0^{\theta_1} \sqrt{\frac{(dx/d\theta)^2 + (dy/d\theta)^2}{2g(y_0 - y)}}\, d\theta \\ &= \frac{A}{2}\int_0^{\theta_1} \sqrt{\frac{2(1 - \cos\theta)}{gA(1 - \cos\theta)}}\, d\theta \\ &= \sqrt{\frac{A}{2g}}\theta_1. \end{aligned} \tag{4.1.34}$$

By constraints

$$A(\theta_1 - \sin\theta_1) = 2(x - x_1), \quad A(1 - \cos\theta_1) = -2(y_1 - y_0), \tag{4.1.35}$$

we can uniquely determine values of A and θ_1.

(2) (Brachistochrone problem through the earth) Let A and B be two fixed given points on the surface of the earth, and we let γ be a *plane* curve connecting A and B passing through the earth's interior.

Suppose that a tunnel can be dug through the earth from A to B along the path γ, and we then consider the time of motion T required for a bead to slide without friction through the tunnel from A to B,

$$T \doteqdot \int_\gamma \frac{ds}{v}, \tag{4.1.36}$$

where s measures arc length along γ, ds/dt is the rate of change of arc length with respect to time t during the motion, and the instantaneous speed of motion v is given by $v = ds/dt$. We see the particular tunnel γ that yields the least value for T.

By conservation of energy, we have

$$\frac{1}{2}mv^2 + mg\frac{r^2}{2\rho} = mg\frac{\rho}{2}, \tag{4.1.37}$$

where r is the distance of the bead from the center of the earth, ρ is the radius of the earth, and g is the acceleration of the earth's gravity at the surface of the earth. From (4.1.37), we have

$$v = \sqrt{\frac{g(\rho^2 - r^2)}{\rho}}. \tag{4.1.38}$$

We place a Cartesian (x, y)-coordinate plane with the origin at the center of the earth and with the positive x-axis passing through the point A, and we then let r and θ be the usual plane polar coordinates of the point (x, y), defined by

$$x = r \cdot \cos\theta, \quad y = r \cdot \sin\theta. \tag{4.1.39}$$

We then represent the curve γ in terms of polar coordinates as

$$\gamma : r = R(\theta) = \begin{cases} x = R(\theta) \cdot \cos\theta, \\ y = R(\theta) \cdot \sin\theta, \end{cases} \quad \theta \in [0, \theta_1], \tag{4.1.40}$$

where θ_1 is the fixed central angle determined by the given points A and B and is given by

$$\rho\theta_1 = S_{AB}, \tag{4.1.41}$$

and S_{AB} is the known arc length between the given points A and B.

Since

$$\begin{aligned} ds &= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\ &= \sqrt{(R'(\theta) \cdot \cos\theta - R(\theta) \cdot \sin\theta)^2 + (R'(\theta) \cdot \sin\theta + R(\theta) \cdot \cos\theta)^2} d\theta \\ &= = \sqrt{R(\theta)^2 + R'(\theta)^2} d\theta, \end{aligned}$$

it follows from (4.1.36) and (4.1.38) that

$$T = \sqrt{\frac{\rho}{g}} \int_0^{\theta_1} \sqrt{\frac{R(\theta)^2 + R'(\theta)^2}{\rho^2 - R(\theta)^2}} d\theta \doteqdot \int_0^{\theta_1} F(R(\theta), R'(\theta)) d\theta. \tag{4.1.42}$$

We now seek to minimize the functional (4.1.42) among all $R(\theta) \in \mathscr{C}^1[0,\theta_1]$ which satisfy the constraints

$$R(0) = R(\theta_1) = \rho. \tag{4.1.43}$$

From (4.1.22), we have

$$F(R(\theta), R'(\theta)) - R'(\theta)F_z(R(\theta), R'(\theta)) = C \tag{4.1.44}$$

for some constant C. Consequently,

$$\sqrt{\frac{R(\theta)^2 + R'(\theta)^2}{\rho^2 - R(\theta)^2}} - \frac{R'(\theta)^2}{\sqrt{[\rho^2 R(\theta)^2][R(\theta)^2 + R'(\theta)^2]}} = \sqrt{\frac{g}{\rho}}C, \tag{4.1.45}$$

which can be simplified as

$$[R'(\theta)]^2 = \frac{\rho^2}{r_1^2}\frac{R(\theta)^2 - r_1^2}{\rho^2 - R(\theta)^2}R(\theta)^2, \tag{4.1.46}$$

where

$$r_1 = \frac{\rho}{\sqrt{1 + (\rho C^2/g)}}. \tag{4.1.47}$$

Hence

$$R'(\theta) = \begin{cases} -\frac{\rho}{r_1}R(\theta)\sqrt{\frac{R(\theta)^2 - r_1^2}{\rho^2 - R(\theta)^2}}, & \theta \in [0, \theta_1/2], \\ \frac{\rho}{r_1}R(\theta)\sqrt{\frac{R(\theta)^2 - r_1^2}{\rho^2 - R(\theta)^2}}, & \theta \in [\theta_1/2, \theta_1]. \end{cases} \tag{4.1.48}$$

Letting

$$R(\theta)^2 = \frac{\rho^2 + r_1^2}{2} + \frac{\rho^2 - r_1^2}{2}\cos\left[\frac{2\rho\varphi(\theta)}{\rho - r_1}\right], \tag{4.1.49}$$

the equation (4.1.48) can be written as

$$\varphi'(\theta) = \frac{(\rho^2 + r_1^2) + (\rho^2 - r_1^2)\cos[2\rho\varphi(\theta)/(\rho - r_1)]}{r_1(\rho + r_1)[1 - \cos[2\rho\varphi(\theta)/(\rho - r_1)]]}, \tag{4.1.50}$$

or

$$\left\{-1 + \frac{2\rho^2}{(\rho^2 + r_1^2) + (\rho^2 - r_1^2)\cos[2\rho\varphi/(\rho - r_1)]}\right\}\varphi'(\theta) = \frac{\rho - r_1}{r_1}.$$

By the initial condition $\varphi(0) = 0$, we get

$$2\rho^2\int_0^\varphi \frac{d\varphi}{(\rho^2 + r_1^2) + (\rho^2 - r_1^2)\cos[2\rho\varphi/(\rho - r_1)]} = \varphi + \frac{\rho - r_1}{r_1}\theta. \tag{4.1.51}$$

Using the formula[3]

$$\int \frac{d\varphi}{a + b\cos\varphi} = \frac{2}{\sqrt{a^2 - b^2}}\arctan\frac{\sqrt{a^2 - b^2}\tan(\varphi/2)}{a + b}, \quad a^2 > b^2,$$

---

[3]Using $x = \tan(\varphi/2)$ and $\cos\varphi = \frac{1-x^2}{1+x^2}$, we have

$$\begin{aligned}\int \frac{d\varphi}{a + b\cos\varphi} &= \frac{2}{a - b}\int \frac{dx}{\left(\sqrt{\frac{a+b}{a-b}}\right)^2 + x^2} \\ &= \frac{2}{a - b}\sqrt{\frac{a - b}{a + b}}\tan^{-1}\left(\sqrt{\frac{a - b}{a + b}}\tan\frac{\varphi}{2}\right).\end{aligned}$$

we arrive at

$$\frac{r_1}{\rho}\tan\frac{\rho\varphi}{\rho-r_1}=\tan\left(\theta+\frac{r_1\varphi}{\rho-r_1}\right). \tag{4.1.52}$$

Another initial condition $\varphi(\theta_1)=(\rho-r_1)\pi/\rho$, we obtain from (4.1.52) that

$$\theta_1+\frac{r_1}{\rho}\pi=\pi, \quad r_1=\rho\left(1-\frac{\theta_1}{\pi}\right)=\rho-\frac{S_{AB}}{\pi}. \tag{4.1.53}$$

As an exercise we can show that

$$\gamma:\begin{cases} x=\frac{\rho+r_1}{2}\cos\varphi+\frac{\rho-r_1}{2}\cos\frac{\rho+r_1}{\rho-r_1}\varphi, \\ y=\frac{\rho+r_1}{2}\sin\varphi-\frac{\rho-r_1}{2}\sin\frac{\rho+r_1}{\rho-r_1}\varphi, \end{cases} \quad \varphi\in[0,\theta_1], \tag{4.1.54}$$

and hence

$$T_{\min}=\theta_1\sqrt{\frac{\rho(\rho+r_1)}{g(\rho-r_1)}}=\sqrt{\frac{2\pi S_{AB}-S_{AB}^2}{\rho g}}. \tag{4.1.55}$$

Example 4.5. Let $P_0=(x_0,y_0)$ and $P_1=(x_1,y_1)$ be two given points in the plane with $x_0<x_1$ and $y_0,y_1>0$, and let γ be any curve which connects P_0 and P_1 given as

$$\gamma: y=Y(x), \quad x\in[x_0,x_1].$$

The area of the surface obtained by rotating γ about the x-axis is given by

$$A(Y):=2\pi\int_{x_0}^{x_1}Y(x)\sqrt{1+Y'(x)^2}dx.$$

Assume that $Y\in\mathscr{C}^1[x_0,x_1]$ with $Y(x_0)=y_0$ and $Y(x_1)=y_1$ minimizes the area functional. Since $F(y,z)=2\pi y\sqrt{1+z^2}$, it follows from (4.1.22) that

$$C=2\pi Y(x)\sqrt{1+Y'(x)^2}-Y'(x)\frac{2\pi Y(x)Y'(x)}{\sqrt{1+Y'(x)^2}}=\frac{2\pi Y(x)}{\sqrt{1+Y'(x)^2}}.$$

A general solution, called **catenary**, of the above ordinary differential equation has the form

$$Y(x)=a\cdot\cosh\left(\frac{x-b}{a}\right)$$

for some suitable constants a and b, where $\cosh(t)=(e^t+e^{-t})/2$ is the hyperbolic cosine function. The resulting surface of revolution is called a **catenoid**.

(2) (Queen Dido's problem) Let $P_0=(x_0,0)$ and $P_1=(x_1,0)$ be two fixed points on the x-axis with $x_0<x_1$, and let ℓ be any given fixed length satisfying $x_1-x_0<\ell<\frac{\pi}{2}(x_1-x_0)$. Let γ be any curve of length ℓ connecting P_0 and P_1 given as

$$\gamma: y=Y(x), \quad x\in[x_0,x_1]$$

with $Y(x)\geq 0$. We will show that a suitable circular arc encloses the greatest area with the x-axis among all C^2-curve of length ℓ. Consider

$$\begin{aligned}\mathscr{J}(Y) &:= \int_{x_0}^{x_1}Y(x)dx,\\ \mathscr{K}_0(Y) &:= Y(x_0),\\ \mathscr{K}_1(Y) &:= Y(x_1),\\ \mathscr{K}_2(Y) &:= \int_{x_0}^{x_1}\sqrt{1+Y'(x)^2}dx.\end{aligned}$$

The first functional gives the area of Y enclosed with the x-axis, while the last functional gives the length of Y. The problem now is to find a maximal vector in $\mathscr{D}[\mathcal{K}_0 = 0, \mathcal{K}_1 = 0, \mathcal{K}_2 = \ell]$ for $\mathcal{J}$ where $\mathscr{D} = \mathscr{C}^2[x_0, x_1]$. Note that

$$\begin{aligned}
\delta\mathcal{J}(Y;\Delta Y) &= \int_{x_0}^{x_1} \Delta Y(x)dx, \\
\delta\mathcal{K}_0(Y;\Delta Y) &= \Delta Y(x_0), \\
\delta\mathcal{K}_1(Y;\Delta Y) &= \Delta Y(x_1), \\
\delta\mathcal{K}_2(Y;\Delta Y) &= \int_{x_0}^{x_1} \frac{Y'(x)}{\sqrt{1+Y'(x)^2}}\Delta Y'(x)dx.
\end{aligned}$$

Let Y be a extremum vector in $\mathscr{D}[\mathcal{K}_0 = 0, \mathcal{K}_1 = 0, \mathcal{K}_2 = \ell]$. To apply Theorem 3.10, we should prove the determinant

$$\det := \det \begin{pmatrix} \delta\mathcal{K}_0(Y;\Delta Y_0) & \delta\mathcal{K}_0(Y;\Delta Y_1) & \delta\mathcal{K}_0(Y;\Delta Y_2) \\ \delta K_1(Y;\Delta Y_0) & \delta\mathcal{K}_1(Y;\Delta Y_1) & \delta\mathcal{K}_1(Y;\Delta Y_2) \\ \delta K_2(Y;\Delta Y_0) & \delta\mathcal{K}_2(Y;\Delta Y_1) & \delta\mathcal{K}_2(Y;\Delta Y_2) \end{pmatrix}$$

does not vanish identically for all vectors $\Delta Y_0, \Delta Y_1, \Delta Y_2 \in \mathscr{C}^2[x_0, x_1]$. Choosing

$$\Delta Y_1(x) = \int_{x_0}^{x} \sqrt{1+Y'(x)^2}dx, \quad \Delta Y_2(x) = \Delta Y_1(x) + 1,$$

then $\Delta Y_1'(x) = \Delta Y_2'(x) = \sqrt{1+Y'(x)^2}$ for any $x \in [x_0, x_1]$ and

$$\delta\mathcal{K}_2(Y;\Delta Y_1) = \delta\mathcal{K}_2(Y;\Delta Y_2) = \int_{x_0}^{x_1} Y'(x)dx = Y(x_1) - Y(x_0) = 0$$

since $\mathcal{K}_0(Y) = \mathcal{K}_1(Y) = 0$. Hence we have

$$\begin{aligned}
\det &= -\delta\mathcal{K}_2(Y;\Delta Y_0)\cdot\delta\mathcal{K}_1(Y;\Delta Y_1) \\
&= -\int_{x_0}^{x_1} \frac{Y'(x)}{\sqrt{1+Y'(x)^2}}\Delta Y_0'(x)dx \cdot \int_{x_0}^{x_1} \sqrt{1+Y'(x)^2}dx
\end{aligned}$$

for any vector $\Delta Y_0 \in \mathscr{C}^2[x_0, x_1]$. If $\det \equiv 0$, then we must have

$$\int_{x_0}^{x_1} \frac{Y'(x)}{\sqrt{1+Y'(x)^2}}\Delta Y_0'(x)dx = 0$$

from which $Y'(x) \equiv 0$ and then $\ell = \mathcal{K}_2(Y) = x_1 - x_0 < \ell$, a contradiction. Therefore, det does not vanish identically. By Theorem 3.10,

$$\delta\mathcal{J}(Y;\Delta Y) = \lambda_0\delta\mathcal{K}_0(Y;\Delta Y) + \lambda_1\delta\mathcal{K}_1(Y;\Delta Y) + \lambda_2\delta\mathcal{K}_2(Y;\Delta Y)$$

for some constants λ_0, λ_1, and λ_2. Thus

$$\int_{x_0}^{x_1} \Delta Y(x)dx = \lambda_0\Delta Y(x_0) + \lambda_1\Delta Y(x_1) + \lambda_2\int_{x_0}^{x_1} \frac{Y'(x)}{\sqrt{1+Y'(x)^2}}\Delta Y'(x)dx.$$

Using the identity

$$\frac{d}{dx}\left[\frac{Y'(x)}{\sqrt{1+Y'(x)^2}}\Delta Y(x)\right] = \frac{Y'(x)}{\sqrt{1+Y'(x)^2}}\Delta Y'(x) + \frac{d}{dx}\left[\frac{Y'(x)}{\sqrt{1+Y'(x)^2}}\right]\Delta Y(x)$$

yields

$$\int_{x_0}^{x_1} \left\{1 + \lambda_2\frac{d}{dx}\left[\frac{Y'(x)}{\sqrt{1+Y'(x)^2}}\right]\right\}\Delta Y(x)dx = 0$$

for any vector $\Delta Y \in \mathscr{C}^2[x_0, x_1]$ with $\Delta Y(x_0) = \Delta Y(x_1) = 0$. By Du Bois-Reymond's lemma, we get

$$1 + \lambda_2 \frac{d}{dx}\left[\frac{Y'(x)}{\sqrt{1+Y'(x)^2}}\right].$$

It is not hard to prove that

$$Y'(x)^2 = \frac{(x-a)^2}{c^2-(x-a)^2}$$

for some positive constants a and b. Integrating on both sides yields

$$(x-\bar{a})^2 + (y-\bar{b})^2 = \bar{c}^2$$

for suitable positive constants $\bar{a}, \bar{b}$, and $\bar{c}$.

(3) The following inequality

$$\int_0^1 Y'(x)^2 dx \ge \frac{\pi^2}{4}\int_0^1 Y(x)^2 dx \tag{4.1.56}$$

for any $Y \in \mathscr{C}^1[0,1]$ with $Y(0) = 0$ and $Y(1) = 1$. By approximation, we may assume that $Y \in \mathscr{C}^2[0,1]$. We now minimize the functional

$$\mathcal{J}(Y) := \frac{\int_0^1 Y'(x)^2 dx}{\int_0^1 Y(x)^2 dx} \tag{4.1.57}$$

in the subset $\mathscr{D}[\mathcal{K}_0 = 0, \mathcal{K}_1 = 1]$, where $\mathscr{D} = \mathscr{C}^2[0,1]$ and

$$\mathcal{K}_0(Y) := Y(0), \quad \mathcal{K}(1) := Y(1).$$

If Y is an extremum vector in $\mathscr{D}[\mathcal{K}_0 = 0, \mathcal{K}_1 = 1]$ for $\mathcal{J}$, then

$$\begin{aligned} & \frac{1}{2}\left(\int_0^1 Y(x)^2 dx\right)^2 \delta\mathcal{J}(Y;\Delta Y) \\ = & \left(\int_0^1 Y(x)^2 dx\right)\left[\int_0^1 Y'(x)\Delta Y'(x)dx\right] - \left(\int_0^1 Y'(x)^2 dx\right)\left[\int_0^1 Y(x)\Delta Y(x)\right]; \end{aligned}$$

by integration by parts, we have

$$\int_0^1 Y'(x)\Delta Y'(x)dx = Y'(1)\Delta Y(1) - Y'(0)\Delta Y(0) - \int_0^1 Y''(x)\Delta y(x)dx,$$

and hence

$$\begin{aligned} & \frac{1}{2}\left(\int_0^1 Y(x)^2 dx\right)^2 \delta\mathcal{J}(Y;\Delta Y) \\ = & \left(\int_0^1 Y(x)^2 dx\right) Y'(1)\Delta Y(1) - \left(\int_0^1 Y(x)^2 dx\right) Y'(0)\Delta Y(0) \\ & - \int_0^1 \left[\left(\int_0^1 Y(x)^2 dx\right) Y''(x) + \left(\int_0^1 Y'(x)^2 dx\right) Y(x)\right]\Delta Y(x) dx. \end{aligned}$$

By Theorem 3.10, we have

$$\delta\mathcal{J}(Y;\Delta Y) = \lambda_0 \delta\mathcal{K}(Y;\Delta Y) + \lambda_1 \delta\mathcal{K}_1(Y;\Delta Y).$$

Letting $\Delta Y(0) = \Delta Y(1) = 0$ yields

$$\int_0^1 \left[\left(\int_0^1 Y(x)^2 dx \right) Y''(x) + \left(\int_0^1 Y'(x)^2 dx \right) Y(x) \right] \Delta Y(x) dx = 0.$$

Consequently,

$$\left(\int_0^1 Y(x)^2 dx \right) Y''(x) + \left(\int_0^1 Y'(x)^2 dx \right) Y(x) = 0, \quad x \in [0,1]. \tag{4.1.58}$$

Multiplying by $Y(x)$ and then integrating over $[0,1]$ on both sides of (4.1.58), we arrive at

$$\begin{aligned} 0 &= \left(\int_0^1 Y'(x)^2 dx \right) \left(\int_0^1 Y(x)^2 dx \right) + \left(\int_0^1 Y(x)^2 dx \right) \left(\int_0^1 Y(x) Y''(x) dx \right) \\ &= \left(\int_0^1 Y'(x)^2 dx \right) \left(\int_0^1 Y(x)^2 dx \right) \\ &\quad + \left(\int_0^1 Y(x)^2 dx \right) \left[Y(1)Y'(1) - Y(0)Y'(0) - \int_0^1 Y'(x)^2 dx \right] \\ &= Y'(1) \left(\int_0^1 Y(x)^2 dx \right). \end{aligned}$$

Since $Y(1) = 1$, it follows that the integral of $Y(x)^2$ over $[0,1]$ is nonzero and hence

$$Y'(1) = 1. \tag{4.1.59}$$

Solving the ODE with the initial conditions $Y(0) = 0$ and $Y(1) = 1$, we obtain

$$Y(x) = \frac{\sin(ax)}{\sin a}, \quad a := \sqrt{\frac{\int_0^1 Y'(x)^2 dx}{\int_0^1 Y(x)^2 dx}}. \tag{4.1.60}$$

From (4.1.59) and

$$Y'(x) = \frac{a}{\sin a} \cos(ax),$$

we have

$$1 = Y'(1) = \frac{a}{\sin a} \cos(a) \Longrightarrow a = \frac{\pi}{2} + k\pi, \quad k = 0, 1, \cdots,$$

since $a > 0$. Then

$$\begin{aligned} \mathcal{J}(Y) &= \left(\frac{\pi}{2} + k\pi \right)^2 \frac{\int_0^1 \cos(\pi x/2)^2 dx}{\int_0^1 \sin(\pi x/2)^2 dx} = \left(\frac{\pi}{2} + k\pi \right)^2 \frac{\int_0^{\pi/2} \cos(t)^2 dt}{\int_0^{\pi/2} \sin(t)^2 dt} \\ &= \left(\frac{\pi}{2} + k\pi \right)^2 \geq \frac{\pi^2}{4}, \end{aligned}$$

and the minimum of $\mathcal{J}$ is

$$\mathcal{J}_{\min} = \mathcal{J}\left(\sin\left(\frac{\pi}{2} x \right) \right) = \frac{\pi^2}{4}.$$

4.2. **Geodesic curves.** We consider a given fixed surface S in $\mathbf{R}^3$, and we let P_0 and P_1 be any given fixed points on S.

> **Problem:** Find a curve γ which has the shortest length among all curves which lie on the surface S and connect P_0 and P_1.

Any such a curve giving the minimum distance between two fixed points of S is called a **geodesic curve**.

Example 4.6. (Geodesics on a right circular cylinder) Let x, y, z be the usual Cartesian coordinates in $\mathbf{R}^3$ and then cylindrical coordinates r, θ, u are defined by

$$x = r \cdot \cos\theta, \quad y = r \cdot \sin\theta, \quad z = u.$$

We assume that the central axis of the cylinder S coincide with the z-axis and then the surface S can be given parametrically as

$$S : \begin{cases} x = a \cdot \cos\theta, \\ y = a \cdot \sin\theta, \qquad \theta \in [0, 2\pi], \quad u \in \mathbf{R}. \\ z = u, \end{cases}$$

Here a is a fixed positive constant which gives the value of the radius of the cylinder. Any curve γ lying on the surface of the cylinder S can be parametrically represented as

$$\gamma : \begin{cases} x = a \cdot \cos\theta, \\ y = a \cdot \sin\theta, \qquad \theta \in [\theta_0, \theta_1] \\ z = U(\theta), \end{cases} \tag{4.2.1}$$

for some suitable function $U(\theta)$. The constants θ_0 and θ_1 are the θ-coordinates of the endpoints $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$ of γ (we may assume that $\theta_0 \leq \theta_1$), which satisfy

$$x_0 = a \cdot \cos\theta_0, \quad y_0 = a \cdot \sin\theta_0, \quad x_1 = a \cdot \cos\theta_1, \quad y_1 = a \cdot \sin\theta_1. \tag{4.2.2}$$

Without loss of generality, we furthermore assume that

$$0 \leq \theta_1 - \theta_0 \leq \pi. \tag{4.2.3}$$

The length L of (4.2.1) is given by

$$L(\gamma) \doteqdot \int_\gamma \left|\frac{d\gamma}{d\theta}\right| = \int_{\theta_0}^{\theta_1} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2} d\theta. \tag{4.2.4}$$

Then

$$L(U) = L(\gamma) = \int_{\theta_0}^{\theta_1} \sqrt{a^2 + U'(\theta)^2} d\theta = \int_{\theta_0}^{\theta_1} F(U'(\theta)) d\theta, \tag{4.2.5}$$

where

$$F(w) \doteqdot \sqrt{a^2 + w^2}. \tag{4.2.6}$$

The problem of finding the geodesic curve connecting $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$ on the cylinder can be reduced to the problem of finding the function $U = U(\theta)$ which minimizes the length functional $L(U)$ of (4.2.5) subject to the constraints

$$U(\theta_0) = z_0, \quad U(\theta_1) = z_1. \tag{4.2.7}$$

By (4.1.24), the Euler-Lagrange equation becomes

$$C = F_w(U'(\theta)) = \frac{U'(\theta)}{\sqrt{a^2 + U'(\theta)^2}} \tag{4.2.8}$$

for some constant C, and then

$$U(\theta) = A\theta + B \tag{4.2.9}$$

for some constants A and B. In this case the curve γ is called a **circular helix**. By (4.2.7), we have

$$A = \frac{z_1 - z_0}{\theta_1 - \theta_0}, \quad B = \frac{\theta_1 z_0 - \theta_0 z_1}{\theta_1 - \theta_0} \tag{4.2.10}$$

and

$$L_{\min} = \int_{\theta_0}^{\theta_1} \sqrt{a^2 + A^2} d\theta = \sqrt{a^2(\theta_1 - \theta_0)^2 + (z_1 - z_0)^2}. \tag{4.2.11}$$

On the other hand, from (4.2.2) we have

$$\cos(\theta_1 - \theta_0) = \cos\theta_1 \cos\theta_0 + \sin\theta_1 \sin\theta_0 = \frac{x_0 x_1 + y_0 y_1}{a^2}, \quad \theta_1 - \theta_0 \in [0, \pi].$$

Example 4.7. (Geodesics on a sphere) Consider spherical polar coordinates r, θ, ϕ by

$$\begin{aligned} x &= r \cdot \sin\phi \cdot \cos\theta, \\ y &= r \cdot \sin\phi \cdot \sin\theta, \\ z &= r \cdot \cos\phi, \end{aligned}$$

and then the surface of a sphere S of radius a centered at the origin can be given parametrically as

$$S : \begin{cases} x = a \cdot \sin\phi \cdot \cos\theta, \\ y = a \cdot \sin\phi \cdot \sin\theta, \\ z = a \cdot \cos\phi \end{cases}, \quad \phi \in [0, \pi], \quad \theta \in [0, 2\pi]. \tag{4.2.12}$$

Any curve γ on S can be written as

$$\gamma : \begin{cases} x = a \cdot \sin\phi \cdot \cos\Theta(\phi), \\ y = a \cdot \cos\phi \cdot \sin\Theta(\phi), \\ z = a \cdot \cos\phi, \end{cases}, \quad \phi \in [\phi_0, \phi_1] \tag{4.2.13}$$

for some suitable function $\Theta(\phi)$ on γ. The endpoints $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$ of γ satisfy

$$z_0 = \sqrt{x_0^2 + y_0^2 + z_0^2} \cdot \cos\phi_0, \quad z_1 = \sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \cos\phi_1, \quad \phi_0, \phi_1 \in [0, \pi]. \tag{4.2.14}$$

The length of γ is

$$L(\Theta) = L(\gamma) = a \int_{\phi_0}^{\phi_1} \sqrt{1 + \Theta'(\phi)^2 \sin^2\phi} d\theta = \int_{\phi_0}^{\phi_1} F(\phi, \Theta'(\phi)) d\theta \tag{4.2.15}$$

where

$$F(\phi, w) = a\sqrt{1 + w^2 \sin^2\phi}. \tag{4.2.16}$$

The problem of finding the geodesic curve connecting $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$ on the sphere can now be reduced to the problem of finding the function $\Theta(\phi)$ which minimizes the length functional $L(\Theta)$ of (4.2.15) subject to constraints

$$\tan\Theta(\phi_0) = \frac{y_0}{x_0}, \quad \tan\Theta(\phi_1) = \frac{y_1}{x_0}. \tag{4.2.17}$$

By (4.1.24), the Euler-Lagrange equation becomes

$$C = F_w(\phi, \Theta'(\phi)) = a\frac{\Theta'(\phi)\sin^2\phi}{\sqrt{1+\Theta'(\phi)^2\sin^2\phi}} \tag{4.2.18}$$

and then

$$\Theta'(\phi) = \frac{A}{\sin\phi\sqrt{\sin^2\phi - A^2}} \tag{4.2.19}$$

for some constant A. Writing

$$A \doteqdot \sin\alpha \tag{4.2.20}$$

yields

$$\Theta'(\phi) = \frac{\sin\alpha}{\sin\phi\sqrt{\sin^2\phi - \sin^2\alpha}}. \tag{4.2.21}$$

Introducing

$$\tan\phi = \frac{1}{u}, \tag{4.2.22}$$

we have

$$\frac{d\Theta}{du} = \Theta'(\phi)\frac{d\phi}{du} = \frac{-\tan\alpha}{\sqrt{1-u^2\tan^2\alpha}}. \tag{4.2.23}$$

Hence

$$\Theta + \beta = \cos^{-1}(u\cdot\tan\alpha) \tag{4.2.24}$$

where β is a constant of integration.

4.3. **Problems with variable end points.** We consider the general problem of minimizing or maximizing the functional

$$\mathcal{J}(x_1, Y) = \int_{x_0}^{x_1} F(x, Y(x), Y'(x))dx \tag{4.3.1}$$

among all curves

$$\gamma : y = Y(x), \quad x \in [x_0, x_1] \tag{4.3.2}$$

which satisfy the initial constraint

$$Y(x_0) = y_0 \tag{4.3.3}$$

and the terminal constraint

$$\Phi(x_1, Y(x_1)) = 0 \tag{4.3.4}$$

where $\Phi = \Phi(x, y)$ is a given function.

We take the vector space $\mathscr{X}$ as the set of all pairs (x_1, Y), where $x_1 \in \mathbf{R}$ and $Y \in \mathscr{C}^1_0(\mathbf{R})$. If $(x_1, Y), (x_1^*, Y^*) \in \mathscr{X}$, then we define

$$(x_1, Y) + (x_1^*, Y^*) \doteqdot (x_1 + x_1^*, Y + Y^*),$$

which again gives a vector in $\mathscr{X}$. Similarly,we define the product $a(x_1, Y)$ by

$$a(x_1, Y) = (ax_1, aY)$$

for any $a \in \mathbf{R}$ and any vector $(x_1, Y) \in \mathscr{X}$. Thus the vector space $\mathscr{X}$ is well-defined. We equip $\mathscr{X}$ with the norm $||\cdot||_{\mathscr{X}}$ defined by

$$||(x_1, Y)||_{\mathscr{X}} \doteqdot |x_1| + ||Y||_{\mathscr{C}^1(\mathbf{R})} = |x_1| + \max_{x\in\mathbf{R}} |Y(x)| + \max_{x\in\mathbf{R}} |Y'(x)| \tag{4.3.5}$$

for any vector $(x_1, Y) \in \mathscr{X}$.

The extremum problem for the functional (4.3.1) is to seek a vector (x_1, Y) in some given open set $\mathscr{D} \subset \mathscr{X}$ that will maximize or minimize in $\mathscr{D}$ the functional (4.3.1), where the admissible vectors (x_1, Y) are also required to satisfy (4.3.3) and (4.3.4).

If we define functionals $\mathcal{K}_0$ and $\mathcal{K}_1$ on $\mathscr{D}$ by

$$\mathcal{K}_0(x_1, Y) = Y(x_0), \tag{4.3.6}$$

$$\mathcal{K}_1(x_1, Y) = \Phi(x_1, Y(x_1)), \tag{4.3.7}$$

then the extremum problem is to find extremum vectors in $\mathscr{D}[\mathcal{K}_0 = y_0, \mathcal{K}_1 = 0]$ for the functional (4.3.1).

Since

$$\delta\mathcal{J}(x_1, Y; \Delta x_1, \Delta Y) = \frac{d}{d\epsilon}\mathcal{J}(x_1 + \epsilon\Delta x_1, Y + \epsilon\Delta Y)\Big|_{\epsilon=0},$$

it follows that

$$\begin{aligned} &\delta\mathcal{J}(x_1, Y; \Delta x_1, \Delta Y) \\ = \ & \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{x_0}^{x_1+\epsilon\Delta x_1} F(x, Y(x) + \epsilon\Delta Y(x), Y'(x) + \epsilon\Delta Y'(x))dx \\ = \ & F(x_1, Y(x_1), Y'(x_1))\Delta x_1 \\ & + \int_{x_0}^{x_1} \left[F_y(x, Y(x), Y'(x))\Delta Y(x) + F_w(x, Y(x), Y'(x))\Delta Y'(x)\right] dx \end{aligned} \tag{4.3.8}$$

for any vector $(\Delta x_1, \Delta Y) \in \mathscr{X}$. Similarly, we have

$$\delta\mathcal{K}_0(x_1, Y; \Delta x_1, \Delta Y) = \Delta Y(x_0), \tag{4.3.9}$$

and

$$\begin{aligned} \delta\mathcal{K}_1(x_1, Y; \Delta x_1, \Delta Y) &= \frac{d}{d\epsilon}\mathcal{K}_1(x_1 + \epsilon\Delta x_1, Y + \epsilon\Delta Y)\Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \Phi(x_1 + \epsilon\Delta x_1, Y(x_1 + \epsilon\Delta x_1) + \epsilon\Delta Y(x_1 + \epsilon\Delta x_1)) \\ &= \Phi_x(x_1, Y(x_1))\Delta x_1 \\ &\quad + \Phi_y(x_1, Y(x_1))\left[Y'(x_1)\Delta x_1 + \Delta Y(x_1)\right] \end{aligned} \tag{4.3.10}$$

for any vector $(\Delta x_1, \Delta Y) \in \mathscr{X}$.

Suppose now that the vector (x_1, Y) is a local extremum vector in $\mathscr{D}[\mathcal{K}_0 = y_0, \mathcal{K}_1 = 0]$ for $\mathcal{J}$. Then Theorem 3.11 implies that

$$\delta\mathcal{J}(x_1, Y; \Delta x_1, \Delta Y) = \lambda_0\delta\mathcal{K}_0(x_1, Y; \Delta x_1, \Delta Y) + \lambda_1\delta\mathcal{K}_1(x_1, Y; \Delta x_1, \Delta Y) \tag{4.3.11}$$

for suitable constants λ_0 and λ_1, for all numbers Δx_1, and for all functions $\Delta Y \in \mathscr{C}^1[x_0, x_1] \subset \mathscr{C}^1_0(\mathbf{R})$.

From (4.3.8), (4.3.9), (4.3.10), and (4.3.11), we have

$$\begin{aligned} & F(x_1, Y(x_1), Y'(x_1))\Delta x_1 \\ & + \int_{x_0}^{x_1} [F_y(x, Y(x), Y'(x))\Delta Y(x) + F_z(x, Y(x), Y'(x))\Delta Y'(x)]\, dx \\ = \; & \lambda_0 \Delta Y(x_0) + \lambda_1 \Phi_z(x_1, Y(x_1))\Delta x_1 + \lambda_1 \Phi_y(x_1, Y(x_1))\, [Y'(x_1)\Delta x_1 + \Delta Y(x_1)] \end{aligned}$$

and then, by integration by parts,

$$\begin{aligned} & \int_{x_0}^{x_1} \left[F_y(x, Y(x), Y'(x)) - \frac{d}{dx} F_z(x, Y(x), Y'(x))\right] \Delta Y(x) dx \\ & + F_z(x, Y(x), Y'(x))\Delta Y(x)\Big|_{x_0}^{x_1} \\ = \; & \lambda_0 \Delta Y(x_0) + \lambda_1 \Phi_y(x_1, Y(x_1))\Delta Y(x_1) \\ & + \{-F(x_1, Y(x_1), Y'(x_1)) + \lambda_1 [\Phi_x(x_1, Y(x_1)) + Y'(x_1)\Phi_y(x_1, Y(x_1))]\}\, \Delta x_1. \end{aligned}$$

Thus

$$\begin{aligned} & \int_{x_0}^{x_1} \left[F_y(x, Y(x), Y'(x)) - \frac{d}{dx} F_z(x, Y(x), Y'(x))\right] \Delta Y(x) dx \\ = \; & [\lambda_0 + F_z(x_0, y_0, Y'(x_0))]\, \Delta Y(x_0) \\ & + [\lambda_1 \Phi_y(x_1, Y(x_1)) - F_z(x_1, Y(x_1), Y'(x_1))]\, \Delta Y(x_1) \\ & + \{-F(x_1, Y(x_1), Y'(x_1)) \\ & + \lambda_1 [\Phi_x(x_1, Y(x_1)) + Y'(x_1)\Phi_y(x_1, Y(x_1))]\}\, \Delta x_1, \end{aligned} \tag{4.3.12}$$

which holds for all vectors $(\Delta x_1, \Delta Y) \in \mathscr{X}$ if (x_1, Y) is a local extremum vector in $\mathscr{D}[\mathcal{K}_0 = y_0, \mathcal{K}_1 = 0]$ for $\mathcal{J}$.

(1) If we take $\Delta x_1 = 0$ and consider functions $\Delta Y(x)$ which vanish at the end points x_0 and x_1, we find

$$\int_{x_0}^{x_1} \left[F_y(x, Y(x), Y'(x)) - \frac{d}{dx} F_z(x, Y(x), Y'(x))\right] \Delta Y(x) = 0 \tag{4.3.13}$$

which must hold for all continuously differentiable functions $\Delta Y(x)$ on $[x_0, x_1]$ satisfying $\Delta Y(x_0) = \Delta Y(x_1) = 0$. By Du Bois-Reymond's lemma, we still get (4.1.12), i.e.,

$$F_y(x, Y(x), Y'(x)) - \frac{d}{dx} F_z(x, Y(x), Y'(x)) = 0. \tag{4.3.14}$$

Plugging (4.3.14) into (4.3.12) we conclude that

$$\begin{aligned} 0 \; = \; & [\lambda_0 + F_z(x_0, y_0, Y'(x_0))]\, \Delta Y(x_0) \\ & + [\lambda_1 \Phi_y(x_1, Y(x_1)) - F_z(x_1, Y(x_1), Y'(x_1))]\, \Delta Y(x_1) \\ & + \{-F(x_1, Y(x_1), Y'(x_1)) \\ & + \lambda_1 [\Phi_x(x_1, Y(x_1)) + Y'(x_1)\Phi_y(x_1, Y(x_1))]\}\, \Delta x_1 \end{aligned} \tag{4.3.15}$$

which holds for all $\Delta x_1 \in \mathbf{R}$ and all continuously differentiable functions $\Delta Y(x)$ on $[x_0, x_1]$.

(2) Taking first $\Delta x_1 = 0$ and

$$\Delta Y(x) = \frac{x_1 - x}{x_1 - x_0},$$

we have

$$\lambda_0 + F_z(x_0, y_0, Y'(x_0)) = 0. \tag{4.3.16}$$

Taking then $\Delta x_1 = 0$ and

$$\Delta Y(x) = \frac{x - x_0}{x_1 - x_0},$$

we have

$$\lambda_1 \Phi_y(x_1, Y(x_1)) - F_z(x_1, Y(x_1), Y'(x_1)) = 0. \tag{4.3.17}$$

Finally, we choose $\Delta x_1 = 1$ and use (4.3.16) and (4.3.17), we conclude that

$$F(x_1, Y(x_1), Y'(x_1)) = \lambda_1 \left[\Phi_x(x_1, Y(x_1)) + Y'(x_1)\Phi_y(x_1, Y(x_1))\right]. \tag{4.3.18}$$

Eliminating λ_1 in (4.3.18) by using (4.3.17), we arrive at

$$\begin{aligned} & \Phi_y(x_1, Y(x_1)) F(x_1, Y(x_1), Y'(x_1)) \\ = \; & F_z(x_1, Y(x_1), Y'(x_1)) \left[\Phi_x(x_1, Y(x_1)) + Y'(x_1)\Phi_y(x_1, Y(x_1))\right], \end{aligned} \tag{4.3.19}$$

which is then a **natural boundary condition** which must hold at the variable endpoint $x = x_1$ for any local extremum vector (x_1, Y).

(3) (Geometric interpretation of (4.3.19)) Consider the function

$$F(x, y, z) \doteqdot f(x, y)\sqrt{1 + z^2} \tag{4.3.20}$$

for some given function $f(x, y)$. Since

$$F_z(x, y, z) = f(x, y)\frac{z}{\sqrt{1 + z^2}},$$

it follows that (except possibly if $f(x_1, Y(x_1)) = 0$)

$$\begin{aligned} & \Phi_y(x_1, Y(x_1)) \frac{Y'(x_1)}{\sqrt{1 + Y'(x_1)^2}} \\ = \; & \frac{Y'(x_1)}{\sqrt{1 + Y'(x_1)^2}} \left[\Phi_x(x_1, Y(x_1)) + Y'(x_1)\Phi_y(x_1, Y(x_1))\right] \end{aligned}$$

and then

$$\Phi_y(x_1, Y(x_1)) = Y'(x_1)\Phi_x(x_1, Y(x_1)). \tag{4.3.21}$$

From this we claim that the extremum curve γ of (4.3.2) must intersect the given curve $C : \Phi(x, y) = 0$ orthogonally. Indeed, the slope y_C' of the curve C is given as

$$y_C' = -\frac{\Phi_x}{\Phi_y}$$

and hence

$$Y'(x_1) y_C'(x_1) + 1 = 0. \tag{4.3.22}$$

Example 4.8. We consider the minimizing the functional

$$\mathcal{J}(x_1, Y) = \int_5^{x_1} \frac{\sqrt{1 + Y'(x)^2}}{Y(x)} dx \tag{4.3.23}$$

among all curves γ given as

$$\gamma : y = Y(x), \quad x \in [5, x_1]$$

which join the point $P_0 = (5, 5)$ to the line C defined as

$$C : y = x - 5. \tag{4.3.24}$$

Then

$$F(x, y, z) = \frac{\sqrt{1+z^2}}{y}, \quad F_z(x, y, z) = \frac{z}{y\sqrt{1+z^2}}. \tag{4.3.25}$$

Since F is independent on the first argument x, then (4.3.14) reduces to (4.1.22) and therefore

$$Y(x)^2[1 + Y'(x)^2] = A^2 \tag{4.3.26}$$

for some constant A. The differential equation (4.3.26) gives us

$$\int \frac{Y(x)Y'(x)dx}{\sqrt{A^2 - Y(x)^2}} = \pm \int dx$$

which implies that $-\sqrt{A^2 - Y(x)^2} = \pm(x + B)$ and

$$Y(x) = \sqrt{A^2 - (x - B)^2} \tag{4.3.27}$$

for another constant B. Here we have taken the positive square root in (4.3.27) so as to make the condition $Y(5) = 5$ possible at the point P_0. Since $Y(5) = 5$, we have

$$A^2 = 25 + (5 - B)^2; \tag{4.3.28}$$

on the other hand, (4.3.24) yields

$$Y(x_1) = x_1 - 5; \tag{4.3.29}$$

finally, the natural boundary condition (4.3.19) or (4.3.21) implies that

$$Y'(x_1) = -1. \tag{4.3.30}$$

Now, (4.3.27)–(4.3.30) gives us

$$\begin{aligned} Y'(x) &= \frac{B - x}{\sqrt{A^2 - (x - B)^2}} = \frac{B - x}{Y(x)}, \\ Y'(x_1) &= \frac{B - x_1}{x_1 - 5} = -1; \end{aligned}$$

thus $B = 5$ and $A^2 = 25$. Consequently,

$$Y(x) = \sqrt{25 - (x - 5)^2}, \quad x_1 = 5 + \frac{5}{\sqrt{2}}. \tag{4.3.31}$$

From (4.3.31) and (4.3.23), we have

$$\begin{aligned} \mathcal{J}_{\min} &= \int_5^{5+\frac{5}{\sqrt{2}}} \frac{\sqrt{1 + \frac{(x-5)^2}{25-(x-5)^2}}}{\sqrt{25 - (x - 5)^2}} dx = 5 \int_5^{5+\frac{5}{\sqrt{2}}} \frac{dx}{x(10 - x)} \\ &= \frac{1}{2} \int_5^{5+\frac{5}{\sqrt{2}}} \left(\frac{1}{x} - \frac{1}{x - 10} \right) dx = \frac{1}{2} \ln \left| \frac{x}{x - 10} \right| \Big|_5^{5+\frac{5}{\sqrt{2}}} \\ &= \frac{1}{2} \ln \frac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \frac{1}{2} \ln(3 + 2\sqrt{2}). \end{aligned}$$

Example 4.9. (1) (James Bernoulli's brachistochrone problem) In Example 4.4, if we require

$$Y(x_0) = y_0, \quad C : x = x_1.$$

By (4.1.33) we have

$$\gamma : \begin{cases} x = x_0 + \frac{A}{2}(\theta - \sin\theta), \\ y = y_0 - \frac{A}{2}(1 - \cos\theta), \end{cases} \quad \theta \in [0, \theta_1]. \tag{4.3.32}$$

Since $\Phi(x, y) = x - x_1$, the natural boundary condition (4.3.19) and the expression of $F_z(x, Y(x), Y'(x))$ in Example 4.4 show that

$$0 = Y'(x_1)$$

or $\frac{dy}{d\theta}|_{\theta_1} = 0$. By (4.3.32) it yields

$$\theta_1 = \pi, \quad A = \frac{2(x_1 - x_0)}{\pi}. \tag{4.3.33}$$

Plugging (4.3.33) into (4.1.34), the minimum time is given by

$$T_{\min} = \sqrt{\frac{(x_1 - x_0)\pi}{g}} \tag{4.3.34}$$

where we assume that $x_1 \geq x_0$.

(2) Let $P_0 = (0, y_0)$ be a given fixed point on the y-axis, and let $P_1 = (x_1, 0)$ represent any variable point on the x-axis, with $x_1 > 0$ and $y_0 > 0$. Let γ be any curve connecting P_0 and P_1 given as $\gamma : y = Y(x)$, $0 \leq x \leq x_1$. The area enclosed between γ and the coordinate axes is given by

$$A(x_1, Y) := \int_0^{x_1} Y(x)dx, \tag{4.3.35}$$

while the surface area generated when γ is rotated about the x-axis is given by

$$\mathcal{J}(x_1, Y) := 2\pi \int_0^{x_1} Y(x)\sqrt{1 + Y'(x)^2}dx. \tag{4.3.36}$$

We consider the problem

> Given a positive constant A, find a curve in $\mathscr{D}[\mathcal{K}_0 = y_0, \mathcal{K}_1 = 0, \mathcal{K}_2 = A]$ for $\mathcal{J}$, where $\mathscr{D} = \mathscr{C}^1(\mathbf{R}^+)$ and
>
> $$\begin{aligned} \mathcal{K}_0(x_1, Y) &:= Y(0), \\ \mathcal{K}_1(x_1, Y) &:= Y(x_1), \\ \mathcal{K}_2(x_1, Y) &:= A(x_1, Y). \end{aligned}$$

Let

$$F(y, z) := 2\pi y\sqrt{1 + z^2}, \quad F_z(y, z) = \frac{2\pi yz}{\sqrt{1 + z^2}}.$$

The variation of $\mathcal{J}(x_1, Y)$ is given by

$$\begin{aligned} \delta\mathcal{J}(x_1, Y; \Delta x_1, \Delta Y) &= F(Y(x_1), Y'(x_1))\Delta x_1 + \int_0^{x_1} [F_y(Y(x), Y'(x))\Delta Y(x) \\ &\quad + F_z(Y(x), Y'(x))\Delta Y'(x)]\, dx \\ &= F(Y(x_1), Y'(x_1))\Delta x_1 + F_z(Y(x), y'(x))\Delta Y(x)\Big|_0^{x_1} \\ &\quad + \int_0^{x_1} \left[F_y(Y(x), Y'(x)) - \frac{d}{dx}F_z(Y(x), Y'(x))\right] \Delta Y(x)dx. \end{aligned}$$

By Theorem 3.10, we have

$$\delta\mathcal{J}(x_1, Y; \Delta x_1, \Delta Y) = \sum_{i=0}^{2} \lambda_i \delta\mathcal{K}_i(x_1, Y; \Delta x_1, \Delta Y)$$

for some constants λ_i, $i = 0, 1, 2$. Consequently, by letting $\Delta x_1 = \Delta Y(x_1) = \Delta Y(0) = 0$,

$$F_y(Y(x), Y'(x)) - \frac{d}{dx}F_z(Y(x), Y'(x)) = \lambda_2 \tag{4.3.37}$$

which implies

$$\frac{d}{dx}\left[F(Y(x), Y'(x)) - Y'(x)F_z(Y(x), y'(x)) - \lambda_2 Y(x)\right] = 0$$

and hence

$$F(Y(x), Y'(x)) - y'(x)F_z(Y(x), Y'(x)) = \lambda_2 Y(x) + C \tag{4.3.38}$$

for some constant. Therefore

$$\frac{Y(x)}{\sqrt{1 + Y'(x)^2}} = \lambda_2 Y(x) + C \Longrightarrow C = 0,$$

and

$$\lambda^2 = \frac{1}{\sqrt{1 + Y'(x)^2}} \Longrightarrow Y(x) = ax + b \tag{4.3.39}$$

for some constants a and b. By initial conditions, we conclude that

$$\frac{x}{x_1} + \frac{y}{y_0} = 1, \quad y = Y(x). \tag{4.3.40}$$

(3) Among all curves γ that have length ℓ and begin and end on the parabola $y = x^2$, it is desired to find such a curve that bounds the greatest possible area between itself and the given parabola. If γ^* is any such extremum curve, then γ^* must be an appropriate arc of the circle of radius r centered at the point $(0, -b)$, where r is determined by the equation

$$2r\left(\sin\frac{\ell}{2r}\right)^2 = \cos\frac{\ell}{2r}, \quad 0 < \frac{\ell}{2r} < \pi \tag{4.3.41}$$

and where b is then given as

$$b = \frac{\sqrt{1 + 16r^2} - 1}{8}. \tag{4.3.42}$$

Using the Euler-Lagrange multiplier theorem with three constraints, we can show that any extremum curve γ given as $\gamma : y = Y(x)$, $x_0 \leq x \leq x_1$, where x_0, x_1 are variable points, must satisfy

$$\frac{d}{dx}\left[\frac{Y'(x)}{\sqrt{1 + Y'(x)^2}}\right] = \text{constant}, \quad x_0 < x < x_1 \tag{4.3.43}$$

and the natural boundary conditions (see (4.3.19) and (4.3.21))

$$2x_0 Y'(x_0) = 2x_1 Y'(x_1) = -1 \tag{4.3.44}$$

along with the specified constraints

$$Y(x_0) = x_0^2, \quad Y(x_1) = x_1^2 \tag{4.3.45}$$

and

$$\int_{x_0}^{x_1} \sqrt{1+Y'(x)^2}dx = \ell. \tag{4.3.46}$$

Equation(4.3.43) can be integrated to give

$$(x+a)^2 + (y+b) = r^2, \quad y = Y(x), \tag{4.3.47}$$

for suitable constants a, b, r. From (4.3.44), (4.3.45), and (4.3.47), we obtain

$$x_i^2 + 2ax_i - b = 0, \quad i = 0, 1. \tag{4.3.48}$$

Since $x_0 < x_1$, it follows from (4.3.48) that

$$x_0 + a = -\sqrt{b+a^2}, \quad x_2 + a = \sqrt{b+a^2}, \tag{4.3.49}$$

and hence, from (4.3.45) and (4.3.47),

$$x_0 = -x_1 < 0, \quad a = 0, \quad x_1 = \sqrt{b} = -x_0. \tag{4.3.50}$$

Hence, (4.3.47) becomes

$$x^2 + (y+b)^2 = r^2, \tag{4.3.51}$$

from which we get

$$b + (b+b)^2 = r^2 \Longrightarrow b = \frac{\sqrt{1+r^2}-1}{8}.$$

Finally, (4.3.41) follows from (4.3.46):

$$\ell = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{rdx}{\sqrt{r^2-x^2}} = 2r\int_0^{\sqrt{b}} \frac{dx}{\sqrt{r^2-x^2}} \Longrightarrow \left(\sin\frac{\ell}{2r}\right)^2 = \frac{b}{r^2},$$

and

$$b = r^2\left(\sin\frac{\ell}{2r}\right)^2 = \frac{\sqrt{1+16r^2}-1}{8} \Longrightarrow 2r\left(\sin\frac{\ell}{2r}\right)^2 = \frac{\sqrt{1+16r^2}-1}{4r};$$

so

$$\frac{\sqrt{1+16r^2}-1}{8r^2} = 1 - \left(\cos\frac{\ell}{2r}\right)^2 \Longrightarrow \left(\cos\frac{\ell}{2r}\right)^2 = \left(\frac{\sqrt{1+16r^2}-1}{4r}\right)^2$$

and therefore

$$2r\left(\sin\frac{\ell}{2r}\right)^2 = \cos\left(\frac{\ell}{2r}\right).$$

4.4. **Functionals involving several unknown functions.** Consider the extremum problem of a functional $\mathscr{J}$ of the form

$$\mathscr{J} = \int_{x_0}^{x_1} F(x, Y_1(x), \cdots, Y_n(x), Y_1'(x), \cdots, Y_n'(x))dx, \tag{4.4.1}$$

which depends on n unknown functions $Y_1, \cdots, Y_n \in \mathscr{C}^1[x_0, x_1]$, with constraints

$$Y_i(x_0) = a_i, \quad Y_i(x_1) = b_i, \quad i = 1, \cdots, n, \tag{4.4.2}$$

where $a_1, \cdots, a_n, b_1, \cdots, b_n$ are given constants, and $F = F(x, y_1, \cdots, y_n, z_1, \cdots, z_n)$.

Set

$$\mathbf{Y}(x) = (Y_1(x), \cdots, Y_n(x)), \tag{4.4.3}$$

and

$$\mathbf{Y}(x_0) = \mathbf{a} = (a_1, \cdots, a_n), \quad \mathbf{Y}(x_1) = \mathbf{b} = (b_1, \cdots, b_n).$$

Then (4.4.1) can be written as

$$\mathscr{J}(\mathbf{Y}) = \int_{x_0}^{x_1} F(x, \mathbf{Y}(x), \mathbf{Y}'(x))dx. \tag{4.4.4}$$

We take the domain $\mathscr{D}$ of the functional $\mathscr{J}$ to be the entire vector space $\mathscr{X}$ which consists of all vector functions $\mathbf{Y} = (Y_1, \cdots, Y_n)$ whose components $Y_i \in \mathscr{C}^1[x_0, x_1]$ for all $i = 1, \cdots, n$. We equip $\mathscr{X}$ with the norm $||\cdot||_{\mathscr{X}}$ defined by

$$||\mathbf{Y}||_{\mathscr{X}} = \sum_{i=1}^{n} \left(\max_{x \in [x_0, x_1]} |Y_i(x)| + \max_{x \in [x_0, x_1]} |Y_i'(x)| \right)$$

for any vector $\mathbf{Y} = (Y_1, \cdots, Y_n) \in \mathscr{X}$.

If we define functionals $\mathcal{K}_i$ and $\mathcal{L}_i$ on $\mathscr{X}$ by

$$\mathcal{K}_i(\mathbf{Y}) = Y_i(x_0), \quad \mathcal{L}_i(\mathbf{Y}) = Y_i(x_1), \quad i = 1, \cdots, n, \tag{4.4.5}$$

then the fixed endpoint extremum problem is to find local extremum vectors in $\mathscr{D}[\mathcal{K}_i = a_i, \mathcal{L}_i = b_i, i = 1, \cdots, n]$ for the functional $\mathscr{J}$ of (4.4.4), where $\mathscr{D} = \mathscr{X}$. Note that

$$\delta\mathcal{K}_i(\mathbf{Y}; \Delta\mathbf{Y}) = \Delta Y_i(x_0), \tag{4.4.6}$$

$$\delta\mathcal{L}_i(\mathbf{Y}; \Delta\mathbf{Y}) = \Delta Y_i(x_1) \tag{4.4.7}$$

for any vector $\Delta\mathbf{Y} = (\Delta Y_1, \cdots, \Delta Y_n) \in \mathscr{X}$. From

$$\mathscr{J}(\mathbf{Y} + \epsilon\Delta\mathbf{Y}) = \int_{x_0}^{x_1} F(x, \mathbf{Y}(x) + \epsilon\Delta\mathbf{Y}(x), \mathbf{Y}'(x) + \epsilon\Delta\mathbf{Y}'(x))dx,$$

we have

$$\begin{aligned} \delta\mathscr{J}(\mathbf{Y}; \Delta\mathbf{Y}) &= \sum_{i=1}^{n} \int_{x_0}^{x_1} [F_{y_i}(x, \mathbf{Y}(x), \mathbf{Y}'(x))\Delta Y_i(x) \\ &\quad + F_{z_i}(x, \mathbf{Y}(x), \mathbf{Y}'(x))\Delta Y_i'(x)]\, dx \end{aligned}$$

which can be written as

$$\begin{aligned} &\delta\mathscr{J}(\mathbf{Y}; \Delta\mathbf{Y}) \\ = &\sum_{i=1}^{n} \int_{x_0}^{x_1} \left[F_{y_i}(x, \mathbf{Y}(x), \mathbf{Y}'(x)) - \frac{d}{dx} F_{z_i}(x, \mathbf{Y}(x), \mathbf{Y}'(x)) \right] \Delta Y_i(x)dx \\ + &\sum_{i=1}^{n} [F_{w_i}(x_1, \mathbf{Y}(x_1), \mathbf{Y}'(x_1))\Delta Y_i(x_1) - F_{z_i}(x_0, \mathbf{Y}(x_0), \mathbf{Y}'(x_0))\Delta Y_i(x_0)] \end{aligned} \tag{4.4.8}$$

for any vector $\Delta\mathbf{Y} = (\Delta Y_1, \cdots, \Delta Y_n) \in \mathscr{X}$. By Theorem 3.10 (where the second case is only true), we get

$$\delta\mathscr{J}(\mathbf{Y}; \Delta\mathbf{Y}) = \sum_{i=1}^{n} [\lambda_i \delta\mathcal{K}_i(\mathbf{Y}; \Delta\mathbf{Y}) + \mu_i \delta\mathcal{L}_i(\mathbf{Y}; \Delta\mathbf{Y})] \tag{4.4.9}$$

for some constants $\lambda_1, \cdots, \lambda_n, \mu_1, \cdots, \mu_n$ and all vectors $\mathbf{\Delta Y} = (\Delta Y_1, \cdots, \Delta Y_n) \in \mathscr{X}$. Plugging (4.4.5) and (4.4.8) into (4.4.9) yields

$$\begin{aligned} &\sum_{i=1}^{n} \int_{x_0}^{x_1} \left[F_{y_i}(x, \mathbf{Y}(x), \mathbf{Y}'(x)) - \frac{d}{dx} F_{z_i}(x, \mathbf{Y}(x), \mathbf{Y}'(x)) \right] \Delta Y_i(x) dx \\ (4.4.10) = \quad &\sum_{i=1}^{n} \left[F_{z_i}(x_0, \mathbf{a}, \mathbf{Y}'(x_0)) + \lambda_i \right] \Delta Y_i(x_0) \\ &+ \sum_{i=1}^{n} \left[-F_{z_i}(x_1, \mathbf{b}, \mathbf{Y}'(x_1)) + \mu_i \right] \Delta Y_i(x_1). \end{aligned}$$

If j is any fixed integer $(1 \le j \le n)$ we can choose each $\Delta Y_i \equiv 0$ for all $i \ne j$ so that

$$\begin{aligned} &\int_{x_0}^{x_1} \left[F_{y_j}(x, \mathbf{Y}(x), \mathbf{Y}'(x)) - \frac{d}{dx} F_{z_j}(x, \mathbf{Y}(x), \mathbf{Y}'(x)) \right] \Delta Y_j(x) dx \\ = \quad &\left[F_{z_j}(x_0, \mathbf{a}, \mathbf{Y}'(x_0)) + \lambda_j \right] \Delta Y_j(x_0) + \left[-F_{z_j}(x_1, \mathbf{b}, \mathbf{Y}'(x_1)) + \mu_j \right] \Delta Y_j(x_1) \end{aligned}$$

which holds for every function $\Delta Y_j \in \mathscr{C}^1[x_0, x_1]$. By Du Bois-Reymond's lemma and the argument used to derive (4.3.14), we can conclude that the n extremum functions $Y_1(x), \cdots, Y_n(x)$ must satisfy

$$F_{y_i}(x, \mathbf{Y}(x), \mathbf{Y}'(x)) - \frac{d}{dx} F_{z_i}(x, \mathbf{Y}(x), \mathbf{Y}'(x)) = 0 \tag{4.4.11}$$

for all $x \in [x_0, x_1]$ and $i = 1, \cdots, n$.

Example 4.10. Consider $n = 2$ and

$$F(x, y_1, y_2, z_1, z_2) = -2y_1^2 + 2y_1 y_2 - z_1^2 + z_2^2. \tag{4.4.12}$$

We seek to minimize or maximize the functional

$$\mathcal{J}(\mathbf{Y}) = \int_{x_0}^{x_1} F(x, Y_1(x), Y_2(x), Y_1'(x), Y_2'(x)) dx, \quad \mathbf{Y} = (Y_1, Y_2),$$

among all vectors $\mathbf{Y} = (Y_1, Y_2)$ with $Y_i \in \mathscr{C}^1[x_0, x_1]$, subject to the fixed endpoint conditions

$$Y_1(x_0) = a_0, \; Y_2(x_0) = a_2, \; Y_1(x_1) = b_1, \; Y_2(x_1) = b_2. \tag{4.4.13}$$

Since

$$\begin{aligned} F_{y_1}(x, y_1, y_2, z_1, z_2) &= -4y_1 + 2y_2, \\ F_{y_2}(x, y_1, y_2, z_1, z_2) &= 2y_1, \\ F_{z_1}(x, y_1, y_2, z_1, z_2) &= -2z_1, \\ F_{z_2}(x, y_1, y_2, z_1, z_2) &= 2z_2, \end{aligned}$$

it follows from (4.4.11) that

$$\begin{cases} 0 = -2Y_1(x) + Y_2(x) + Y_1''(x), \\ 0 = Y_1(x) - Y_2''(x). \end{cases} \tag{4.4.14}$$

Consequently,

$$Y_2^{(4)}(x) - 2Y_2''(x) + Y_2(x) = 0. \tag{4.4.15}$$

Introducing

$$u(x) \doteqdot Y_2''(x) - Y_2(x), \tag{4.4.16}$$

we have

$$\begin{aligned} u''(x) &= Y_2^{(4)}(x) - Y_2''(x) &= 2Y_2''(x) - Y_2(x) - Y_2''(x) \\ &= Y_1''(x) - Y_2(x) &= u(x). \end{aligned}$$

Hence

$$Y_2''(x) - Y_2(x) = u(x) = Ae^{x} + Be^{-x} \tag{4.4.17}$$

for some constants A and B. If the right-hand side of (4.4.17) is zero, then we have $Y_2(x) = Ce^x + De^{-x}$. Therefore, we may consider

$$Y_2(x) = C(x)e^x + D(x)e^{-x} \tag{4.4.18}$$

for suitable functions $C(x)$ and $D(x)$. From

$$Y_2''(x) = \left[C''(x) + 2C'(x) + C(x)\right]e^x + \left[D''(x) - 2D'(x) + D(x)\right]e^{-x},$$

we arrive at

$$C''(x) + 2C'(x) = A, \quad D''(x) - 2D'(x) = B. \tag{4.4.19}$$

Solving (4.4.19) implies

$$C(x) = \frac{A}{2}x + C_0 + \frac{C_1}{2}e^{-2x}, \tag{4.4.20}$$

$$D(x) = -\frac{B}{2}x + D_0 + \frac{D_1}{2}e^{2x}, \tag{4.4.21}$$

for arbitrary constants C_0, C_1, D_0, D_1. Substituting (4.4.20) and (4.4.21) into (4.4.18), we find that

$$Y_2(x) = \left(C_0 + \frac{D_1}{2}\right)e^x + \left(\frac{C_1}{2} + D_0\right)e^{-x} + \frac{x}{2}\left(Ae^x - Be^{-x}\right), \tag{4.4.22}$$

and

$$Y_2(x) = \left(C_0 + \frac{D_0}{2} + A\right)e^x + \left(\frac{C_1}{2} + D_0 + B\right)e^{-x} + \frac{x}{2}\left(Ae^x - Be^{-x}\right). \tag{4.4.23}$$

Example 4.11. (Hamilton's principle of stationary action) Netwon's second law of motion states that the total force $\mathbf{F} = (F_1, F_2, F_3)$ which acts on a particle located at a position $\mathbf{y} = (y_1, y_2, y_3)$ will cause a motion of the particle along a curve γ given as

$$\gamma : \mathbf{y} = \mathbf{Y}(t), \tag{4.4.24}$$

in accordance with the vector equation

$$m\mathbf{Y}'' = \mathbf{F} \tag{4.4.25}$$

where m is the mass of the particle and t denotes the time.

For simplicity, we assume that the total force $\mathbf{F}$ is the negative of the gradient of a real-valued function $V = V(\mathbf{y})$ as

$$\mathbf{F} = -\nabla V, \tag{4.4.26}$$

where V is called the potential associated with a particle of mass m located at $\mathbf{y}$ in the presence of the force $\mathbf{F}$. For example, the Newtonian gravitational force exerted on a particle of mass m at $\mathbf{y}$ due to a body of mass M located at the origin can be given as

$$\mathbf{F} = -\frac{GMm\mathbf{y}}{||\mathbf{y}||^3} = -\frac{GMm\mathbf{y}}{(y_1^2 + y_2^2 + y_3^2)^{3/2}} = -\nabla V, \tag{4.4.27}$$

where

$$V = -\frac{GMm}{||\mathbf{y}||} = -\frac{GMm}{(y_1^2 + y_2^2 + y_3^2)^{1/2}}. \tag{4.4.28}$$

Here G is the Newtonian gravitational constant.

The kinetic energy of such a particle moving along a path γ given as (4.4.24) is denoted as T and is defined by

$$T = \frac{1}{2}m||\mathbf{Y}'||^2 = \frac{1}{2}m\left[Y_1'(t)^2 + Y_2'(t)^2 + Y_3'(t)^2\right]. \tag{4.4.29}$$

From (4.4.26), (4.4.26), and (4.4.29), we have

$$\frac{d}{dt}(T+V) = m\langle \mathbf{Y}', \mathbf{Y}''\rangle + \langle \nabla V, \mathbf{Y}'\rangle = \langle \mathbf{Y}', m\mathbf{Y}'' + \nabla V\rangle = 0;$$

thus the total energy $T + V$ is conserved during the motion of a particle in the presence of a force $\mathbf{F} = -\nabla V$.

We consider the motion of a particle beginning at a fixed point $\mathbf{a} = (a_1, a_2, a_3)$ at time $t_{\mathbf{a}}$ and ending at a point $\mathbf{b} = (b_1, b_2, b_3)$ at time $t_{\mathbf{b}}$. Then

$$\gamma : \mathbf{y} = \mathbf{Y}(t), \quad t \in [t_{\mathbf{a}}, t_{\mathbf{b}}] \tag{4.4.30}$$

and the vector function $\mathbf{Y} = \mathbf{Y}(t)$ is required to satisfy the endpoint conditions

$$\mathbf{Y}(t_{\mathbf{a}}) = \mathbf{a}, \quad \mathbf{Y}(t_{\mathbf{b}}) = \mathbf{b}. \tag{4.4.31}$$

For any continuously differentiable vector function $\mathbf{Y} = \mathbf{Y}(t)$, consider the action of a motion of a particle of mass m along such a path γ, following Hamilton, as

$$\mathcal{A}(\mathbf{Y}) = \int_{t_{\mathbf{a}}}^{t_{\mathbf{b}}} \left[\frac{1}{2}m||\mathbf{Y}'(t)||^2 - V(\mathbf{Y}(t))\right] dt = \int_{t_{\mathbf{a}}}^{t_{\mathbf{b}}} L(\mathbf{Y}(t), \mathbf{Y}'(t))dt, \tag{4.4.32}$$

where

$$L(\mathbf{y}, \mathbf{z}) = T(\mathbf{z}) - V(\mathbf{y}) = \frac{1}{2}m||\mathbf{z}||^2 - V(\mathbf{y}) \tag{4.4.33}$$

is called the **Lagrangian function** of the motion.

Hamilton's principle of least action (for conservative forces) amounts to the assertion that from among all (actual, determined by (4.4.31), and hypothetical, determined by (4.4.31) and continuously differentiable) motions which begin at $\mathbf{a}$ at time $t_{\mathbf{a}}$ and end at $\mathbf{b}$ at time $t_{\mathbf{b}}$ the particle will actually experience that motion which minimizes the action; i.e., the actual motion will correspond to the function $\mathbf{Y} = \mathbf{Y}(t)$ which minimizes the action functional (4.4.32) among all continuously differentiable functions $\mathbf{Y} = \mathbf{Y}(t)$ which satisfy (4.4.31).

Then the Euler-Lagrange equation implies

$$L_{y_i}(\mathbf{Y}(t), \mathbf{Y}'(t)) = \frac{d}{dt}L_{z_i}(\mathbf{Y}(t), \mathbf{Y}'(t)), \quad i = 1, 2, 3 \tag{4.4.34}$$

for $t \in (t_{\mathbf{a}}, t_{\mathbf{b}})$. Since

$$\begin{aligned} L_{y_i}(\mathbf{y}, \mathbf{z}) &= -\frac{\partial}{\partial y_i}V(\mathbf{y}) &= F_i(\mathbf{y}), \\ L_{z_i}(\mathbf{Y}, \mathbf{z}) &= \frac{1}{2}m\frac{\partial}{\partial z_i}||\mathbf{z}||^2 &= mz_i, \end{aligned}$$

it follows that

$$F_i(\mathbf{Y}(t)) = \frac{d}{dt}mY_i'(t) = mY_i''(t); \tag{4.4.35}$$

thus $\mathbf{F} = m\mathbf{Y}''$.

As an application of Hamilton's principle, we consider the vibration of beads on an elastic string.

Example 4.12. (Vibration problem)For simplicity, we shall consider a case involving only two beads. The beads are attached to a light elastic string of length 4ℓ which is stretched at a large tension τ between two fixed points. One bead of mass m is located at the position 2ℓ, and a heavier bead of mass $2m$ is located at the position 3ℓ.

We consider only transverse vibrations in a fixed vertical plane, and we let $Y_1 = Y_1(t)$ and $Y_2 = Y_2(t)$ be the perpendicular displacements of the two beads from the equilibrium position of the string.

The kinetic energy of motion of the first bead of mass m is given as

$$T_1 = \frac{1}{2} m Y_1'(t)^2,$$

while the kinetic energy of motion of the second bead of mass $2m$ is

$$T_2 = m Y_2'(t)^2.$$

If we neglect the mass of the light string, then the total kinetic energy of the vibrating system is

$$T = m\left\{\frac{1}{2}Y_1'(t)^2 + Y_2'(t)^2\right\} \tag{4.4.36}$$

We assume that

$$\left\{\begin{array}{c}\text{potential energy of the}\\ \text{system due to the stretching}\\ \text{of the elastic string}\end{array}\right\} \text{ is proportional to } \left\{\begin{array}{c}\text{the amount by}\\ \text{which the string has}\\ \text{been stretched}\end{array}\right\}.$$

The left half of the string is stretched by

$$\begin{aligned}\sqrt{(2\ell)^2 + Y_1(t)^2} - 2\ell &= 2\ell\left(\sqrt{1 + \left(\frac{Y_1(t)}{2\ell}\right)^2} - 1\right)\\ &\approx 2\ell \cdot \frac{1}{2}\left(\frac{Y_1(t)}{2\ell}\right)^2\\ &= \frac{Y_1(t)^2}{4\ell}\end{aligned}$$

since we considered only small vibrations. Similarly, the stretch of the portion of the string between the two beads and the remaining piece of string are

$$\frac{[Y_2(t) - Y_1(t)]^2}{2\ell}, \quad \text{and} \quad \frac{Y_2(t)^2}{2\ell},$$

respectively. Therefore

$$V = \tau\left\{\frac{Y_1(t)^2}{4\ell} + \frac{[Y_2(t) - Y_1(t)]^2}{2\ell} + \frac{Y_2(t)^2}{2\ell}\right\}. \tag{4.4.37}$$

The action of the motion during a given time interval $[t_0, t_1]$ is given by

$$\mathcal{A}(\mathbf{Y}) = \int_{t_0}^{t_1} L(\mathbf{Y}(t), \mathbf{Y}'(t))dt, \tag{4.4.38}$$

where

$$L(\mathbf{y},\mathbf{z}) = m\left(\frac{1}{2}z_1^2 + z_2^2\right) - \tau\left[\frac{y_1^2}{4\ell} + \frac{(y_2-y_1)^2}{2\ell} + \frac{y_2^2}{2\ell}\right]. \tag{4.4.39}$$

Since

$$\begin{aligned} L_{y_1}(\mathbf{y},\mathbf{z}) &= -\tau\left(\frac{y_1}{2\ell} - \frac{y_2-y_1}{\ell}\right), \\ L_{y_2}(\mathbf{y},\mathbf{z}) &= -\tau\left(\frac{y_2}{\ell} + \frac{y_2-y_1}{\ell}\right), \\ L_{z_1}(\mathbf{y},\mathbf{z}) &= mz_1, \\ L_{z_2}(\mathbf{y},\mathbf{z}) &= 2mz_2, \end{aligned}$$

we conclude from (4.4.34) that

$$Y_1''(t) = \frac{\tau}{2\ell m}\left(-3Y_1(t) + 2Y_2(t)\right), \tag{4.4.40}$$

$$Y_2''(t) = \frac{\tau}{2\ell m}\left(Y_1(t) - 2Y_2(t)\right). \tag{4.4.41}$$

To solving the system (4.4.40)–(4.4.41), consider

$$U_\alpha \doteqdot Y_1 + \alpha Y_2. \tag{4.4.42}$$

Then

$$\begin{aligned} U_\alpha'' &= \frac{\tau}{2\ell m}\left(-3Y_1 + 2Y_2 + \alpha Y_1 - 2\alpha Y_2\right) \\ &= \frac{\tau(\alpha-3)}{2\ell m}\left[Y_1 + \frac{2(1-\alpha)}{\alpha-3}Y_2\right]; \end{aligned}$$

choosing α such that

$$\alpha = \frac{2(1-\alpha)}{\alpha-3},$$

which gives us $\alpha = 2$ or -1, we have

$$U_\alpha'' = \frac{\tau(\alpha-3)}{2\ell m}U_\alpha. \tag{4.4.43}$$

Let

$$\omega \doteqdot \frac{\tau}{2\ell m}.$$

If $\alpha = 2$, then

$$U_2'' = -\omega^2 U_2, \quad U_2(t) = a_1 \cdot \cos[\omega(t-\theta_1)]; \tag{4.4.44}$$

if $\alpha = -1$, then

$$U_{-1}'' = -4\omega^2 U_{-1}, \quad U_{-1}(t) = a_2 \cdot \cos[2\omega(t-\theta_2)]. \tag{4.4.45}$$

Using (4.4.42), (4.4.44), and (4.4.45), we can solve $Y_1(t)$ and $Y_2(t)$.

4.5. **Functionals involving higher-order derivatives.** Consider the functional

$$\mathcal{J}(Y)=\int_{x_0}^{x_1} F(x,Y(x),Y'(x),Y''(x))dx, \tag{4.5.1}$$

where the function $F=F(x,y,z,w)$ is a specified given function defined for all points (x,y,z,w) in some open set in $\mathbf{R}^4$, and $Y\in\mathscr{C}^2[x_0,x_1]$. We use the norm

$$||Y||_{\mathscr{C}^2[x_0,x_1]}=\sum_{i=0}^{2}\max_{x\in[x_0,x_1]}\left|Y^{(i)}(x)\right|$$

on the vector space $\mathscr{C}^2[x_0,x_1]$. We also assume that the functional (4.5.1) is defined in some open subset $\mathscr{D}\subset\mathscr{C}^2[x_0,x_1]$.

We now consider the problem of minimizing or maximizing the functional (4.5.1) subject to the constraints

$$Y(x_0)=y_0,\quad Y(x_1)=y_1,\quad Y'(x_0)=m_0 \tag{4.5.2}$$

for given numbers y_0,y_1, and m_0. If we define functionals $\mathcal{K}_1,\mathcal{K}_2$, and $\mathcal{K}_3$ by

$$\mathcal{K}_1(Y)=Y(x_0),\quad \mathcal{K}_2(Y)=Y(x_1),\quad \mathcal{K}_3(Y)=Y'(x_0). \tag{4.5.3}$$

for any function $Y\in\mathscr{C}^2[x_0,x_1]$, then the problem is to find extremum vectors in $\mathscr{D}[\mathcal{K}_1=y_0,\mathcal{K}_2=y_1,\mathcal{K}_3=m_0]$ for the functional (4.5.1). Note that

$$\begin{array}{rcl}\delta\mathcal{K}_1(Y;\Delta Y)&=&\Delta Y(x_0),\\ \delta\mathcal{K}_2(Y;\Delta Y)&=&\Delta Y(x_1),\\ \delta\mathcal{K}_3(Y;\Delta Y)&=&\Delta Y'(x_0)\end{array} \tag{4.5.4}$$

for any function $\Delta Y\in\mathscr{C}^2[x_0,x_1]$, and also

$$\begin{array}{rcl}\delta\mathcal{J}(Y;\Delta Y)&=&\displaystyle\int_{x_0}^{x_1}[F_y(x,Y(x),Y'(x),Y''(x))\Delta Y(x)\\ &&+F_z(x,Y(x),Y'(x),Y''(x))\Delta Y'(x)\\ &&+F_w(x,Y(x),Y'(x),Y''(x))\Delta Y''(x)]\,dx.\end{array} \tag{4.5.5}$$

To apply Theorem 3.10, we need to show that

$$\det\doteqdot\det\begin{pmatrix}\delta\mathcal{K}_1(Y;\Delta Y_1)&\delta\mathcal{K}_1(Y;\Delta Y_2)&\delta\mathcal{K}_1(Y;\Delta Y_3)\\ \delta\mathcal{K}_2(Y;\Delta Y_1)&\delta\mathcal{K}_2(Y;\Delta Y_2)&\delta\mathcal{K}_2(Y;\Delta Y_3)\\ \delta\mathcal{K}_3(Y;\Delta Y_1)&\delta\mathcal{K}_3(Y;\Delta Y_2)&\delta\mathcal{K}_3(Y;\Delta Y_3)\end{pmatrix}$$

does not vanish identically. Indeed, we take

$$\Delta Y_1(x)=1,\quad \Delta Y_2(x)=\frac{x-x_0}{x_1-x_0},\quad \Delta Y_3(x)=\left(\frac{x_1-x}{x_1-x_0}\right)^2,$$

then

$$\det=\begin{vmatrix}1&0&1\\1&1&0\\0&\frac{1}{x_1-x_0}&\frac{-2}{x_1-x_0}\end{vmatrix}=\frac{-1}{x_1-x_0}<0.$$

Consequently, if $Y=Y(x)$ is a local extremum vector in $\mathscr{D}[\mathcal{K}_1=y_0,\mathcal{K}_2=y_1,\mathcal{K}_3=m_0]$, where $\mathscr{D}=\mathscr{C}^2[x_0,x_1]$, for $\mathcal{J}$, there exist constants $\lambda_1,\lambda_2,\lambda_3$ such that

$$\delta\mathcal{J}(Y;\Delta Y)=\lambda_1\delta\mathcal{K}_1(Y;\Delta Y)+\lambda_2\delta\mathcal{K}_2(Y;\Delta Y)+\lambda_3\delta\mathcal{K}_3(Y;\Delta Y) \tag{4.5.6}$$

holds for all vectors $\Delta Y\in\mathscr{C}^2[x_0,x_1]$.

Plugging (4.4.4) and (4.5.5) into (4.5.6) yields

$$\begin{aligned} & \lambda_1 \Delta Y(x_0) + \lambda_2 \Delta Y(x_1) + \lambda_3 \Delta Y'(x_0) \\ = & \int_{x_0}^{x_1} [F_y(x, Y(x), Y'(x), Y''(x)) \Delta Y(x) \\ & + F_z(x, Y(x), Y'(x), Y''(x)) \Delta Y'(x) + F_w(x, Y(x), Y'(x), Y''(x)) \Delta Y''(x)] \, dx. \end{aligned} \tag{4.5.7}$$

From

$$\begin{aligned} & \int_{x_0}^{x_1} F_z(x, Y(x), Y'(x), Y''(x)) \Delta Y'(x) dx \\ = & F_z(x_1, Y(x_1), Y'(x_1), Y''(x_1)) \Delta Y(x_1) \\ & - F_z(x_0, Y(x_0), Y'(x_0), Y''(x_0)) \Delta Y(x_0) \\ & - \int_{x_0}^{x_1} \Delta Y(x) \frac{d}{dx} F_z(x, Y(x), Y'(x), Y''(x)) dx \end{aligned}$$

and

$$\begin{aligned} & \int_{x_0}^{x_1} F_w(x, Y(x), Y'(x), Y''(x)) \Delta Y''(x) dx \\ = & \int_{x_0}^{x_1} F_w(x, Y(x), Y'(x), Y''(x)) d\Delta Y'(x) \\ = & F_w(x_1, Y(x_1), Y'(x_1), Y''(x_1)) \Delta Y'(x_1) \\ & - F_w(x_0, Y(x_0), Y'(x_0), Y''(x_0)) \Delta Y'(x_0) \\ & - \int_{x_0}^{x_1} \frac{d}{dx} F_w(x, Y(x), Y'(x), Y''(x)) \Delta Y'(x) dx \\ = & F_w(x_1, Y(x_1), Y'(x_1), Y''(x_1)) \Delta Y'(x_1) \\ & - F_w(x_0, Y(x_0), Y'(x_0), Y''(x_0)) \Delta Y'(x_0) \\ & - \Delta Y(x_1) \frac{d}{dx} F_w(x_1, Y(x_1), Y'(x_1), Y''(x_1)) \\ & + \Delta Y(x_0) \frac{d}{dx} F_w(x_0, Y(x_0), Y'(x_0), Y''(x_0)) \\ & + \int_{x_0}^{x_1} \Delta Y(x) \frac{d^2}{dx^2} F_w(x, Y(x), Y'(x), Y''(x)) dx. \end{aligned}$$

Hence (4.5.6) can be written as

$$\begin{aligned} & \int_{x_0}^{x_1} \left[F_y(x, Y(x), Y'(x), Y''(x)) - \frac{d}{dx} F_z(x, Y(x), Y'(x), Y''(x)) \right. \\ & \left. + \frac{d^2}{dx^2} F_w(x, Y(x), Y'(x), Y''(x)) \right] \Delta Y(x) dx \\ = & \left[\lambda_1 + F_z(x_0, Y(x_0), Y'(x_0), Y''(x_0)) - \frac{d}{dx} F_w(x_0, Y(x_0), Y'(x_0), Y''(x_0)) \right] \Delta Y(x_0) \\ & + \left[\lambda_2 - F_z(x_1, Y(x_1), Y'(x_1), Y''(x_1)) + \frac{d}{dx} F_w(x_1, Y(x_1), Y'(x_1), Y''(x_1)) \right] \Delta Y(x_1) \\ & + \left[\lambda_3 + F_w(x_0, Y(x_0), Y'(x_0), Y''(x_0)) \right] \Delta Y'(x_0) \\ & - F_w(x_1, Y(x_1), Y'(x_1), Y''(x_1)) \Delta Y'(x_1). \end{aligned}$$

In particular,

$$\begin{aligned}\int_{x_0}^{x_1}\Big[F_y(x,Y(x),Y'(x),Y''(x)) - \frac{d}{dx}F_z(x,Y(x),Y'(x),Y''(x))\\ + \frac{d^2}{dx^2}F_w(x,Y(x),Y'(x),Y''(x))\Big]\Delta Y(x)dx &= 0, \qquad (4.5.8)\end{aligned}$$

for all functions $\Delta Y \in \mathscr{C}^2[x_0,x_1]$ which satisfy $\Delta Y(x_0) = \Delta Y'(x_0) = \Delta Y(x_1) = \Delta Y'(x_1) = 0$. By Du Bois-Reymond's lemma, we have

$$\begin{aligned}F_y(x,Y(x),Y'(x),Y''(x)) - \frac{d}{dx}F_z(x,Y(x),Y'(x),Y''(x))\\ + \frac{d^2}{dx^2}F_w(x,Y(x),Y'(x),Y''(x)) &= 0, \qquad (4.5.9)\end{aligned}$$

and then

$$\begin{aligned}&\Big[\lambda_1 + F_z(x_0,Y(x_0),Y'(x_0),Y''(x_0)) - \frac{d}{dx}F_w(x_0,Y(x_0),Y'(x_0),Y''(x_0))\Big]\Delta Y(x_0)\\ &+ \Big[\lambda_2 - F_z(x_1,Y(x_1),Y'(x_1),Y''(x_1)) + \frac{d}{dx}F_w(x_1,Y(x_1),Y'(x_1),Y''(x_1))\Big]\Delta Y(x_1)\\ &+ [\lambda_3 + F_w(x_0,Y(x_0),Y'(x_0),Y''(x_0))]\Delta Y'(x_0)\\ &- F_w(x_1,Y(x_1),Y'(x_1),Y''(x_1))\Delta Y'(x_1) = 0.\end{aligned}$$

Choosing ΔY to be function such that

$$\Delta Y(x_0) = \Delta Y(x_1) = \Delta Y'(x_0) = 0, \quad \Delta Y'(x_1) \neq 0,$$

we find

$$F_w(x_1,y_1,Y'(x_1),Y''(x_1)) = 0. \qquad (4.5.10)$$

Example 4.13. Consider

$$F(x,y,z,w) = \frac{w^2}{(1+z^2)^{5/2}}, \quad x_0 = 0, \quad x_1 = a \qquad (4.5.11)$$

and

$$Y(0) = b, \quad Y(a) = 0, \quad Y'(0) = 0. \qquad (4.5.12)$$

By (4.5.9), we get

$$5\frac{d}{dx}\left(\frac{Y'(x)Y''(x)^2}{[1+Y'(x)^2]^{7/2}}\right) + 2\frac{d^2}{dx^2}\left(\frac{Y''(x)}{[1+Y'(x)^2]^{5/2}}\right) = 0. \qquad (4.5.13)$$

The condition (4.5.10) now becomes

$$Y''(a) = 0. \qquad (4.5.14)$$

Integrating (4.5.13) yields

$$\frac{d}{dx}\left(\frac{Y''(x)}{[1+Y'(x)^2]^{5/2}}\right) + \frac{5}{2}\frac{Y'(x)Y''(x)^2}{[1+Y'(x)^2]^{7/2}} = A \qquad (4.5.15)$$

for some suitable constant A. The equation (4.5.15) can be written as

$$Y'''(x) - \frac{5}{2}\frac{Y'(x)Y''(x)^2}{1+Y'(x)^2} = A[1+Y'(x)^2]^{5/2}. \qquad (4.5.16)$$

4.6. **Functionals involving several independent variables.** Consider the functional

$$\mathcal{J}(Z) = \iint_R F(x, y, Z(x,y), Z_x(x,y), Z_y(x,y))dxdy, \quad Z \in \mathscr{C}^1(R), \tag{4.6.1}$$

for some function $F = F(x, y, z, w, u)$, where R is some given fixed bounded open region in the (x, y)-plane. We assume that

$$Z(x,y) = \phi(x,y), \quad (x,y) \in \partial R, \tag{4.6.2}$$

for some given fixed continuous function defined on the boundary of R.

We now consider the problem of minimizing or maximizing the functional (4.6.1) among all functions Z defined on R which satisfy the boundary constraint (4.6.2). We also assume that the domain $\mathscr{D}$ of $\mathcal{J}$ is some given open subset of the normed vector space $\mathscr{C}^1(R + \partial R)$. We shall use the symbol

$$\mathscr{D}[Z(x,y) = \phi(x,y),\ (x,y) \in \partial R]$$

to denote the subset of $\mathscr{D}$ consisting of all vectors $Z \in \mathscr{D}$ which satisfy the boundary constraint (4.6.2) on ∂R. The extremum problem under consideration is to find local extremum vectors in $\mathscr{D}[Z(x,y) = \phi(x,y),\ (x,y) \in \partial R]$ for $\mathcal{J}$.

If Z^* is a local minimum vector in $\mathscr{D}[Z(x,y) = \phi(x,y),\ (x,y) \in \partial R]$ for $\mathcal{J}$, then

$$\mathcal{J}(Z^*) \le \mathcal{J}(Z^* + U) \tag{4.6.3}$$

for all functions U which vanish on the boundary ∂R and which lie in some ball centered at the zero vector in $\mathscr{C}^1(R + \partial R)$. Let $\mathscr{C}_0^1(R + \partial R)$ be the subspace of $\mathscr{C}^1(R + \partial R)$ consisting of all functions of class $\mathscr{C}^1$ on $R + \partial R$ which vanish on the boundary ∂R. Consider a new functional $\mathcal{J}_0$ defined by

$$\mathcal{J}_0(U) \doteqdot \mathcal{J}(Z^* + U) \tag{4.6.4}$$

for all vectors U in some ball $\mathscr{D}_0 = B_\rho(0) \subset \mathscr{C}_0^1(R + \partial R)$. Then (4.6.3) can be written as

$$\mathcal{J}_0(0) \le \mathcal{J}_0(U), \quad U \in \mathscr{D}_0. \tag{4.6.5}$$

By Theorem 2.3, we have

$$\lim_{\epsilon \to 0} \frac{\mathcal{J}_0(\epsilon \Delta U) - \mathcal{J}_0(0)}{\epsilon} = \delta \mathcal{J}_0(0; \Delta U) = 0 \tag{4.6.6}$$

for all vectors $\Delta U \in \mathscr{C}_0^1(R + \partial R)$. Thus

$$\delta \mathcal{J}(Z^*; \Delta Z) = \lim_{\epsilon \to 0} \frac{\mathcal{J}(Z^* + \epsilon \Delta Z) - \mathcal{J}(Z^*)}{\epsilon} = 0 \tag{4.6.7}$$

for all vectors $\Delta Z \in \mathscr{C}_0^1(R + \partial R)$.

Note that

$$\begin{aligned} \delta \mathcal{J}(Z; \Delta Z) \quad = \quad & \iint_R [F_z(x, y, Z(x,y), Z_x(x,y), Z_y(x,y))\Delta Z(x,y) \\ & + F_w(x, y, Z(x,y), Z_x(x,y), Z_y(x,y))\Delta Z_x(x,y) \\ & + F_u(x, y, Z(x,y), Z_x(x,y), Z_y(x,y))\Delta Z_y(x,y)]\, dxdy \end{aligned} \tag{4.6.8}$$

for any vector $Z \in \mathscr{D}$ and any vector $\Delta Z \in \mathscr{C}^1(R + \partial R)$. Here $\Delta Z_x = \partial \Delta Z / \partial x$ and $\Delta Z_y = \partial \Delta Z / \partial y$. The Green theorem states that

(4.6.9)

$$\iint_R f_x(x,y)dxdy = \oint_{\partial R} f(x,y)N_x ds, \quad \iint_R f_y(x,y)dxdy = \oint_{\partial R} f(x,y)N_y ds$$

for any function $f \in \mathscr{C}^1(R+\partial R)$, where $\mathbf{N} = (N_x, N_y)$ denotes the exterior-directed unit normal vector on ∂R. Consequently,

$$\iint_R F_w \Delta Z_x dxdy = -\iint_R \left(\frac{\partial}{\partial x}F_w\right)\Delta Z dxdy + \oint_{\partial R} F_w \Delta Z N_x ds, \tag{4.6.10}$$

$$\iint_R F_u \Delta Z_y dxdy = -\iint_R \left(\frac{\partial}{\partial y}F_u\right)\Delta Z dxdy + \oint_{\partial R} F_u \Delta Z N_y ds. \tag{4.6.11}$$

Plugging (4.6.10) and (4.6.11) into (4.6.8), we get

$$\begin{aligned} \delta\mathcal{J}(Z;\Delta Z) = & \iint_R \Big[-\frac{\partial}{\partial x}F_w(x,y,Z(x,y),Z_x(x,y),Z_y(x,y)) \\ & +F_z(x,y,Z(x,y),Z_x(x,y),Z_y(x,y)) \\ & -\frac{\partial}{\partial y}F_u(x,y,Z(x,y),Z_x(x,y),Z_y(x,y))\Big]\Delta Z(x,y)dxdy \\ & + \oint_{\partial R}[F_w(x,y,Z(x,y),Z_x(x,y),Z_y(x,y))N_x \\ & +F_u(x,y,Z(x,y),Z_x(x,y),Z_y(x,y))N_y]\,\Delta Z(x,y)ds. \end{aligned} \tag{4.6.12}$$

By (4.6.7), we find that for any extremum vector $Z = Z(x,y)$ we have

$$\begin{aligned} 0 = & \iint_R \Big[-\frac{\partial}{\partial x}F_w(x,y,Z(x,y),Z_x(x,y),Z_y(x,y)) \\ & +F_z(x,y,Z(x,y),Z_x(x,y),Z_y(x,y)) \\ & -\frac{\partial}{\partial y}F_u(x,y,Z(x,y),Z_x(x,y),Z_y(x,y))\Big]\Delta Z(x,y)dxdy \end{aligned} \tag{4.6.13}$$

for all functions $\Delta Z \in \mathscr{C}_0^1(R+\partial R)$. According to Du Bois-Reymond's lemma, we get the following Euler-Lagrange equation

$$\begin{aligned} 0 = & F_z(x,y,Z(x,y),Z_x(x,y),Z_y(x,y)) - \frac{\partial}{\partial x}F_w(x,y,Z(x,y),Z_x(x,y),Z_y(x,y)) \\ & - \frac{\partial}{\partial y}F_u(x,y,Z(x,y),Z_x(x,y),Z_y(x,y)). \end{aligned} \tag{4.6.14}$$

Example 4.14. (Minimal surface) If we take F to be defined by

$$F(x,y,z,w,u) \doteqdot \sqrt{1+w^2+u^2}, \tag{4.6.15}$$

then

$$\mathcal{J}(Z) = \iint_R \sqrt{1+Z_x(x,y)^2+Z_y(x,y)^2}dxdy \tag{4.6.16}$$

gives the surface area of the graph of Z in $\mathbf{R}^3$. The graph of any extremum function Z for the surface area functional (), subject to the boundary condition (4.6.2), is called a **minimal surface**. In this case, (4.6.14) becomes

$$\frac{\partial}{\partial x}\left(\frac{Z_x}{\sqrt{1+Z_x^2+Z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{Z_y}{\sqrt{1+Z_x^2+Z_y^2}}\right) = 0, \tag{4.6.17}$$

or

$$(1+Z_y^2)Z_{xx} - 2Z_xZ_yZ_{xy} + (1+Z_x^2)Z_{yy} = 0. \tag{4.6.18}$$

The above equation is called the **minimal surface equation**.

Example 4.15. (Vibrating string) In Example 4.12, we considered the small transverse vibrations of a continuous string and now we consider a vibrating elastic string of uniform cross section.

- Let m denote the total mass of the string.
- Let ℓ denote the length of the quiet string at rest in its equilibrium positions.
- We take the quiet string to coincide with the interval $[0, \ell]$ along the x-axis, and we suppose that the vibrations occurs in the (x, z)-plane.
- We assume that the string can be given parametrically as

$$\gamma : z = Z(x,t), \quad x \in [0, \ell] \tag{4.6.19}$$

for some suitable function $Z = Z(x,t)$, and impose the constraints

$$Z(0,t) = Z(\ell, t) = 0 \tag{4.6.20}$$

for all t.
- The time interval is given by $[t_0, t_1]$.
- We shall assume that the potential energy V due to the stretching of the elastic string is proportional to the amount by which the string has been stretched. Hence

$$V = \tau \left[\int_0^\ell \sqrt{1 + Z_x(x,t)^2} dx - \int_0^\ell dx \right] = \tau \int_0^\ell \left[\sqrt{1 + Z_x(x,t)^2} - 1 \right] dx \tag{4.6.21}$$

for some given positive proportionality factor τ which gives a measure of the tension in the string. When Z_x is small, the quantity (4.6.21) can be approximated by

$$V_0 = \frac{\tau}{2} \int_0^\ell Z_x(x,t)^2 dx. \tag{4.6.22}$$

- Finally, we shall consider only the case of transversal motions in which each piece of the string vibrates up and down. In this case, the kinetic energy T is given by

$$T = \frac{m}{2\ell} \int_0^\ell Z_t(x,t)^2 dx. \tag{4.6.23}$$

Indeed, if we divide the initial interval $[0, \ell]$ into n subintervals given as $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$ with $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = \ell$. Then the initial (at time t_0) mass m_i of the ith piece of string is

$$m_i = \frac{x_i - x_{i-1}}{\ell} m,$$

from which the mass m_i of the ith piece of string is constant during the entire motion (because of the transversal motions of the string where each piece vibrates up and down), with

$$m_i = \frac{x_i - x_{i-1}}{\ell} m, \quad t \in [t_0, t_1]$$

and for $i = 1, \cdots, n$. Therefore, where $\bar{x}_i \in (x_{i-1}, x_i)$,

$$\begin{aligned} T &= \lim_{n\to\infty} \sum_{i=1}^{n} T_i = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} m_i Z_t(\bar{x}_i, t)^2 \\ &= \lim_{n\to\infty} \frac{m}{2\ell} \sum_{i=1}^{n} (x_i - x_{i-1}) Z_t(\bar{x}_i, t)^2 = \frac{m}{2\ell} \int_0^{\ell} Z_t(x,t)^2 dx. \end{aligned}$$

From (4.6.22) and (4.6.23), the action functional $\mathcal{A}_0$ for this model is

$$\mathcal{A}_0 = \int_{t_0}^{t_1} (T - V_0) dt = \iint_R F(Z_x(x,t), Z_t(x,t)) dx dt, \tag{4.6.24}$$

where

$$F(p,q) = \frac{1}{2}\left(\frac{m}{\ell} q^2 - \tau p^2\right), \quad R = \{(x,t) \in \mathbf{R}^2 : x \in (0,\ell),\ t \in (t_0, t_1)\}. \tag{4.6.25}$$

Hamilton's principle asserts that the actual motion of the string is described by a function $Z = Z(x,t)$ which furnishes a minimum value to the action functional (4.6.24) among all continuously differentiable functions which coincide with $Z(x,t)$ at $t = t_0$ an at $t = t_1$ and which vanish at the end points $x = 0$ and $x = \ell$ (for $t \in [t_0, t_1]$). By (4.6.14), we get

$$\frac{\partial}{\partial x} F_p(Z_x(x,t), Z_t(x,t)) + \frac{\partial}{\partial t} F_q(Z_x(x,t), Z_t(x,t)) = 0, \tag{4.6.26}$$

and then

$$\rho Z_{tt} = \tau Z_{xx} \tag{4.6.27}$$

for $\rho = m/\ell$.

In some case, both $\rho = \rho(x)$ and $\tau = \tau(x)$ become nonnegative functions of x for $x \in [0,\ell]$. In this case, we can prove that, as (4.6.27), the small vibrations of the string are given by

$$\rho(x) Z_{tt}(x,t) = \frac{\partial}{\partial x}\left[\tau(x) Z_x(x,t)\right]. \tag{4.6.28}$$

Remark 4.16. Let the boundary of the plane region R be divided into two disjoint (distinct) parts $\partial_1 R$ and $\partial_2 R$, and let ϕ be a given function defined on $\partial_1 R$. If $Z(x,y)$ minimizes or maximizes the functional $\mathcal{J}$ of (4.6.1) among all functions satisfying the boundary condition $Z(x,y) = \phi(x,y)$ for all $(x,y) \in \partial_1 R$, then on $\partial_2 R$, the extremum function $Z4$ must satisfy the **natural boundary condition**

$$\begin{aligned} &F_w(x,y,Z(x,y),Z_x(x,y),Z_y(x,y))N_x \\ &+ F_u(x,y,Z(x,y),Z_x(x,y),Z_y(x,y))N_y = 0, \quad (x,y) \in \partial_2 R, \end{aligned} \tag{4.6.29}$$

where $\mathbf{N} = (N_x, N_y)$ denotes the exterior-directed unit normal vector on ∂R. In this case any such extremum function Z must satisfy the same Euler-Lagrange equation (4.6.14) throughout entire region R.

Remark 4.17. Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{p} = (p_1, p_2, p_3)$ denote arbitrary points in $\mathbf{R}^3$, and let $F = F(x_1, x_2, x_3, u, p_1, p_2, p_3) = F(\mathbf{x}, u, \mathbf{p})$ be a given function of the seven real variables $x_1, x_2, x_3, u, p_1, p_2, p_3$. Define a functional

$$\mathcal{J}(U) := \int_R F(\mathbf{x}, U(\mathbf{x}), \nabla U(\mathbf{x})) d\mathbf{x} \tag{4.6.30}$$

for any real-valued function $U = U(\mathbf{x})$ of class $\mathscr{C}^1$ on a given fixed open set $R \subset \mathbf{R}^3$, where ∇U is the gradient of U given as $\nabla U(\mathbf{x}) = (\partial U/\partial x_1, \partial U/\partial x_2, \partial U/\partial x_3) = (U_{x_1}, U_{x_2}, U_{x_3})$, and where the integral is a volume integral taken over the region R with $d\mathbf{x} = dx_1 dx_2 dx_3$. Given a function $\psi \in \mathscr{C}(\partial R)$, consider the set

$$\mathscr{D} := \{U \in \mathscr{C}^1(R) : U(x,y) = \phi(x,y) \text{ for all } (x,y) \in \partial R\}.$$

If U is a local extremum function in $\mathscr{D}$ for the functional $\mathscr{J}$, then

$$F_u(\mathbf{x}, U(\mathbf{x}), \nabla U(\mathbf{x})) - \sum_{1 \le i \le 3} \frac{\partial}{\partial x_i} F_{p_i}(\mathbf{x}, U(\mathbf{x}), \nabla U(\mathbf{x})) = 0 \tag{4.6.31}$$

for all points $\mathbf{x} \in R$.

5. Application II: Sturm-Liouville eigenvalues

5.1. **Sturm-Liouville problems.** The equation (4.6.28) is a special case of the following

$$\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W(x)\right] + q(x)W(x) = -\lambda\rho(x)W(x), \tag{5.1.1}$$

where the function $W = W(x)$, $x \in (x_0, x_1)$, satisfies certain endpoint conditions at $x = x_0$ and $x = x_1$. The functions $\tau(x), q(x)$, and $\rho(x)$ are given functions, and λ is a parameter called a **eigenvalue**.

The Sturm-Liouville problem is to study the eigenvalues and we shall see below that those eigenvalues can be obtained as the solutions of certain extremum problems.

Example 5.1. We solve the equation (4.6.28) subject to the fixed endpoint conditions

$$Z(0,t) = Z(\ell,t) = 0, \quad t \ge 0 \tag{5.1.2}$$

and the initial conditions

$$Z(x,0) = \phi(x), \quad Z_t(x,0) = \psi(x), \quad x \in [0,\ell]. \tag{5.1.3}$$

If $Z(x,t) = X(x)T(t)$, then

$$\frac{1}{T(t)}\frac{d^2}{dt^2}T(t) = \frac{1}{\rho(x)X(x)}\frac{d}{dx}\left[\tau(x)\frac{d}{dx}X(x)\right] \tag{5.1.4}$$

for all $x \in (0,\ell)$ and $t > 0$. Thus, for some constant λ, we have

$$\frac{d^2}{dt^2}T(t) = -\lambda T(t), \quad t > 0, \tag{5.1.5}$$

$$\frac{d}{dx}\left[\tau(x)\frac{d}{dx}X(x)\right] = -\lambda\rho(x)X(x), \quad x \in (0,\ell). \tag{5.1.6}$$

The endpoint conditions (5.1.2) reduce to

$$X(0) = 0, \quad X(\ell) = 0. \tag{5.1.7}$$

To avoid the trivial solution to (5.1.6), we impose the additional condition

$$\int_0^\ell \rho(x)X(x)^2 dx > 0. \tag{5.1.8}$$

If τ and ρ are positive constants, then (5.1.6) becomes

$$X''(x) = -\frac{\lambda}{c^2}X(x), \quad c^2 = \frac{\tau}{\rho} \tag{5.1.9}$$

with general solution given as

$$X(x) = A \cdot \sin\left(\sqrt{\lambda}\frac{x}{c}\right) + B \cdot \cos\left(\sqrt{\lambda}\frac{x}{c}\right) \tag{5.1.10}$$

for arbitrary constants of integration A and B. The boundary condition (5.1.7) implies that $B = 0$, and then the remaining boundary condition $X(\ell) = 0$ implies that

$$A \cdot \sin\left(\sqrt{\lambda}\frac{\ell}{c}\right) = 0. \tag{5.1.11}$$

Consequently, $A = 0$ or $\sqrt{\lambda}\ell/c = n\pi$ for some integer n. By (5.1.8), we must have

$$\lambda_n = \left(\frac{n\pi c}{\ell}\right)^2 = \left(\frac{n\pi}{\ell}\right)^2 \frac{\tau}{\rho}, \quad n \in \mathbf{N}. \tag{5.1.12}$$

Thus we find the nontrivial solution given as

$$X_n(x) = \sin\frac{n\pi x}{\ell} \tag{5.1.13}$$

or any (nonzero) constant multiple of this solution. By (5.1.5), we get the most general solution given by

$$T_n(t) = a_n \cdot \cos(\sqrt{\lambda_n}t) + b_n \cdot \sin\sqrt{\lambda_n}t = a_n \cdot \cos\frac{n\pi ct}{\ell} + b_n \cdot \sin\frac{n\pi ct}{\ell} \tag{5.1.14}$$

for arbitrary constants of integration a_n and b_n. The resulting product solution $Z = X_nT_n$ becomes

$$Z_n(x,t) = X_n(x)T_n(t) = \sin\frac{n\pi x}{\ell}\left[a_n \cdot \cos\frac{n\pi ct}{\ell} + b_n \cdot \sin\frac{n\pi ct}{\ell}\right]. \tag{5.1.15}$$

clearly that the special solution (5.1.15) will in general not satisfy the initial conditions (5.1.3). However, we can form sums

$$Z(x,t) = \sum_{n=1}^{\infty} Z_n(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{\ell}\left[a_n \cdot \cos\frac{n\pi ct}{\ell} + b_n \cdot \sin\frac{n\pi ct}{\ell}\right], \tag{5.1.16}$$

with the constants a_n and b_n related as

$$\phi(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{\ell}a_n, \quad \frac{\ell}{n\pi c}\psi(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{\ell}b_n.$$

By Fourier transform, a_n and b_n are determined by

$$a_n = \frac{2}{\ell}\int_0^{\ell} \phi(x)\sin\frac{n\pi x}{\ell}dx, \quad b_n = \frac{2}{nc\pi}\int_0^{\ell} \psi(x)\sin\frac{n\pi x}{\ell}dx \tag{5.1.17}$$

with $c = \sqrt{\tau/\rho}$.

The nth special product solution $Z_n(x,t)$ represents the nth fundamental vibration for the given string. The eigenfunction $X_n(x) = \sin(n\pi x/\ell)$ gives the sharp or form of this fundamental vibration at each t, while the function $T_n(t) = a_n \cdot \cos(\sqrt{\lambda_n}t) + b_n \cdot \sin\sqrt{\lambda_n}t$ causes each point of the string to vary periodically with time with a period $2\pi/\sqrt{\lambda_n} = 2\ell\sqrt{\rho/\tau}/n$.

5.2. **Rayleigh quotient and the lowest eigenvalue.** We shall now consider the Sturm-Liouville problem for the following equation

$$\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W(x)\right]+q(x)W(x)=-\lambda\rho(x)W(x),\quad x\in(x_0,x_1). \tag{5.2.1}$$

We assume that

- the given functions τ and ρ are positive,
- both ρ and q are continuous,
- τ is continuously differentiable, and
- $W(x)$ satisfies the boundary condition

$$W(x_0)=W(x_1)=0. \tag{5.2.2}$$

It can be showed that this Sturm-Liouville problem always has infinitely many eigenvalues

$$\lambda_1<\lambda_2<\cdots<\cdots,\quad \lim_{k\to\infty}\lambda_k=+\infty. \tag{5.2.3}$$

Furthermore, corresponding to each eigenvalue λ_n there is an eigenfunction $W_n(x)$ which is a nontrivial solution to the given boundary value problem, and any two eigenfunctions which correspond to the same eigenvalue must be constant multiples of each other.

Multiplying $W(x)$ on both sides of (5.2.1), we get

$$\lambda\rho(x)W(x)^2=-W(x)\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W(x)\right]-q(x)W(x)^2$$

and then

$$\lambda\int_{x_0}^{x_1}\rho(x)W(x)^2dx=\int_{x_0}^{x_1}\left\{\tau(x)\left[\frac{d}{dx}W(x)\right]^2-q(x)W(x)^2\right\}dx.$$

For convenience, we introduce two functionals

$$\mathcal{D}(W)\doteqdot\int_{x_0}^{x_1}\left[\tau(x)w'(x)^2-q(x)W(x)^2\right]dx, \tag{5.2.4}$$

$$\mathcal{H}(W)\doteqdot\int_{x_0}^{x_1}\rho(x)W(x)^2dx \tag{5.2.5}$$

for any vector $W\in\mathscr{C}_0^1[x_0,x_1]$, the vector space of all continuously differentiable functions on the interval $[x_0,x_1]$ which vanish at the endpoints. The **Rayleigh quotient** now is defined as

$$\mathcal{R}(W)\doteqdot\frac{\mathcal{D}(W)}{\mathcal{H}(W)} \tag{5.2.6}$$

for any nonzero vector $W\in\mathscr{C}_0^1[x_0,x_1]$.

If λ is any eigenvalue with corresponding eigenfunction W, then

$$\lambda=\mathcal{R}(W).$$

Proposition 5.2. (Rayleigh principle) *The lowest eigenvalue λ_1 is equal to the minimum value of the Rayleigh quotient $\mathcal{R}$. Namely,*

$$\lambda_1=\min_{W\in\mathscr{C}_0^1[x_0,x_1]}\mathcal{R}(W). \tag{5.2.7}$$

Proof. (Sketch) Since τ, ρ are positive and ρ, q are continuous, it follows that

$$\begin{aligned}\mathcal{D}(W) &\geq -||q||_{\mathscr{C}[x_0,x_1]} \int_{x_0}^{x_1} W(x)^2 dx, \\ \mathcal{H}(W) &\leq ||\rho||_{\mathscr{C}[x_0,x_1]} \int_{x_0}^{x_1} W(x)^2 dx\end{aligned}$$

and then $\mathcal{R}(W)$ is bounded below by $-||q||_{\mathscr{C}[x_0,x_1]}/||\rho||_{\mathscr{C}[x_0,x_1]}$, from which we can show that $\mathcal{R}$ achieves a minimum value on $\mathscr{C}_0^1[x_0, x_1]$.

If $W_1 \in \mathscr{C}_0^1[x_0, x_1]$ is a minimum vector for the functional $\mathscr{R}$, then

$$\delta\mathcal{R}(W_1; \Delta W) = 0$$

for all vectors $\Delta W \in \mathscr{C}_0^1[x_0, x_1]$. From the formula

$$\delta\mathcal{R}(W; \Delta W) = \frac{\delta\mathcal{D}(W; \Delta W) - \mathcal{R}(W)\delta\mathcal{H}(W; \Delta W)}{\mathcal{H}(W)},$$

we obtain

$$\delta\mathcal{D}(W_1; \Delta W) = \mathcal{R}(W_1)\delta\mathcal{H}(W_1; \Delta W)$$

for all vectors $\Delta W \in \mathscr{C}_0^1[x_0, x_1]$. On the other hand,

$$\begin{aligned}\delta\mathcal{D}(W; \Delta W) &= 2\int_{x_0}^{x_1} \left[\tau(x)W'(x)\Delta W'(x) - q(x)W(x)\Delta W(x)\right] dx \\ &= -2\int_{x_0}^{x_1} \left\{\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W(x)\right] + q(x)W(x)\right\} \Delta W(x) dx, \\ \delta\mathcal{H}(W; \Delta W) &= 2\int_{x_0}^{x_1} \rho(x)W(x)\Delta W(x) dx\end{aligned}$$

for all functions $\Delta W \in \mathscr{C}_0^1[x_0, x_1]$. Hence we obtain

$$\int_{x_0}^{x_1} \left\{\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W_1(x)\right] + q(x)W_1(x) + \lambda^*\rho(x)W_1(x)\right\} \Delta W(x) dx = 0$$

for all functions $\Delta W \in \mathscr{C}_0^1[x_0, x_1]$, where

$$\lambda^* \doteqdot \mathcal{R}(W_1) = \frac{\mathcal{D}(W_1)}{\mathcal{H}(W_1)}.$$

By Du Bois-Reymond's lemma, W_1 satisfy the following differential equation

$$\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W_1(x)\right] + q(x)W_1(x) = -\lambda^*\rho(x)W_1(x)$$

for $x \in (x_0, x_1)$. Thus, $\lambda^* = \mathcal{R}(W_1)$ is an eigenvalue and W_1 is the corresponding eigenfunction (up to an arbitrary multiplicative constant).

If λ is any eigenvalue with corresponding eigenfunction W, then

$$\lambda^* = \mathcal{R}(W_1) \leq \mathcal{R}(W) = \lambda,$$

since W_1 is a minimum vector in $\mathscr{C}_0^1[x_0, x_1]$ for $\mathcal{R}$. Hence $\lambda^* = \lambda_1$. □

Example 5.3. Consider the Sturm-Liouville problem

$$W''(x) - xW(x) = -\lambda W(x), \quad x \in (0, 1)$$

with $W(0) = W(1) = 0$. According to (5.2.4)–(5.2.6), we have

$$\mathcal{R}(W) = \frac{\int_0^1 [W'(x)^2 + xW(x)^2]dx}{\int_0^1 W(x)^2 dx} = \frac{\int_0^1 W'(x)^2 dx}{\int_0^1 W(x)^2 dx} + \frac{\int_0^1 xW(x)^2 dx}{\int_0^1 W(x)^2 dx}$$

and then the lowest eigenvalue λ_1 satisfies

$$\min_{W \in \mathscr{C}_0^1[0,1]} \frac{\int_0^1 W'(x)^2 dx}{\int_0^1 W(x)^2 dx} \leq \lambda_1 = \min_{W \in \mathscr{C}_0^1[0,1]} \mathcal{R}(W) \leq 1 + \min_{W \in \mathscr{C}_0^1[0,1]} \frac{\int_0^1 W'(x)^2 dx}{\int_0^1 W(x)^2 dx}.$$

If μ_1 is the lowest eigenvalue for the problem

$$W''(x) = -\mu W(x), \quad x \in (0,1)$$

with $W(0) = W(1) = 0$, then

$$\mu_1 \leq \lambda_1 \leq 1 + \mu_1.$$

Solving the above equation yields $\sqrt{\mu_n} = n\pi$ for $n \in \mathbf{N}$. Consequently, $\sqrt{\mu_1} = \pi$ and then $\mu_1 = \pi^2$. Hence $\pi^2 \leq \lambda_1 \leq 1 + \pi^2$.

5.3. **Rayleigh-Ritz method and the lowest eigenvalue.** In this section we shall describe another method, called Rayleigh-Ritz method, which is also based on Rayleigh's principle and which yields upper bounds on the lowest eigenvalue.

We shall consider the following Sturm-Liouville problem

$$\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W(x)\right] + q(x)W(x) = -\lambda\rho(x)W(x), \quad x \in (x_0, x_1),$$

with $W(x_0) = W(x_1) = 0$.

the idea of Rayleigh-Ritz method is as follows:

(a) Consider the problem of minimizing $\mathcal{R} = \mathcal{D}/\mathcal{H}$ over all functions $W \in \mathscr{C}_0^1[x_0, x_1|\psi_1, \cdots, \psi_n]$ the subspace spanned by a given collection of fixed functions $\psi_1, \cdots, \psi_n \in \mathscr{C}_0^1[x_0, x_1]$. Then any such a function W can be written as

$$W = \sum_{i=1}^n c_i \psi_i$$

for arbitrary constants $c_1, \cdots, c_n$. This simpler problem of minimizing the Rayleigh quotient $\mathcal{R}$ involves only a suitable choice for the n constants $c_1, \cdots, c_n$.

(b) Let $\mathbf{c} = (c_1, \cdots, c_n)$ and define two real-valued functions by

$$d(\mathbf{c}) = \mathcal{D}\left(\sum_{i=1}^n c_i\psi_i\right), \quad h(\mathbf{c}) = \mathcal{H}\left(\sum_{i=1}^n c_i\psi_i\right)$$

for any n-tuple $\mathbf{c} = (c_1, \cdots, c_n) \in \mathbf{R}^n$. Then

$$\mathcal{R}(W_\mathbf{c}) = \frac{d(\mathbf{c})}{h(\mathbf{c})}$$

for any nonzero function $W_\mathbf{c} = \sum_{i=1}^n c_i\psi_i$, and so the problem of minimizing the Rayleigh quotient over $\mathscr{C}_0^1[x_0, x_1|\psi_1, \cdots, \psi_n]$ is equivalent to the problem of minimizing the ratio $d(\mathbf{c})/h(\mathbf{c})$ over $\mathbf{R}^n$.

(c) If $\mathbf{c}^* = (c_1^*, \cdots, c_n^*)$ is a minimum vector in $\mathbf{R}^n$ for the ratio $d(\mathbf{c})/h(\mathbf{c})$, then

$$\frac{\partial}{\partial c_k}\left[\frac{d(\mathbf{c})}{h(\mathbf{c})}\right]\Big|_{\mathbf{c}=\mathbf{c}^*} = 0, \quad k = 1, \cdots, n$$

from which we find that

$$\frac{\partial}{\partial c_k} d(\mathbf{c}^*) = r^* \frac{\partial}{\partial c_k} h(\mathbf{c}^*) \tag{5.3.1}$$

for $k = 1, \cdots, n$, where $r^* = d(\mathbf{c}^*)/h(\mathbf{c}^*)$. By the definition, we have

$$d(\mathbf{c}) = \sum_{1\le i,j\le n} a_{ij} c_i c_j, \quad h(\mathbf{c}) = \sum_{1\le i,j\le n} b_{ij} c_i c_j, \tag{5.3.2}$$

where

$$a_{ij} = \int_{x_0}^{x_1} \left[\tau(x)\psi_i'(x)\psi_j'(x) - q(x)\psi_i(x)\psi_j(x)\right] dx, \tag{5.3.3}$$

$$b_{ij} = \int_{x_0}^{x_1} \rho(x)\psi_i(x)\psi_j(x) dx. \tag{5.3.4}$$

(d) From (5.3.1) and (5.3.2), we have

$$\sum_{j=1}^{n} a_{kj} c_j^* = r^* \sum_{j=1}^{n} b_{kj} c_j^* \tag{5.3.5}$$

or

$$(A - r^* B)\mathbf{c}^* = 0, \tag{5.3.6}$$

where $A = (a_{ij})_{1\le i,j\le n}$ and $B = (b_{ij})_{1\le i,j\le n}$. Hence, $\mathbf{c}^*$ is nonzero if and only if the matrix $A - r^*B$ is singular–that is, if and only if r^* satisfies the equation

$$\det(A - rB) = 0. \tag{5.3.7}$$

Each solution r of (5.3.7) is an eigenvalue for the matrix equation (5.3.6), and the smallest eigenvalue r_1 is the minimum value of the Rayleigh quotient $\mathcal{D}/\mathcal{H}$ over $\mathscr{C}_0^1[x_0, x_1|\psi_1, \cdots, \psi_n]$.

$$\left\{\begin{array}{c}\text{The smallest eigenvalue}\\ \text{of } \det(A - rB) = 0\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\text{The minimum value of the Rayleigh}\\ \text{quotient } \mathcal{D}/\mathcal{H} \text{ over } \mathscr{C}_0^1[x_0, x_1|\psi_1, \cdots, \psi_n]\end{array}\right\}$$

Consequently,

$$\lambda_1 \le r_1. \tag{5.3.8}$$

Example 5.4. Consider the equation

$$\tau W''(x) = -\lambda\rho W(x), \quad x \in (0, \ell)$$

with $W(0) = W(\ell) = 0$. Here τ and ρ are given positive constants. From (5.1.12), we have

$$\lambda_1 = \frac{\pi^2\tau}{\ell^2\rho} \approx \frac{9.87\tau}{\ell^2\rho} < \frac{10\tau}{\ell^{23}\rho}. \tag{5.3.9}$$

To use the Rayleigh-Ritz method, we define

$$\psi_1(x) = x(\ell - x), \quad \psi_2(x) = x(\ell^2 - x^2), \quad x \in [0, \ell].$$

Since

$$
\begin{aligned}
A &= \tau \begin{pmatrix} \int_0^\ell (\ell-2x)^2 dx & \int_0^\ell (\ell-2x)(\ell^2-3x^2)dx \\ \int_0^\ell (\ell-2x)(\ell^2-3x^2)dx & \int_0^\ell (\ell^2-3x^2)^2 dx \end{pmatrix} \\
&= \tau \ell^3 \begin{pmatrix} \frac{1}{3} & \frac{\ell}{2} \\ \frac{\ell}{2} & \frac{4}{5}\ell^2 \end{pmatrix}, \\
B &= \tau \begin{pmatrix} \int_0^\ell x^2(\ell-x)^2 dx & \int_0^\ell x^2(\ell-x)(\ell^2-x^2)dx \\ \int_0^\ell x^2(\ell-x)(\ell^2-x^2)dx & \int_0^\ell x^2(\ell^2-x^2)^2 dx \end{pmatrix} \\
&= \frac{\rho\ell^5}{5} \begin{pmatrix} \frac{1}{6} & \frac{\ell}{42} \\ \frac{\ell}{4} & \frac{8}{21}\ell^2 \end{pmatrix},
\end{aligned}
$$

it follows that (5.3.7) becomes

$$\rho^2\ell^4 r^2 - 52\rho\ell^2\tau r + 420\tau^2 = 0$$

and then the smallest eigenvalue r_1 over $\mathscr{C}_0^1[0,\ell|\psi_1,\psi_2]$ is

$$r_1 = \frac{10\tau}{\ell^2\rho}. \tag{5.3.10}$$

5.4. **Rayleigh quotient and Higher eigenvalues.** The lowest eigenvalue λ_1 for the Sturm-Liouville problem

$$\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W(x)\right] + q(x)W(x) = -\lambda\rho(x)W(x), \quad x \in (x_0, x_1) \tag{5.4.1}$$

with $W(x_0) = W(x_1) = 0$, can be characterized as

$$\lambda_1 = \min_{W\in\mathscr{C}_0^1[x_0,x_1]} \mathcal{R}(W) = \min_{W\in\mathscr{C}_0^1[x_0,x_1]} \frac{\mathcal{D}(W)}{\mathcal{H}(W)}, \tag{5.4.2}$$

where

$$
\begin{aligned}
\mathcal{D}(W) &= \int_{x_0}^{x_1} \left[\tau(x)W'(x)^2 - q(x)W(x)^2\right] dx, \\
\mathcal{H}(W) &= \int_{x_0}^{x_1} \rho(x)W(x)^2 dx.
\end{aligned}
$$

Proposition 5.5. *The second eigenvalue λ_2 can be characterized as*

$$\lambda_2 = \min\left\{\frac{\mathcal{D}(W)}{\mathcal{H}(W)} : W \in \mathscr{C}_0^1[x_0,x_1] \text{ with } \int_{x_0}^{x_1} \rho(x)W_1(x)W(x)dx = 0\right\} \tag{5.4.3}$$

where W_1 is the eigenfunction corresponding to the lowest eigenvalue λ_1.

Proof. If W_2 is a solution to the right extremum problem, then by Theorem 3.5, we have

$$\delta\mathcal{D}(W_2;\Delta W) - \mathcal{R}(W_2)\delta\mathcal{H}(W_2;\Delta H) = C\int_{x_0}^{x_1} \rho(x)W_1(x)\Delta W(x)dx$$

for all vectors $\Delta W \in \mathscr{C}_0^1[x_0,x_1]$. Thus

$$\int_{x_0}^{x_1}\left\{\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W_2(x)\right] + q(x)W_1(x) + \lambda^*\rho(x)W_2(x) + \mu\rho(x)W_1(x)\right\}\Delta W(x)dx = 0$$

for some suitable constant μ and for all vectors $\Delta W \in \mathscr{C}_0^1[x_0,x_1]$, where

$$\lambda^* = \mathcal{R}(W_2) = \frac{\mathcal{D}(W_2)}{\mathcal{H}(W_2)}.$$

By Du Bois-Reymond's lemma, W_2 satisfies the following differential equation

$$\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W_2(x)\right]+q(x)W_2(x)=-\lambda^*\rho(x)W_2(x)-\mu\rho(x)W_1(x)$$

for $x\in(x_0,x_1)$. Multiplying by W_1 on both sides and using the constraint yields

$$\begin{aligned}
&-\mu\int_{x_0}^{x_1}\rho(x)W_1(x)^2dx\\
=&\int_{x_0}^{x_1}\left\{\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W_2(x)\right]W_1(x)+q(x)W_2(x)W_1(x)\right\}dx\\
=&-\int_{x_0}^{x_1}\tau(x)\frac{d}{dx}W_2(x)\frac{d}{dx}W_1(x)dx+\int_{x_0}^{x_1}q(x)W_2(x)W_1(x)dx\\
=&\int_{x_1}^{x_2}\left\{\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W_1(x)\right]+q(x)W_1(x)\right\}W_2(x)dx\\
=&-\lambda_1\int_{x_0}^{x_1}\rho(x)W_1(x)W_2(x)dx \quad = \quad 0
\end{aligned}$$

so that $\mu=0$ and thus W_1 satisfies

$$\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W_2(x)\right]+q(x)W_2(x)=-\lambda^*\rho(x)W_2(x)$$

for $x\in(x_0,x_1)$.

Hence $\lambda^*=\mathcal{R}(W_2)$ is an eigenvalue for the Sturm-Liouville problem and W_2 (up to any constant multiplicative constants) is the corresponding eigenfunction. Now we can claim that

$$\lambda^*=\lambda_2. \tag{5.4.4}$$

Before proving this, we need the following

Lemma 5.6. *Let W^* and W^{**} be any two eigenfunctions for the Sturm-Liouville problem*

$$\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W(x)\right]+q(x)W(x)=-\lambda\rho(x)W(x),\quad x\in(x_0,x_1)$$

with $W(x_0)=W(x_1)=0$, corresponding to distinct eigenvalues λ^* and λ^{**}. Then W^* and W^{**} must satisfy the orthogonality condition

$$\int_{x_0}^{x_1}\rho(x)W^*(x)W^{**}(x)dx=0.$$

Proof. From

$$\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W^*(x)\right]+q(x)W^*(x)=-\lambda^*\rho(x)W^*(x),$$

we have

$$\begin{aligned}
&-\lambda^*\int_{x_0}^{x_1}\rho(x)W^*(x)W^{**}(x)dx\\
=&\int_{x_0}^{x_1}\frac{d}{dx}\left[\tau(x)\frac{d}{dx}W^*(x)\right]W^{**}(x)dx+\int_{x_0}^{x_1}q(x)W^*(x)W^{**}(x)dx\\
=&-\int_{x_0}^{x_1}\tau(x)\frac{d}{dx}W^*(x)\frac{d}{dx}W^{**}(x)dx+\int_{x_0}^{x_1}q(x)W^*(x)W^{**}(x)dx;
\end{aligned}$$

similarly,

$$
\begin{aligned}
&-\lambda^{**}\int_{x_0}^{x_1}\rho(x)W^*(x)W^{**}(x)dx\\
=\quad&-\int_{x_0}^{x_1}\tau(x)\frac{d}{dx}W^*(x)\frac{d}{dx}W^{**}(x)dx+\int_{x_0}^{x_1}q(x)W^*(x)W^{**}(x)dx.
\end{aligned}
$$

Therefore

$$(\lambda^*-\lambda^{**})\int_{x_0}^{x_1}\rho(x)W^*(x)W^{**}(x)dx=0,$$

from which we conclude that

$$\int_{x_0}^{x_1}\rho(x)W^*(x)W^{**}(x)dx=0$$

since $\lambda^*\neq\lambda^{**}$. □

We return to proof (5.4.4). If $\lambda\neq\lambda_1$ is any eigenvalue with corresponding eigenfunction W, then, by Lemma 5.6,

$$\int_{x_0}^{x_1}\rho(x)W_1(x)W(x)dx=0.$$

By the definition (5.4.3), we have

$$\lambda^*=\mathcal{R}(W_2)\leq\mathcal{R}(W)=\lambda$$

for any eigenvalue $\lambda\neq\lambda_1$. Hence $\lambda^*=\lambda_2$. □

If We already have the first $n-1$ eigenvalues $0<\lambda_1<\lambda_2<\cdots<\lambda_{n-1}$ with corresponding eigenfunctions $W_1,W_2,\cdots,W_{n-1}$, then we can show that the nth eigenvalue λ_n is characterized as

$$\lambda_n=\min\left\{\mathcal{R}(W):W\in\mathscr{C}_0^1[x_0,x_1],\ \int_{x_0}^{x_1}\rho(x)W_i(x)W(x)dx=0,\ i=1,\cdots,n-1\right\}.$$

5.5. **The Courant minimax principle.** Given $\phi_1,\cdots,\phi_{n-1}\in\mathscr{C}_0^1[x_0,x_1]$, set

(5.5.1)
$$C(\phi_1,\cdots,\phi_{n-1})=\min\left\{\mathcal{R}(W):W\in\mathscr{C}_0^1[x_0,x_1],\ \int_{x_0}^{x_1}\phi_i(x)W(x)dx=0,\ 1\leq i\leq n-1\right\}.$$

If, in particular, we take

$$\phi_i(x)=\rho(x)W_i(x)$$

for $i=1,\cdots,n-1$, where $W_1,\cdots,W_{n-1}$ are the first $(n-1)$ eigenfunctions, then

$$C(\rho W_1,\cdots,\rho W_{n-1})=\lambda_n.$$

Lemma 5.7. *For any $\phi_1,\cdots,\phi_{n-1}\in\mathscr{C}_0^1[x_0,x_1]$ we have*

$$C(\phi_1,\cdots,\phi_{n-1})\leq\lambda_n. \tag{5.5.2}$$

Proof. By (5.5.1) and (5.5.2), it suffices to find a function $W^*\in\mathscr{C}_0^1[x_0,x_1]$ such that

$$\mathcal{R}(W^*)\leq\lambda_n. \tag{5.5.3}$$

We may consider

$$W^*=\sum_{j=1}^{n}c_jW_j \tag{5.5.4}$$

for suitable constants $c_1, \cdots, c_n$ which must be determined so that W^* satisfies

$$0 = \int_{x_0}^{x_1} \phi_i(x) W^*(x) dx = \sum_{j=1}^{n} \gamma_{ij} c_j, \quad i = 1, \cdots, n-1,$$

where

$$\gamma_{ij} = \int_{x_0}^{x_1} \phi_i(x) W_j(x) dx.$$

but it is well known that such a system of $n-1$ linear homogeneous equations in n unknowns always has a nontrivial solution. Hence it is always find suitable constants $c_1, \cdots, c_n$ not all zero, such that the resulting function W^* satisfies the given $n-1$ constraints.

By (5.5.4), we get

$$\mathcal{R}(W^*) = \frac{\mathcal{D}(\sum_{j=1}^{n} c_j W_j)}{\mathcal{H}(\sum_{j=1}^{n} c_j W_j)} = \frac{\sum_{1 \le i,j \le n} c_i c_j \alpha_{ij}}{\sum_{1 \le i,j \le n} c_i c_j \beta_{ij}},$$

where

$$\begin{aligned} \alpha_{ij} &= \int_{x_0}^{x_1} \left[\tau(x) W_i'(x) W_j'(x) - q(x) W_i(x) W_j(x) \right] dx \\ &= -\int_{x_0}^{x_1} \left\{ \frac{d}{dx} \left[\tau(x) \frac{d}{dx} W_i(x) \right] + q(x) W_i(x) \right\} W_j(x) dx, \\ \beta_{ij} &= \int_{x_0}^{x_1} \rho(x) W_i(x) W_j(x) dx \quad = \quad \delta_{ij} \beta_{ij} \end{aligned}$$

by Lemma 5.6. Since W_i is the eigenfunction of the Sturm-Liouville problem corresponding to the eigenvalue λ_i, it follows that

$$\alpha_{ij} = \lambda_i \beta_{ij}.$$

Hence

$$\mathcal{R}(W^*) = \frac{\sum_{1 \le i,j \le n} c_i c_j \lambda_i \beta_{ij}}{\sum_{1 \le i,j \le n} c_i c_j \beta_{ij}} = \frac{\sum_{i=1}^{n} c_i^2 \lambda_i \beta_{ii}}{\sum_{i=1}^{n} c_i^2 \beta_{ii}},$$

where

$$\beta_{ii} = \int_{x_0}^{x_1} \rho(x) W_i(x)^2 dx > 0.$$

From $0 < \lambda_1 < \cdots < \lambda_n$, we get

$$\mathcal{R}(W^*) \le \frac{\sum_{i=1}^{n} c_i^2 \lambda_n \beta_{ii}}{\sum_{i=1}^{n} c_i^2 \beta_{ii}} = \lambda_n.$$

Thus we proved (5.5.3). □

Theorem 5.8. (Courant's minimax principle) *The nth eigenvalue λ_n is equal to the maximum value of the expression $C(\phi_1, \cdots, \phi_{n-1})$ over all possible functions $\phi_1, \cdots, \phi_{n-1} \in \mathscr{C}_0^1[x_0, x_1]$. That is*

$$\lambda_n = \max_{\phi_1, \cdots, \phi_n \in \mathscr{C}_0^1[x_0, x_1]} C(\phi_1, \cdots, \phi_n). \tag{5.5.5}$$

The minimax principle is well suited for the purpose of comparing the eigenvalues of different Sturm-Liouville problems. Let λ_n and λ_n^* denote the respective eigenvalues for the problems

$$\begin{aligned} \frac{d}{dx}\left[\tau(x)\frac{d}{dx}W(x)\right] + q(x)W(x) &= -\lambda\rho(x)W(x), \\ \frac{d}{dx}\left[\tau^*(x)\frac{d}{dx}W(x)\right] + q^*(x)W(x) &= -\lambda\rho^*(x)W(x) \end{aligned}$$

with $\lambda_1 < \cdots < \lambda_n < \cdots$ and $\lambda_1^* < \cdots < \lambda_n^* < \cdots$, where $x \in (x_0, x_1)$ and $W(x_0) = W(x_1) = 0$. Recall the corresponding Rayleigh quotients

$$\begin{aligned} \mathcal{R}(W) &= \frac{\int_{x_0}^{x_1}[\tau(x)W'(x)^2 - q(x)W(x)^2]dx}{\int_{x_0}^{x_1}\rho(x)W(x)^2dx}, \\ \mathcal{R}^*(W) &= \frac{\int_{x_0}^{x_1}[\tau^*(x)W'(x)^2 - q^*(x)W(x)^2]dx}{\int_{x_0}^{x_1}\rho^*(x)W(x)^2dx}. \end{aligned}$$

Here $\tau(x), q(x), \rho(x), \tau^*(x), q^*(x)$, and $\rho^*(x)$ are given functions defined on $[x_0, x_1]$.

Lemma 5.9. *If*

$$\mathcal{R}(W) \le \mathcal{R}^*(W) \tag{5.5.6}$$

for all $W \in \mathscr{C}_0^1[x_0, x_1]$, *then*

$$\lambda_n \le \lambda_n^* \tag{5.5.7}$$

for all $n \in \mathbf{N}$.

Proof. If (5.5.6) holds, then

$$C(\phi_1, \cdots, \phi_{n-1}) \le C^*(\phi_1, \cdots, \phi_{n-1})$$

for all $\phi_1, \cdots, \phi_{n-1} \in \mathscr{C}_0^1[x_0, x_1]$. By Theorem 5.8, (5.5.7) follows. □

Remark 5.10. If

$$\tau \le \tau^*, \quad q \ge q^*, \quad \rho \ge \rho^*, \quad x \in [x_0, x_1], \tag{5.5.8}$$

then (5.5.6) is valid.

If, in particular, we take

$$\begin{aligned} \tau^* &\doteqdot \max_{x\in[x_0,x_1]} \tau(x) = ||\tau||_{\mathscr{C}[x_0,x_1]} = \tau_M, \\ q^* &\doteqdot \min_{x\in[x_0,x_1]} q(x) = q_m, \\ \rho^* &\doteqdot \min_{x\in[x_0,x_1]} \rho(x) = \rho_m, \end{aligned} \tag{5.5.9}$$

then (5.5.8) holds, and we have $\lambda_n \le \lambda_n^*$. Since the given problem for λ^* becomes

$$W''(x) = -\frac{q_m + \lambda^*\rho_m}{\tau_M}W(x), \quad x \in (x_0, x_1)$$

with $W(x_0) = W(x_1) = 0$. The general form is

$$W(x) = a_n \sin\left[\sqrt{\frac{q_m + \lambda_n^*\rho_m}{\tau_M}}(x - x_0)\right],$$

and the eigenvalues λ_n^* and eigenvalues $W_n^*(x)$ are given as

$$\begin{aligned} \lambda_n^* &= -\frac{q_m}{\rho_m} + \frac{n^2\pi^2}{(x_1-x_0)^2}\frac{\tau_M}{\rho_m}, \\ W_m^*(x) &= \sin\left[\sqrt{\frac{q_m+\lambda_n^*\rho_m}{\tau_M}}(x-x_0)\right], \quad x\in[x_0,x_1]. \end{aligned}$$

According to Lemma 5.9, we get

$$\lambda_n \le -\frac{q_m}{\rho_m} + \frac{n^2\pi^2}{(x_1-x_0)^2}\frac{\tau_M}{\rho_m} \tag{5.5.10}$$

for $n \in \mathbf{N}$.

If we define

$$\tau_m = \min_{x\in[x_0,x_1]} \tau(x), \quad q_M = \max_{x\in[x_0,x_1]} q(x), \quad \rho_M = \max_{x\in[x_0,x_1]} \rho(x),$$

then

$$\lambda_n \ge -\frac{q_M}{\rho_M} + \frac{n^2\pi^2}{(x_1-x_0)^2}\frac{\tau_m}{\rho_M} \tag{5.5.11}$$

for $n \in \mathbf{N}$.

In fact, we can prove

$$\lambda_n = \frac{n^2\pi^2}{(\int_{x_0}^{x_1}\sqrt{\rho(x)/\tau(x)}dx)^2} + E_n \tag{5.5.12}$$

for certain constant E_n which remain bounded.

5.6. **Pólya's conjecture.** We have showed in Example 5.3 that the equation

$$u''(x) + \lambda u(x) = 0 \text{ in } \Omega := (0,1), \quad u = 0 \text{ on } \partial\Omega = \{0,1\}$$

has a countable sequence of eigenvalues λ_j such that

$$\lambda_j = \pi^2 j^2 = 4\pi^2\left(\frac{j}{\omega_1|\Omega|}\right)^{2/1},$$

where $\omega_1 = 2\pi^{1/2}/\Gamma(1/2) = 2$ denotes the volume of the unit disk in $\mathbf{R}^1$ (that is, the length of the interval $(-1,1)$).

We turn to consider higher-dimension cases. Let Ω be a bounded open subset in $\mathbf{R}^n$ and consider the Laplacian equation with Dirichlet boundary condition

$$\Delta u = -\lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{5.6.1}$$

We say λ the **eigenvalue** of the Laplacian equation with Dirichlet boundary condition. As in the Sturm-Liouville problem the problem (5.6.1) has infinitely many eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \cdots, \quad \lim_{j\to\infty}\lambda_j = \infty. \tag{5.6.2}$$

The asymptotic notation $P_j \sim Q_j$ means

$$\lim_{j\to\infty}\frac{P_j}{Q_j} = 1. \tag{5.6.3}$$

The well-known **Weyl's asymptotic formula** states that

Theorem 5.11. (Weyl's asymptotic formula, 1912) *Let Ω be a bounded open subset in $\mathbf{R}^n$. Then*

$$\lambda_j \sim 4\pi^2 \left(\frac{j}{\omega_n|\Omega|}\right)^{2/n} \tag{5.6.4}$$

as $j \to \infty$, where $\omega_n = 2\pi^{n/2}/n\Gamma(n/2)$ denotes the volume of the unit disk in $\mathbf{R}^n$ and $|\Omega|$ is the volume of the domain Ω.

Conjecture 5.12. (Polya, 1960) *Let Ω be a bounded open subset in $\mathbf{R}^n$. Then*

$$\lambda_j \geq 4\pi^2 \left(\frac{j}{\omega_n|\Omega|}\right)^{2/n} \tag{5.6.5}$$

The above conjecture is still *open* even for $n = 2$. The best result up to now is the following Li-Yau's inequality (5.6.9).

Using (5.6.4), one can show that

$$\sum_{1\leq j\leq k} \lambda_j = \frac{n}{n+2}\frac{4\pi^2}{\omega_n^{2/n}}|\Omega|^{-2/n}k^{\frac{n+2}{n}} + C_n\frac{|\partial\Omega|}{|\Omega|^{1+\frac{1}{n}}}k^{1+\frac{1}{n}} + o\left(k^{1+\frac{1}{n}}\right) \tag{5.6.6}$$

as $k \to \infty$, where

$$C_n := \frac{\sqrt{\pi}\Gamma(2+\frac{n}{2})^{1+\frac{1}{n}}}{(1+n)\Gamma(\frac{3}{2}+\frac{n}{2})\Gamma(2)^{\frac{1}{n}}}. \tag{5.6.7}$$

The second term in (5.6.6) was established under some suitable conditions on Ω.

Theorem 5.13. (Li-Yau, 1983) *Let Ω be a bounded open subset in $\mathbf{R}^n$. Then*

$$\sum_{1\leq j\leq k} \lambda_j \geq \frac{n}{n+2}4\pi^2\frac{k^{\frac{n+2}{n}}}{(\omega_n|\Omega|)^{2/n}}. \tag{5.6.8}$$

Consequently,

$$\lambda_j \geq \frac{n}{n+2}4\pi^2\left(\frac{j}{\omega_n|\Omega|}\right)^{2/n}. \tag{5.6.9}$$

In view of the asymptotic formula (5.6.6), Li-Yau's inequality is sharp.

Theorem 5.14. (Lieb, 1980) *Let Ω be a bounded open subset in $\mathbf{R}^n$. Then*

$$\lambda_j \geq C_n'\left(\frac{j}{|\Omega|}\right)^{2/n}, \tag{5.6.10}$$

for some constant C_n' that differs from the constant $4\pi^2/\omega_n^{2/n}$ by a factor.

Theorem 5.15. (Melas, 2002) *Let Ω be a bounded open subset in $\mathbf{R}^n$. Then*

$$\sum_{1\leq j\leq k} \lambda_j \geq \frac{n}{n+2}4\pi^2\frac{k^{\frac{n+2}{n}}}{(\omega_n|\Omega|)^{2/n}} + M_n k\frac{|\Omega|}{I(\Omega)} \tag{5.6.11}$$

for some constant M_n depending only on n, where

$$I(\Omega) := \min_{a\in\mathbf{R}^2}\int_\Omega |x-a|^2dx, \tag{5.6.12}$$

In dimension 2, we have some improvements on Li-Yau's inequality.

(1) Li-Yau inequality:

$$\sum_{1\le j\le k}\lambda_j \ge \frac{2\pi k^2}{|\Omega|}; \tag{5.6.13}$$

(2) Melas inequality:

$$\sum_{1\le j\le k}\lambda_j \ge \frac{2\pi k^2}{|\Omega|}+\frac{|\Omega|}{32I(\Omega)}k; \tag{5.6.14}$$

(3) Kovarik-Vugalter-Weidl inequality (2008): If $\Omega\subset\mathbf{R}^2$ is a bounded open subset with C^2-boundary $\partial\Omega$, then

$$\sum_{1\le j\le k}\lambda_j \ge \frac{2\pi k^2}{|\Omega|}+\alpha c_3 k^{\frac{3}{2}-\epsilon(k)}|\Omega|^{-3/2}C(k,|\Omega|,\partial\Omega)+(1-\alpha)\frac{|\Omega|}{32I(\Omega)}k \tag{5.6.15}$$

for any $\alpha\in[0,1]$, where

$$\epsilon(k)=\frac{2}{\sqrt{\log_2(2\pi k/c_1)}},\ c_1=\sqrt{\frac{3\pi}{14}}10^{-11},\ c_3=\frac{2^{-3}}{9\sqrt{236}}(2\pi)^{5/4}c_1^{1/4}.$$

(4) Geisinger-Laptev-Weidl inequality (2010):

$$\lambda_j \ge \left(\frac{12}{\pi}\right)^{1/n}\frac{n}{(n+3)^{1+\frac{1}{n}}}\left[\frac{\Gamma(\frac{3+n}{2})}{\Gamma(1+\frac{n}{2})}\right]^{2/n}4\pi^2\left(\frac{j}{\omega_n|\Omega|}\right)^{2/n}+\frac{1}{l_0^2}, \tag{5.6.16}$$

where

$$l_0 := \inf_{u\in\mathbf{S}^{n-1}}\sup_{x\in\Omega} l(x,u), \tag{5.6.17}$$

$$l(x,u) := \theta(x,u)+\theta(x,-u), \tag{5.6.18}$$

$$\theta(x,u) := \inf\{t>0: x+tu\notin\Omega\}. \tag{5.6.19}$$

In dimension 2, they improved (5.6.16) as

$$\frac{\lambda_j}{1-\alpha}\ge 10\pi\alpha^{3/2}\frac{j}{|\Omega|}+\frac{15\pi C}{8}\frac{|\partial\Omega|}{|\Omega|}\sqrt{10\pi\alpha^{3/2}\frac{j}{|\Omega|}+\frac{225\pi^2C^2}{256}\frac{|\partial\Omega|^2}{|\Omega|^2}}+\frac{225\pi^2C^2}{128}\frac{|\partial\Omega|^2}{|\Omega|^2}$$

for any bounded *convex* subset $\Omega\subset\mathbf{R}^2$ and any $\alpha\in(0,1)$, where C is a constant given by

$$C\ge\frac{11}{9\pi^2}-\frac{3}{20\pi^4}-\frac{2}{5\pi^2}\ln\left(\frac{4\pi}{3}\right)>0.0642.$$

Using the isoperimetric inequality $|\partial\Omega|\ge 2\pi^{1/2}|\Omega|^{1/2}$ and choosing some suitable constant α, they proved that

$$\lambda_2>\frac{15.03}{|\Omega|}>\frac{4\pi}{|\Omega|},\ \lambda_3>\frac{21.52}{|\Omega|}>\frac{6\pi}{|\Omega|},\ \cdots,\ \lambda_{23}>\frac{144.58}{|\Omega|}>\frac{46\pi^2}{|\Omega|}$$

for any bounded *convex* subset $\Omega\subset\mathbf{R}^2$.

6. Second variation in extremum problems

6.1. **Higher-order variations.** If $\mathcal{J}$ is a functional defined on an open subset $\mathscr{D}$ of a normed vector space $\mathscr{X}$, then the variation of $\mathcal{J}$ at a vector $x \in \mathscr{D}$ has been defined by

$$\delta\mathcal{J}(x;\Delta x) = \frac{d}{d\epsilon}\mathcal{J}(x+\epsilon\Delta x)\Big|_{\epsilon=0}$$

provided that the expression $\mathcal{J}(x+\epsilon\Delta x)$ is differentiable with respect to ϵ at $\epsilon = 0$ for each vector $\Delta x \in \mathscr{X}$. We refer to this variation as the first variation of $\mathcal{J}$ at x. The n**th variation** of $\mathcal{J}$ at a vector $x \in \mathscr{D}$ is defined by

$$\delta^n\mathcal{J}(x;\Delta x) = \frac{d^n}{d\epsilon^n}\mathcal{J}(x+\epsilon\Delta x)\Big|_{\epsilon=0} \tag{6.1.1}$$

provided that the expression $\mathcal{J}(x+\epsilon\Delta x)$ is n times differentiable with respect to ϵ at $\epsilon = 0$ for every vector $\Delta x \in \mathscr{X}$. The nth variation satisfies the **homogeneity relation**

$$\delta^n\mathcal{J}(x;a\Delta x) = a^n\delta^n\mathcal{J}(x;\Delta x) \tag{6.1.2}$$

for any $a \in \mathbf{R}$. In particular, the second variation of $\mathcal{J}$ at x is defined as

$$\delta^2\mathcal{J}(x;\Delta x) = \frac{d^2}{d\epsilon^2}\mathcal{J}(x+\epsilon\Delta x)\Big|_{\epsilon=0}$$

for any vector $\Delta x \in \mathscr{X}$ and satisfies the relation

$$\delta^2\mathcal{J}(x;a\Delta x) = a^2\delta^2\mathcal{J}(x;\Delta x)$$

for any vector $\Delta x \in \mathscr{X}$ and any number $a \in \mathbf{R}$.

Example 6.1. (1) If $\mathcal{J}$ is an ordinary real-valued function defined in (a, b), then

$$\begin{aligned} \frac{d}{d\epsilon}\mathcal{J}(x+\epsilon\Delta x) &= \mathcal{J}'(x+\epsilon\Delta x)\Delta x, \\ \frac{d^2}{d\epsilon^2}\mathcal{J}(x+\epsilon\Delta x) &= \mathcal{J}''(x+\epsilon\Delta x)(\Delta x)^2 \end{aligned}$$

for any suitable numbers $x, \Delta x$, and ϵ, from which

$$\delta^2\mathcal{J}(x;\Delta x) = \mathcal{J}''(x)(\Delta x)^2$$

for any $\Delta x \in \mathbf{R}$.

(2) If $\mathcal{J}$ is a real-valued function defined in some open region of $\mathbf{R}^n$, then

$$\begin{aligned} \frac{d}{d\epsilon}\mathcal{J}(x+\epsilon\Delta x) &= \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\mathcal{J}(x+\epsilon\Delta x)\Delta x_i, \\ \frac{d^2}{d\epsilon^2}\mathcal{J}(x+\epsilon\Delta x) &= \sum_{1\le i,j\le n}\frac{\partial^2}{\partial x_i\partial x_j}\mathcal{J}(x+\epsilon\Delta x)\Delta x_i\Delta x_j \end{aligned}$$

from which we find that

$$\delta^2\mathcal{J}(x;\Delta x) = \sum_{1\le i,j\le n}\frac{\partial^2}{\partial x_i\partial x_j}\mathcal{J}(x)\Delta x_i\Delta x_j$$

for any vector $\Delta x = (\Delta x_1, \cdots, \Delta x_n) \in \mathbf{R}^n$.

(3) If the functional $\mathcal{J}$ has the form $(F = F(t, x, y))$

$$\mathcal{J}(Y) = \int_{t_0}^{t_1} F(t, Y(t), Y'(t))dt$$

for any vector $Y \in \mathscr{C}^1[t_0, t_1]$ (or in some given open subset of $\mathscr{C}^1[t_0, t_1]$ relative to some given norm), then

$$\mathcal{J}(Y + \epsilon\Delta Y) = \int_{t_0}^{t_1} F(t, Y(t) + \epsilon\Delta Y(t), Y'(t) + \epsilon\Delta Y'(t))dt,$$

and

$$\begin{aligned}
\frac{d}{d\epsilon}\mathcal{J}(Y + \epsilon\Delta Y) &= \int_{t_0}^{t_1} [F_x(t, Y(t) + \epsilon\Delta Y(t), Y'(t) + \epsilon\Delta Y'(t))\Delta Y(t) \\
&\quad + F_y(t, Y(t) + \epsilon\Delta Y(t), Y'(t) + \epsilon\Delta Y'(t))\Delta Y'(t)]\, dt, \\
\frac{d^2}{d\epsilon^2}\mathcal{J}(Y + \epsilon\Delta Y) &= \int_{t_0}^{t_1} \left\{F_{xx}(t, Y(t) + \epsilon\Delta Y(t), Y'(t) + \epsilon\Delta Y'(t))[\Delta Y(t)]^2\right. \\
&\quad 2F_{xy}(t, Y(t) + \epsilon\Delta Y(t), Y'(t) + \epsilon\Delta Y'(t))\Delta Y(t)\Delta Y'(t) \\
&\quad \left.+ F_{yy}(t, Y(t) + \epsilon\Delta Y(t), Y'(t) + \epsilon\Delta Y'(t))[\Delta Y'(t)]^2\right\} dt
\end{aligned}$$

Consequently,

$$\begin{aligned}
\delta^2\mathcal{J}(Y; \Delta Y) &= \int_{t_0}^{t_1} \left\{F_{xx}(t, Y(t), Y'(t))[\Delta Y(t)]^2\right. \\
&\quad + 2F_{xy}(t, Y(t), Y'(t))\Delta Y(t)\Delta Y'(t) \\
&\quad \left. F_{yy}(t, Y(t), Y'(t))[\Delta Y'(t)]^2\right\} dt
\end{aligned} \tag{6.1.3}$$

for any vector $\Delta Y \in \mathscr{C}^1[t_0, t_1]$.

Proposition 6.2. *Let $\mathcal{J}$ be a given functional defined on an open subset $\mathscr{D}$ of a normed vector space $\mathscr{X}$ and assume that the expression $\mathcal{J}(x + \epsilon\Delta x)$ is n times continuously differentiable with respect to ϵ for all ϵ near $\epsilon = 0$, for a fixed vector $x \in \mathscr{D}$ and for any vector $\Delta x \in \mathscr{X}$.*

(i) *For all small ϵ, we have*

$$\mathcal{J}(x + \epsilon\Delta x) = \sum_{k=0}^{n} \frac{\epsilon^k}{k!}\delta^k\mathcal{J}(x; \Delta x) + \mathcal{R}_n(x; \Delta x; \epsilon), \tag{6.1.4}$$

where

$$\mathcal{R}_n(x; \Delta x; \epsilon) = \int_0^{\epsilon} \frac{(\epsilon - \sigma)^{n-1}}{(n-1)!}\left[\frac{d^n}{d\sigma^n}\mathcal{J}(x + \sigma\Delta x) - \delta^n\mathcal{J}(x; \Delta x)\right] d\sigma. \tag{6.1.5}$$

(ii) *For all small ϵ, we have*

$$|\mathcal{R}_n(x; \Delta x; \epsilon)| \le \frac{|\epsilon|^n}{n!} \max_{|\sigma| \le |\epsilon|} \left|\frac{d^n}{d\sigma^n}\mathcal{J}(x + \sigma\Delta x) - \delta^n\mathcal{J}(x; \Delta x)\right|. \tag{6.1.6}$$

(iii) *In particular, the second variation of $\mathcal{J}$ can be given as*

$$\delta^2\mathcal{J}(x; \Delta x) = \lim_{\epsilon \to 0} \frac{\mathcal{J}(x + \epsilon\Delta x) + \mathcal{J}(x - \epsilon\Delta x) - 2\mathcal{J}(x)}{\epsilon^2}. \tag{6.1.7}$$

Proof. We notice that (6.1.5) can be written as

$$\begin{aligned}\mathcal{R}_n(x;\Delta x;\epsilon) &= \int_0^\epsilon \frac{(\epsilon-\sigma)^{n-1}}{(n-1)!} d\left[\frac{d^{n-1}}{d\sigma^{n-1}}\mathcal{J}(x+\sigma\Delta x)\right] \\ &\quad - \left[\int_0^\epsilon \frac{\sigma^{n-1}}{(n-1)!} d\sigma\right]\delta^n\mathcal{J}(x;\Delta x) \\ &= \int_0^\epsilon \frac{(\epsilon-\sigma)^{n-1}}{(n-1)!} d\left[\frac{d^{n-1}}{d\sigma^{n-1}}\mathcal{J}(x+\sigma\Delta x)\right] - \frac{\epsilon^n}{n!}\delta^n\mathcal{J}(x;\Delta x).\end{aligned}$$

By integration by part, we furthermore obtain

$$\begin{aligned}\mathcal{R}_n(x;\Delta x;\epsilon) &= -\frac{\epsilon^{n-1}}{(n-1)!}\frac{d^{n-1}}{d\sigma^{n-1}}\mathcal{J}(x+\sigma\Delta x)\Big|_{\sigma=0} \\ &\quad + \int_0^\epsilon \frac{(\epsilon-\sigma)^{n-2}}{(n-2)!}\frac{d^{n-1}}{d\sigma^{n-1}}\mathcal{J}(x+\sigma\Delta x)d\sigma - \frac{\epsilon^n}{n!}\delta^n\mathcal{J}(x;\Delta x) \\ &= -\sum_{k=n-1}^{n}\frac{\epsilon^k}{k!}\delta^k\mathcal{J}(x;\Delta x) + \int_0^\epsilon \frac{(\epsilon-\sigma)^{n-2}}{(n-2)!}\frac{d^{n-1}}{d\sigma^{n-1}}\mathcal{J}(x+\sigma\Delta x)d\sigma.\end{aligned}$$

Keeping this process, we can conclude that

$$\mathcal{R}_n(x;\Delta x;\epsilon) = -\sum_{k=i}^{n}\frac{\epsilon^k}{k!}\delta^k\mathcal{J}(x;\Delta x) + \int_0^\epsilon \frac{(\epsilon-\sigma)^{i-1}}{(i-1)!}\frac{d^i}{d\sigma^i}\mathcal{J}(x+\sigma\Delta x)d\sigma$$

for each $i = 1, \cdots, n-1$. In particular,

$$\begin{aligned}\mathcal{R}_n(x;\Delta x;\epsilon) &= -\sum_{k=1}^{n}\frac{\epsilon^k}{k!}\delta^k\mathcal{J}(x;\Delta x) + \int_0^\epsilon \frac{d}{d\sigma}\mathcal{J}(x+\sigma\Delta x)d\sigma \\ &= -\sum_{k=1}^{n}\frac{\epsilon^k}{k!}\delta^k\mathcal{J}(x;\Delta x) + \mathcal{J}(x) - \mathcal{J}(x+\epsilon\Delta x)\end{aligned}$$

which indicates (6.1.4) and then (6.1.6).

To prove (6.1.7), letting $n = 2$ in (6.1.4) yields

$$\begin{aligned}\mathcal{J}(x+\epsilon\Delta x) &= \mathcal{J}(x) + \epsilon\delta\mathcal{J}(x;\Delta x) + \frac{\epsilon^2}{2}\delta^2\mathcal{J}(x;\Delta x) + \mathcal{R}_2(x;\Delta x;\epsilon), \\ \mathcal{J}(x-\epsilon\Delta x) &= \mathcal{J}(x) - \epsilon\delta\mathcal{J}(x;\Delta x) + \frac{\epsilon^2}{2}\delta^2\mathcal{J}(x;\Delta x) + \mathcal{R}_1(x;\Delta x;-\epsilon).\end{aligned}$$

Thus

$$\frac{\mathcal{J}(x+\epsilon\Delta x)+\mathcal{J}(x-\epsilon\Delta x)-2\mathcal{J}(x)}{\epsilon^2} - \delta^2\mathcal{J}(x;\Delta x) = \frac{\mathcal{R}_2(x;\Delta x;\epsilon)+\mathcal{R}_2(x;\Delta x;-\epsilon)}{\epsilon^2}.$$

However, the estimate (6.1.6) gives

$$\left|\frac{\mathcal{R}_2(x;\Delta x;\epsilon)+\mathcal{R}_2(x;\Delta x;-\epsilon)}{\epsilon^2}\right| \le \max_{|\sigma|\le|\epsilon|}\left|\frac{d^2}{d\sigma^2}\mathcal{J}(x+\sigma\Delta x) - \delta^2\mathcal{J}(x;\Delta x)\right|$$

which tends to 0, because the $d^2\mathcal{J}(x+\sigma\Delta x)/d\sigma^2$ is continuous. □

Remark 6.3. If f is a differentiable function in (a, b), then

$$f'(x) = \lim_{\epsilon\to 0+}\frac{f(x+\epsilon)+f(x-\epsilon)-2f(x)}{2\epsilon} \tag{6.1.8}$$

for any $x \in (a, b)$. However, the converse is not in general true; that means a function satisfying (6.1.8) may not be differentiable. For instance, consider the function $f(x) = |x|$ defined in $(-1, 1)$. Then

$$\frac{f(0+\epsilon) + f(0-\epsilon) - 2f(0)}{\epsilon} = \frac{\epsilon + (-\epsilon) - 0}{2\epsilon} = 0$$

for all $\epsilon > 0$.

6.2. **Necessary conditions for a local extremum.** Let $\mathcal{J}$ be a functional defined on an open subset $\mathscr{D}$ of a normed vector space $\mathscr{X}$. If x^* is a local extremum vector in $\mathscr{D}$ for $\mathcal{J}$ with

$$\delta \mathcal{J}(x^*; \Delta x) = 0 \tag{6.2.1}$$

for every vector $\Delta x \in \mathscr{X}$, and if the expression $\mathcal{J}(x^* + \epsilon \Delta x)$ is twice continuously differentiable near $\epsilon = 0$, then by Proposition 6.2 implies that

$$\mathcal{J}(x^* + \epsilon \Delta x) - \mathcal{J}(x^*) = \frac{\epsilon^2}{2} \delta^2 \mathcal{J}(x^*; \Delta x) + \mathcal{R}_2(x; \Delta x; \epsilon), \tag{6.2.2}$$

where

$$\lim_{\epsilon \to 0} \frac{\mathcal{R}_2(x; \Delta x; \epsilon)}{\epsilon^2} = 0. \tag{6.2.3}$$

If x^* is a local minimum vector in $\mathscr{D}$ for $\mathcal{J}$, then

$$\mathcal{J}(x^* + \epsilon \Delta x) - \mathcal{J}(x^*) \geq 0$$

for all small numbers ϵ and then

$$\delta^2 \mathcal{J}(x^*; \Delta x) + \frac{2}{\epsilon^2} \mathcal{R}_2(x^*; \Delta x; \epsilon) \geq 0$$

for all small nonzero numbers ϵ. Letting $\epsilon \to 0$ yields

$$\delta^2 \mathcal{J}(x^*; \Delta x) \geq 0 \tag{6.2.4}$$

for every vector $\Delta x \in \mathscr{X}$. Similarly, if x^* is a local maximum vector in $\mathscr{D}$ for $\mathcal{J}$, then

$$\delta^2 \mathcal{J}(x^*; \Delta x) \leq 0 \tag{6.2.5}$$

for every vector $\Delta x \in \mathscr{X}$.

Example 6.4. Consider

$$\mathcal{K}(Y) = \int_0^1 [Y(t)]^3 dt$$

for any function $Y \in \mathscr{C}^0[0, 1]$ equipped with the norm $||Y|| = \max_{t \in [0,1]} |Y(t)|$. Then

$$\delta \mathcal{K}(0; \Delta Y) = \delta^2 \mathcal{K}(0; \Delta Y) = 0,$$

but $\mathcal{K}(0) = 0 \geq -1 = \mathcal{K}(-1)$ (which means that $Y^* = 0$ is not a local minimum vector for $\mathcal{K}$), This example shows that conditions (6.2.4) and (6.2.5) are not sufficient.

6.3. **Sufficient conditions for a local extremum.** Let $\mathscr{J}$ be a functional defined on an open subset $\mathscr{D}$ of a normed vector space $\mathscr{X}$, and we consider a vector $x^* \in \mathscr{D}$ for which the first variation of $\mathscr{J}$ vanishes as

$$\delta\mathscr{J}(x^*;\Delta x) = 0 \tag{6.3.1}$$

for every vector $\Delta x \in \mathscr{X}$. We also assume that the second variation of $\mathscr{J}$ is nonnegative at x^* as

$$\delta^2\mathscr{J}(x^*;\Delta x) \geq 0 \tag{6.3.2}$$

for every vector $\Delta x \in \mathscr{X}$. By Example 6.4, x^* need not be a local extremum vector. We seek suitable additional conditions which will be sufficient to guarantee that x^* is in fact a local minimum vector in $\mathscr{D}$ for $\mathscr{J}$.

(1) Suppose that

$$\delta^2\mathscr{J}(x^*;\Delta x) > 0 \tag{6.3.3}$$

for all *nonzero* vector $\Delta x \in \mathscr{X}$. As in Proposition 5.2, we have

$$\mathscr{J}(x^* + \epsilon\Delta x) - \mathscr{J}(x^*) = \frac{\epsilon^2}{2}\delta^2\mathscr{J}(x^*;\Delta x) + \mathcal{R}_2(x^*;\Delta x;\epsilon)$$

where

$$\lim_{\epsilon\to 0}\frac{\mathcal{R}_2(x^*;\Delta x;\epsilon)}{\epsilon^2} = 0.$$

When $\mathscr{X}$ is finite-dimensional, it can be showed that, under conditions (6.3.1), (6.3.2), and (6.3.3), x^* is a local minimum vector for $\mathscr{J}$.

(2) If in addition to (6.3.1) $\mathscr{J}$ satisfies both

$$\mathscr{J}(x^* + \Delta x) - \mathscr{J}(x^*) = \frac{1}{2}\delta^2\mathscr{J}(x^*;\Delta x) + \mathcal{E}_2(x^*;\Delta x), \tag{6.3.4}$$

where

$$\lim_{\Delta x\to 0}\frac{\mathcal{E}_2(x^*;\Delta x)}{||\Delta x||^2_{\mathscr{X}}} = 0, \tag{6.3.5}$$

and

$$\delta^2\mathscr{J}(x^*;\Delta x) \geq p||x||^2_{\mathscr{X}} \tag{6.3.6}$$

for some positive constant $p > 0$ and for all small vectors $\Delta x \in \mathscr{X}$, then

$$\mathscr{J}(x^* + \Delta x) - \mathscr{J}(x^*) \geq ||\Delta x||^2_{\mathscr{X}}\left[\frac{p}{2} + \frac{\mathcal{E}_2(x^*;\Delta x)}{||\Delta x||^2_{\mathscr{X}}}\right] \geq 0$$

for all sufficiently small vectors $\Delta x \in \mathscr{X}$.

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