The Kähler-Ricci flow on singular Calabi-Yau varieties

Dedicated to Professor Shing-Tung Yau

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Abstract In this note, we study the Kähler-Ricci flow on projective Calabi-Yau varieties with log terminal singularities. We prove that the flow has long time existence and it converges to the unique singular Ricci-flat Kähler metric in the initial Kähler class.

1 Introduction

The study of Kähler-Einstein metrics has been an important subject in complex geometry and analysis, following Yau’s celebrated solution to the Calabi Conjecture [Y1]. The existence of Ricci-flat Kähler metrics on Kähler manifolds of vanishing first Chern class is deeply related to mirror symmetry and such Calabi-Yau manifolds have been thoroughly studied in many fields of mathematics and physics [SYZ]. The degeneration of Calabi-Yau manifolds is also a central subject as one changes complex or Kähler structures. In many cases, the degenerated model is again a Calabi-Yau variety with mild singularities and lots of progress has been made in this direction. The existence and uniqueness of singular Ricci-flat Kähler metrics have been proved in [EGZ] on projective Calabi-Yau varieties with log terminal singularities. The behavior of Ricci-flat Kähler metrics is studied in [GW, To1, To2, RZ] on a projective Calabi-Yau manifold when the Kähler classes degenerate to the boundary of the Kähler cone.

The Ricci flow [H] provides a canonical deformation of Kähler metrics toward canonical metrics of Einstein type. Cao [Ca] gives an alternative proof of the existence of Ricci-flat Kähler metrics on a compact Kähler manifold with vanishing first Chern class by the Kähler-Ricci flow. A projective Calabi-Yau variety $X$ is a $\mathbb{Q}$-factorial projective variety with numerically trivial canonical divisor $K_X$. The weak Kähler-Ricci flow is defined in [ST3] on $\mathbb{Q}$-factorial projective varieties with log terminal singularities. The short time existence and uniqueness is also obtained in [ST3] for the weak Kähler-Ricci flow with appropriate initial data. In this paper, we study the limiting behavior of the Kähler-Ricci flow on projective Calabi-Yau varieties with log terminal singularities.

The following theorem is our main result.

Theorem 1.1 Let $X$ be a projective Calabi-Yau variety with log terminal singularities and $\iota : X \to \mathbb{CP}^N$ be a projective embedding of $X$. Let $\omega_0$ be a real smooth semi-positive closed $(1,1)$-form on $X$ equivalent to the pull-back of the Fubini-Study metric by $\iota$. Then the weak Kähler-Ricci flow

$$\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega),$$

$$\omega(0, \cdot) = \omega_0 \quad \text{on} \quad X$$

\[(1.1)\]
has a unique weak solution $\omega(t, \cdot)$ on $[0, \infty) \times X$, where $\omega(t, \cdot) \in C^\infty([0, \infty) \times X_{\text{reg}})$ and $\omega(t, \cdot)$ admits bounded local potentials in $L^\infty$. Furthermore, $\omega(t, \cdot)$ converges to the unique singular Ricci-flat Kähler metric $\omega_{CY} \in [\omega_0]$ in the sense of currents on $X$ and in $C^\infty$-topology on $X_{\text{reg}} = X \setminus X_{\text{sing}}$.

The assumption on the equivalence of $\omega_0$ and $\omega_{FS}$, the pull-back of the Fubini-Study metric by $\iota$, means that there exists $\kappa > 0$ such that

$$\kappa^{-1} \omega_{FS} \leq \omega_0 \leq \kappa \omega_{FS}.$$ 

Theorem 1.1 generalizes the classical result of Cao [Ca]. The long time existence of the weak solution is already proved in [ST3]. We refer the readers to the precise definition of the weak Kähler-Ricci flow in Definition 2.4. The existence and uniqueness of such Ricci flat metrics is proved in [EGZ]. General Calabi-Yau equations are also studied on symplectic manifolds in [We, TW, TWY].

The Kähler-Ricci flow is expected to smooth out initial singular data. In particular, if the Calabi-Yau variety has only crepant singularities, the evolving singular Kähler metrics immediately have bounded scalar curvature.

**Theorem 1.2** Let $X$ be a projective Calabi-Yau variety of complex dimension $n$ with crepant resolution. Let $\omega(t, \cdot)$ be the unique solution of the Kähler-Ricci flow (1.1) starting with a real smooth semi-positive closed $(1, 1)$-form $\omega_0$ on $X$, equivalent to the pull-back of the Fubini-Study metric by the embedding $\iota$. Then for any $\delta > 0$, there exists $C > 0$ such that

$$-\frac{n}{t} \leq S(\omega(t, \cdot)) \leq \frac{C}{t} \quad (1.2)$$

on $X_{\text{reg}}$ for $t > \delta$, where $S(\omega)$ is the scalar curvature of $\omega$.

In particular, the scalar curvature $S(\omega)$ converges to 0 uniformly in $L^\infty(X)$ as $t \to \infty$.

We now give the outline of the current paper. In section 2, we give some background in birational geometry and reduce the Kähler-Ricci flow to the Monge-Ampère flow on singular varieties. In section 3, after deriving the uniform estimates, we prove the existence and convergence of the Monge-Ampère flow on smooth projective manifolds with degenerate data. In section 4, we give the proof of Theorem 1.1 and Theorem 1.2.

2 Preliminaries

2.1 Singular Calabi-Yau varieties

In this section, we will give the definition of projective Calabi-Yau varieties with log terminal singularities. We always assume that the underlying projective varieties are normal. The analysis of holomorphic and plurisubharmonic functions is well studied on normal varieties (cf. [D]). The following definitions are standard in algebraic geometry (see [L]).

**Definition 2.1** A projective variety $X$ is $\mathbb{Q}$-factorial if any $\mathbb{Q}$-Weil divisor on $X$ is $\mathbb{Q}$-Cartier.

**Definition 2.2** Let $X$ be a normal $\mathbb{Q}$-factorial projective variety. Let $\pi : \tilde{X} \to X$ be a resolution of singularities and let $\{E_i\}_{i=1}^p$ be the irreducible components of the exceptional locus $\text{Exc}(\pi)$ of $\pi$ with simple normal crossings. Then there exists a unique collection $a_i \in \mathbb{Q}$ such that
\[ K_{\hat{X}} = \pi^* K_X + \sum_{i=1}^{p} a_i E_i. \]

1. \(X\) is said to have log terminal singularities if \(a_i > -1\) for all \(i\).
2. \(\pi : \hat{X} \to X\) is called a crepant resolution if all \(a_i = 0\).

**Definition 2.3** A projective Calabi-Yau variety is a normal, \(\mathbb{Q}\)-factorial projective variety with log terminal singularities and numerically trivial canonical divisor.

Let \(X\) be a Calabi-Yau variety with numerically trivial canonical divisor \(K_X\). There exists \(\tau \in H^0(X, mK_X)\) for some integer \(m > 0\) and it is nowhere vanishing. \(\Omega := (\tau \wedge \bar{\tau})^m\) is called the Calabi-Yau volume form. Thus \(\text{Ric}(\Omega) = -\sqrt{-1} \partial \bar{\partial} \log \Omega\), is well defined and vanishes on \(X_{\text{reg}}\).

### 2.2 Reduction of the Kähler-Ricci flow to the Monge-Ampère flow

Let us recall the notion of the weak Kähler-Ricci flow on projective varieties with singularities introduced in [ST3].

**Definition 2.4** Let \(X\) be a \(\mathbb{Q}\)-factorial projective Calabi-Yau variety with log terminal singularities. Let \(\omega_0\) be a real semi-positive closed \((1,1)\) form on \(X\) equivalent to the pull-back of the Fubini-Study metric by a projective embedding of \(X\). A family of closed real positive \((1,1)\)-currents \(\omega(t, \cdot) \in [\omega_0]\) on \(X\) for \(t \in [0, \infty)\) is called a solution of the weak Kähler-Ricci flow if the following conditions hold:

1. \(\omega(t, \cdot)\) is smooth on \(X_{\text{reg}}\) for \(t > 0\). Moreover, \(\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi\) for some potential \(\varphi \in C^\infty([0, \infty) \times X_{\text{reg}})\) and \(\varphi(t, \cdot) \in \text{PSH}(X, \omega_0)\) for all \(t \in [0, \infty)\). Furthermore, for any \(T \in (0, \infty)\), \(\varphi \in L^\infty([0, T) \times X)\).
2. 

\[ \begin{align*}
\frac{\partial}{\partial t} \omega &= -\text{Ric}(\omega) \quad \text{on } (0, \infty) \times X_{\text{reg}}, \\
\omega(0, \cdot) &= \omega_0 \quad \text{on } X.
\end{align*} \tag{2.1} \]

The existence and uniqueness of the weak Kähler-Ricci flow is shown in [ST3] to be equivalent to the existence of \(\varphi\) which satisfies the first condition in Definition 2.4 and solves the following parabolic Monge-Ampère equation:

\[ \begin{align*}
\frac{\partial}{\partial t} \varphi &= \log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi}{\Omega} \right)^n \quad \text{on } (0, \infty) \times X_{\text{reg}}, \\
\varphi(0, \cdot) &= 0 \quad \text{on } X,
\end{align*} \tag{2.2} \]

where \(\Omega\) is the Calabi-Yau volume form defined as in Section 2.1 and so \(\text{Ric}(\Omega) = 0\) on \(X_{\text{reg}}\).
3 Degenerate Monge-Ampère flow

3.1 Set-up

Let $X$ be a projective manifold of complex dimension $n$. The following two conditions are introduced in [ST3].

**Condition A.** Let $L \to X$ be a big and semi-ample line bundle over $X$. Let $\omega \in c_1(L)$ be a real smooth semi-positive closed $(1,1)$-form on $X$. We assume that $\omega$ at worst vanishes along a projective subvariety of $X$ to a finite order, i.e., there exists an effective divisor $E_0$ on $X$ such that for any fixed Kähler metric $\vartheta$,

$$\omega \geq C_\vartheta |S_{E_0}|_{h_{E_0}}^2 \vartheta,$$

where $C_\vartheta$ is a positive constant, $S_{E_0}$ is a defining section of $E_0$ and $h_{E_0}$ is a smooth hermitian metric on the line bundle associated to $E_0$.

**Condition B.** Let $\Theta$ be a smooth volume form on $X$. Let $E = \sum_{i=1}^{p} a_i E_i$ and $F = \sum_{j=1}^{q} b_j F_j$ be effective divisors on $X$, where $E_i$ and $F_j$ are irreducible components with simple normal crossings. In addition, we assume $a_i \geq 0$ and $0 < b_j < 1$. Let $\Omega$ be a semi-positive $(n,n)$-form on $X$ such that

$$\Omega = |S_E|_{h_E}^2 |S_F|_{h_F}^{-2} \Theta,$$

where $S_E$ and $S_F$ are the defining sections of $E$ and $F$, $h_E$ and $h_F$ are smooth hermitian metrics on the line bundles associated to $E$ and $F$.

We will always abuse the notation of $\mathbb{Q}$-line bundles and $\mathbb{Q}$-divisors without confusion. Since $L$ is semi-ample and big, by Kodaira’s lemma, there exists an effective $\mathbb{Q}$-divisor $D$ such that $L - \epsilon D$ is ample for sufficiently small positive $\epsilon \in \mathbb{Q}$. We can further assume that the support of $E_0 \cup E \cup F$ is contained in that of $D$. Let $h_D$ be a fixed hermitian metric on the line bundle associated to $[D]$ such that $\omega_{D,\epsilon} := \omega - \epsilon \text{Ric}(h_D) = \omega + \epsilon \sqrt{-1} \partial \overline{\partial} \log h_D > 0$. Let $S_D$ be the defining section of the divisor $D$, and let $|S_D|_{h_D}^2 = |S_D|^2 h_D$ be the norm of the section $S_D$ with respect to the metric $h_D$. Then one can rewrite $\omega_{D,\epsilon} = \omega + \epsilon \sqrt{-1} \partial \overline{\partial} \log |S_D|_{h_D}^2$ on $X \setminus D$. On the other hand, it follows from the simple fact in calculus that any $(n,n)$-form $\Omega$ satisfying **Condition B** is integrable.

Let $\omega_0$ be a real smooth closed $(1,1)$-form on $X$ such that $\omega_0 \geq C \omega$ for some constant $C > 0$. Apparently, $\omega_0$ is semi-positive and big. Moreover, $\omega_{D,\epsilon} := \omega_0 + \epsilon \sqrt{-1} \partial \overline{\partial} \log h_D > 0$ for sufficiently small $\epsilon > 0$. It also holds that $\omega_{D,\epsilon} = \omega_0 + \epsilon \sqrt{-1} \partial \overline{\partial} \log |S_D|_{h_D}^2$ on $X \setminus D$. Let $\Omega$ be the volume form satisfying **Condition B** and let’s also assume that

$$\int_X \Omega = [\omega_0]^n =: V_0$$

after normalization.

Let $\vartheta$ be a fixed Kähler metric on $X$. Let $\omega_s = \omega_0 + s \vartheta$ for $s \in (0,1]$. Let $\Omega_{r_1,r_2} = \frac{|S_E|_{h_E}^2 + r_1}{|S_F|_{h_F}^2 + r_2} \Theta$ be the perturbed smooth positive volume form on $X$. We consider the following family of Monge-Ampère flows:

$$\begin{cases} \\
\partial_t \varphi_{s,r_1,r_2} = \log \frac{(\omega_s + \sqrt{-1} \partial \overline{\partial} \varphi_{s,r_1,r_2})^n}{\Omega_{r_1,r_2}}, \\
\varphi_{s,r_1,r_2}(0,\cdot) = 0. \end{cases} \quad (3.1)$$
Equation (3.1) is exactly the parabolic Monge-Ampère equation studied by Cao [Ca] for $s, r_1, r_2 > 0$ and it is well-known that there exists a unique smooth solution $\varphi_{s,r_1,r_2}$ on $[0, \infty) \times X$.

In order to get the $C^0$ estimates, we need to make the following normalization. Let $V_s = [\omega_s]^n$, $V_{r_1,r_2} = \int_X \Omega_{r_1,r_2}$ and $c_{s,r_1,r_2} = \log \frac{V_s}{V_{r_1,r_2}}$. Let $\tilde{\varphi}_{s,r_1,r_2} = \varphi_{s,r_1,r_2} - tc_{s,r_1,r_2}$. Then equation (3.1) is equivalent to the following family of Monge-Ampère flows:

$$
\begin{align*}
\frac{\partial}{\partial t} \tilde{\varphi}_{s,r_1,r_2} &= \log \left( \frac{\omega_s + \sqrt{-1} \partial \bar{\partial} \varphi_{s,r_1,r_2}}{\Omega_{r_1,r_2}} \right)^n - c_{s,r_1,r_2}, \\
\tilde{\varphi}_{s,r_1,r_2}(0, \cdot) &= 0.
\end{align*}
$$

(3.2)

Obviously, the constants $c_{s,r_1,r_2}$ are uniformly bounded for $s, r_1, r_2 \in (0, 1]$ and they approach 0 as $s, r_1, r_2 \to 0$.

### 3.2 Estimates

Let $\tilde{\omega}_{s,r_1,r_2} = \omega_s + \sqrt{-1} \partial \bar{\partial} \varphi_{s,r_1,r_2}$ and $\tilde{\nabla}_{s,r_1,r_2}, \tilde{\Delta}_{s,r_1,r_2}$ be the gradient and Laplacian operators with respect to the metric $\tilde{\omega}_{s,r_1,r_2}$.

**Lemma 3.1** There exist $p > 1$ and $C > 0$, such that for all $r_1, r_2 \in (0, 1]$, 

$$
\int_X \left( \frac{\Omega_{r_1,r_2}}{\Theta} \right)^p \Theta \leq C.
$$

(3.3)

**Proof** Note that 

$$
\int_X \left( \frac{\Omega_{r_1,r_2}}{\Theta} \right)^p \Theta = \int_X \left( \frac{|S_E|^2_{h_E} + r_1}{|S_F|^2_{h_F} + r_2} \right)^p \Theta \leq C \int_X |S_F|^{-2p} \Theta
$$

for some constant $C > 0$. By choosing $p > 1$ such that $p \cdot b_j < 1$ for all $j$, the lemma follows from the simple calculus fact for $q < 1$:

$$
\int_{[0,1]} x^{-q} dx < \infty.
$$

\( \square \)

**Corollary 3.1** Let $\phi_{s,r_1,r_2}$ be the unique smooth solution satisfying the following Monge-Ampère equation

$$
\begin{align*}
(\omega_s + \sqrt{-1} \partial \bar{\partial} \phi_{s,r_1,r_2})^n &= \frac{V_s}{V_{r_1,r_2}} \Omega_{r_1,r_2}, \\
\max_X \phi_{s,r_1,r_2} &= 0
\end{align*}
$$

(3.4)

for $s, r_1, r_2 > 0$. Then there exists $C > 0$ such that for all $s, r_1, r_2 \in (0, 1]$, 

$$
\| \phi_{s,r_1,r_2} \|_{L^\infty(X)} \leq C.
$$

(3.5)

**Proof** Since $\frac{V_s}{V_{r_1,r_2}} \Omega_{r_1,r_2}$ is uniformly bounded in $L^p(\Theta)$ for some $p > 1$ for $s, r_1, r_2 \in (0, 1]$, the corollary follows from the uniform estimates for degenerate complex Monge-Ampère equations in [Kol1], [EGZ] and [Zh1].

\( \square \)
Proposition 3.1 There exists $C > 0$, such that for all $s, r_1, r_2 \in (0, 1]$, 

$$\|\tilde{\varphi}_{s, r_1, r_2}\|_{L^\infty([0, \infty) \times X)} \leq C. \quad (3.6)$$

Proof Let $\Phi = \tilde{\varphi}_{s, r_1, r_2} - \phi_{s, r_1, r_2}$. Then 

$$\frac{\partial}{\partial t} \Phi = \log \frac{(\omega_s + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_{s, r_1, r_2})^n}{(\omega_s + \sqrt{-1} \partial \bar{\partial} \phi_{s, r_1, r_2})^n} = \log \frac{(\omega_s + \sqrt{-1} \partial \bar{\partial} \Phi)^n}{(\omega_s + \sqrt{-1} \partial \bar{\partial} \phi_{s, r_1, r_2})^n}.$$

Applying the maximum principle, $\frac{\partial}{\partial t} \min_X \Phi \geq 0$, so $\Phi(t, \cdot) \geq \min_X \Phi(0, \cdot) = 0$. Therefore, 

$$\tilde{\varphi}_{s, r_1, r_2} \geq \phi_{s, r_1, r_2} \geq -C$$

by Corollary 3.1. The uniform upper bound for $\tilde{\varphi}_{s, r_1, r_2}$ can be derived in a similar manner. \hfill \Box

Proposition 3.2 There exist $A, C > 0$ such that for $t \in [0, \infty)$ and $s, r_1, r_2 \in (0, 1]$, 

$$\left| \frac{\partial}{\partial t} \varphi_{s, r_1, r_2} \right| \leq C - A \log |S_D|_{h_D}^2.$$

Proof Let 

$$H^+ = -\tilde{\varphi}_{s, r_1, r_2} + A_1(\tilde{\varphi}_{s, r_1, r_2} - \log |S_D|_{h_D}^{2\epsilon}), \quad H^- = -\tilde{\varphi}_{s, r_1, r_2} - A_2(\tilde{\varphi}_{s, r_1, r_2} - \log |S_D|_{h_D}^{2\epsilon}).$$

We fix $\epsilon > 0$ sufficiently small. By choosing $A_1$ sufficiently large, we can assume $H^+(0, \cdot) > -C_1$ by Condition B. Away from $D$, we calculate 

$$(\frac{\partial}{\partial t} - \tilde{\Delta}_{s, r_1, r_2})H^+ = A_1[\tilde{\varphi}_{s, r_1, r_2} - c_{s, r_1, r_2} - \tilde{\Delta}_{s, r_1, r_2}(\tilde{\varphi}_{s, r_1, r_2} - \log |S_D|_{h_D}^{2\epsilon})]$$

$$\geq A_1[\tilde{\varphi}_{s, r_1, r_2} - c_{s, r_1, r_2} - \text{tr} \bar{\varphi}_{s, r_1, r_2}(\bar{\omega}_{s, r_1, r_2} - \omega_{D, \epsilon})]$$

$$\geq A_1(\bar{\varphi}_{s, r_1, r_2} - c_{s, r_1, r_2} - n).$$

Suppose that $H^+$ achieves its minimum at $(t_0, p_0)$ with $p_0 \in X \setminus D, t_0 > 0$. Then it holds by the maximum principle that 

$$\tilde{\varphi}_{s, r_1, r_2}(t_0, p_0) \leq c_{s, r_1, r_2} + n \leq C_2.$$ 

Hence, 

$$-\tilde{\varphi}_{s, r_1, r_2} + A_1(\tilde{\varphi}_{s, r_1, r_2} - \log |S_D|_{h_D}^{2\epsilon}) \geq H^+(t_0, p_0) \geq -C_3.$$ 

If the minimum of $H^+$ is attained at $t = 0$, then

$$-\tilde{\varphi}_{s, r_1, r_2} + A_1(\tilde{\varphi}_{s, r_1, r_2} - \log |S_D|_{h_D}^{2\epsilon}) \geq H^+(0, \cdot) \geq -C_1.$$

Therefore, one direction in the proposition is obtained with the uniform bound of $\tilde{\varphi}_{s, r_1, r_2}$ and the other direction follows from the similar argument applied to $H^-$. \hfill \Box

Lemma 3.2 There exist $C, \alpha > 0$ such that for $t \in [0, \infty)$ and $s, r_1, r_2 \in (0, 1]$, 

$$\text{tr}_g(\bar{\omega}_{s, r_1, r_2}) \leq C |S_D|_{h_D}^{-2\alpha}.$$
**Proof** The proof is a consequence by applying the maximum principle to
\[
\left(\frac{\partial}{\partial t} - \Delta_{s,r_1,r_2}\right)(\log tr_\theta(\tilde{\omega}_{s,r_1,r_2}) - A^2\bar{\varphi}_{s,r_1,r_2} + A|S_D|^2_{L_D})
\]
for sufficiently large \(A\) and the detailed proof can be found in Lemma 3.10 in [ST3]. Although the argument in [ST3] is for short time, it still can be applied here as \(\tilde{\varphi}_{s,r_1,r_2}\) is uniformly bounded. \(\square\)

Let
\[
(\varphi_{s,r_1,r_2})_{i\bar{p}} = (\nabla_{s,r_1,r_2})_{i\bar{p}}\partial_i\partial_{\bar{p}}\varphi_{s,r_1,r_2}
\]
and
\[
T = (\tilde{\omega}_{s,r_1,r_2})\partial_i(\tilde{\omega}_{s,r_1,r_2})^{k\bar{l}}(\tilde{\omega}_{s,r_1,r_2})^{q\bar{r}}(\varphi_{s,r_1,r_2})_{i\bar{p}}(\varphi_{s,r_1,r_2})_{\bar{q}j}.\]
The following third order estimate follows from Lemma 3.4 in [ST3].

**Lemma 3.3** There exist constants \(C, \lambda > 0\), such that for \(t \in [0, \infty)\) and \(s, r_1, r_2 \in (0, 1]\),
\[
T \leq C|S_D|^{-2\lambda}.\]

The following proposition gives us the uniform estimates of the metric \(\tilde{\omega}_{s,r_1,r_2}\) along (3.2).

**Proposition 3.3** For any compact set \(K \subset X \setminus D\) and \(k > 0\), there exists \(C_{K,k} > 0\), such that
\[
\|\tilde{\varphi}_{s,r_1,r_2}\|_{C^k((0,\infty) \times K)} \leq C_{K,k}.\]

**Proof** The proof follows from standard Schauder’s estimates. \(\square\)

**Lemma 3.4** The following monotonicity conditions hold for \(\varphi_{s,r_1,r_2}\) on \([0, \infty) \times X\).

1. \(\varphi_{s,r_1,r_2} \leq \varphi_{s',r_1,r_2}\) for any \(0 < s < s'\);
2. \(\varphi_{s,r_1,r_2} \geq \varphi_{s',r_1,r_2}\) for any \(0 < r_1 < r_1'\);
3. \(\varphi_{s,r_1,r_2} \leq \varphi_{s,r_1',r_2}\) for any \(0 < r_2 < r_2'\).

**Proof** We give a proof for part 1 as below and proofs for part 2 and 3 are similar. Let \(\psi = \varphi_{s',r_1,r_2} - \varphi_{s,r_1,r_2}\). Then \(\psi(0, -) = 0\) and
\[
\frac{\partial}{\partial t}\psi = \log \frac{(\tilde{\omega}_{s,r_2} - (s'-s)\partial + \sqrt{-1}\partial\overline{\partial}\psi)^n}{\tilde{\omega}_{s,r_1,r_2}^{n}} \geq \log \frac{(\tilde{\omega}_{s,r_2} + \sqrt{-1}\partial\overline{\partial}\psi)^n}{\tilde{\omega}_{s,r_1,r_2}^{n}}.
\]
Hence \(\frac{\partial}{\partial t}\min_X \psi \geq 0\) by the maximum principle, yielding \(\psi \geq 0\). \(\square\)

Let
\[
\varphi = \lim_{s \to 0} \lim_{r_2 \to 0} \lim_{r_1 \to 0} \varphi_{s,r_1,r_2}^*,
\]
where \((\lim_{r_1 \to 0} \varphi_{s,r_1,r_2})^*\) denotes the upper semi-continuous envelope of the increasing sequence \(\varphi_{s,r_1,r_2}\) as \(r_1 \to 0\). Therefore, it is an \(\omega_s\)-plurisubharmonic function and is decreasing as \(r_2 \to 0\). Furthermore, \(((\lim_{r_2 \to 0} \lim_{r_1 \to 0} \varphi_{s,r_1,r_2})^*)\) is also an \(\omega_s\)-plurisubharmonic function and decreasing as \(s \to 0\). Hence, \(\varphi\) is an \(\omega_0\)-plurisubharmonic function.
Proposition 3.4 \( \varphi \) is the unique solution to the Monge-Ampère flow

\[
\begin{align*}
\frac{\partial}{\partial t} \varphi &= \log \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi}{\Omega} \quad \text{on } [0, \infty) \times (X \setminus D), \\
\varphi(0, \cdot) &= 0 \quad \text{on } X
\end{align*}
\]

such that \( \varphi \in L^\infty([0, \infty) \times X) \cap C^\infty([0, \infty) \times (X \setminus D)) \) and \( \varphi(t, \cdot) \in PSH(X, \omega_0) \) for each \( t \in [0, \infty) \).

**Proof** We first notice that \( \varphi = \lim_{s \to 0} \lim_{r_2 \to 0} \lim_{r_1 \to 0} \bar{\varphi}_{s, r_1, r_2} \) away from \( D \) by Proposition 3.3. Then the existence can be obtained by passing (3.2) to the limit as \( s, r_1, r_2 \to 0 \) once we have the uniform estimates (Proposition 3.3) and the uniqueness follows from [ST3].

### 3.3 Convergence

In this section, we will prove the convergence of (3.7). We begin by formally defining the generalized Mabuchi K-energy with degenerate and singular data.

**Definition 3.1** Let \( \omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi \) for \( \varphi \in PSH(X, \omega_0) \cap L^\infty(X) \). Suppose that \( \log \frac{\omega^n}{\Omega} \) is well-defined and absolutely integrable. Then the generalized Mabuchi K-energy is defined as

\[
\mu_{\omega_0}(\varphi) = \frac{1}{V_0} \int_X \log \frac{\omega^n}{\Omega} \omega^n.
\]

(3.8)

Note that this generalizes the Mabuchi K-energy for a smooth Kähler metric \( \omega_0 \) and a smooth volume form \( \Omega \). By perturbing the initial metric \( \omega_0 \) and the volume form \( \Omega \) as in the previous section, we have the standard Mabuchi K-energy [Ma] along the flow (3.1):

\[
\mu_{\omega_s}(\varphi_{s, r_1, r_2}) = \frac{1}{V_s} \int_X \log \frac{\tilde{\omega}_s^n}{\Omega_{r_1, r_2}} \tilde{\omega}_s^n \omega_{s, r_1, r_2}.
\]

It is well known that along (3.1), we have

\[
\frac{\partial}{\partial t} \mu_{\omega_s}(\varphi_{s, r_1, r_2}) = -\frac{1}{V_s} \int_X |\nabla_{s, r_1, r_2} \varphi_{s, r_1, r_2}|^2 \tilde{\omega}_s^n \omega_{s, r_1, r_2} \leq 0
\]

for fixed \( s, r_1, r_2 > 0 \), by straightforward calculations.

**Lemma 3.5** There exists \( C > 0 \) such that for \( t > 0, s, r_1, r_2 \in (0, 1] \),

\[
\frac{\partial}{\partial t} \varphi_{s, r_1, r_2}(t, \cdot) \leq C + \frac{C}{t}.
\]

**Proof** Let \( H(t, \cdot) = t \varphi_{s, r_1, r_2} - \varphi_{s, r_1, r_2} - nt - tc_{s, r_1, r_2} \). Then \( H(0, \cdot) \leq C_1 \) for some \( C_1 > 0 \).

\[
(\frac{\partial}{\partial t} - \Delta_{s, r_1, r_2})H(t, \cdot) = \Delta_{s, r_1, r_2} \varphi_{s, r_1, r_2} - n = tr_{s, r_1, r_2}(\tilde{\omega}_{s, r_1, r_2} - \omega_s) - n \leq 0.
\]

It follows that

\[
\frac{\partial}{\partial t} \max_X H(t, \cdot) \leq 0.
\]

Hence,

\[
t(\varphi_{s, r_1, r_2} - n - c_{s, r_1, r_2}) \leq C_1 + \bar{\varphi}_{s, r_1, r_2}.
\]

The lemma is proved as \( \varphi_{s, r_1, r_2} \) is uniformly bounded.

\[\square\]
Lemma 3.5. Furthermore, by Jensen’s inequality

\[ \mu_{\omega_0}(\varphi(0, \cdot)) = \frac{1}{V_0} \int_X \log \frac{\omega^n_0}{\Omega}. \]

Then it is well defined at \( t = 0 \), by Condition A and Condition B as there is at worst \( \log \) pole singularities in the integral. When \( t > 0 \), \( \mu_{\omega_0} (\varphi(t, \cdot)) \) is bounded from above for each \( t \) by Lemma 3.5. Furthermore, by Jensen’s inequality

\[ \mu_{\omega_0} (\varphi) = -\frac{1}{V_0} \int_X \log \frac{\Omega}{\omega^n_0} \geq -\log \left( \frac{1}{V_0} \int_X \Omega \right) = 0. \]

\( \Box \)

Lemma 3.6. Fix \( t \in [0, \infty) \). We have

\[ \lim_{s, r_1, r_2 \to 0} \mu_{\omega_s} (\varphi_{s, r_1, r_2}(t, \cdot)) = \mu_{\omega_0} (\varphi(t, \cdot)). \]

Proof Since there are at worse \( \log \) poles in each integral (Proposition 3.2 and Lemma 3.5), for any \( \delta > 0 \), there exists a tubular neighborhood \( D_\delta \) of \( D \) such that

\[ \frac{1}{V_0} \int_{D_{\delta}} \left| \log \frac{\omega^n}{\Omega} \right| \omega^n \leq \frac{\delta}{4} \quad \text{and} \quad \frac{1}{V_0} \int_{D_{\delta}} \left| \log \frac{\omega^n_s, r_1, r_2}{\Omega_{r_1, r_2}} \right| \omega^n_{s, r_1, r_2} \leq \frac{\delta}{4}. \]

Since \( \varphi_{s, r_1, r_2} \) converges uniformly to \( \varphi \) in \( C^\infty (X \setminus D_\delta) \) by the uniform estimates (Proposition 3.3), there exist \( s_\delta, r_\delta \), such that for any \( s \in (0, s_\delta), r_1, r_2 \in (r_\delta, \infty) \), the following inequality holds:

\[ \left| \frac{1}{V_s} \int_{X \setminus D_\delta} \log \frac{\omega^n_s, r_1, r_2}{\Omega_{r_1, r_2}} \omega^n_{s, r_1, r_2} - \frac{1}{V_0} \int_{X \setminus D_\delta} \log \frac{\omega^n}{\Omega} \omega^n \right| \leq \frac{\delta}{2}, \]

Therefore, we have

\[ -\delta \leq \mu_{\omega_0} (\varphi(t, \cdot)) - \mu_{\omega_s} (\varphi_{s, r_1, r_2}(t, \cdot)) \leq \delta. \]

The lemma follows by letting \( \delta \to 0 \).

\( \Box \)

Let \( \nabla \) and \( \Delta \) be the gradient and Laplacian operators with respect to \( \omega \).

Proposition 3.6. Let \( \varphi \) be the unique solution of (3.7). Then for \( 0 \leq t_1 \leq t_2 \),

\[ \mu_{\omega_0} (\varphi(t_1, \cdot)) - \mu_{\omega_0} (\varphi(t_2, \cdot)) \geq \frac{1}{V_0} \int_{t_1}^{t_2} \int_{X \setminus D} |\nabla \varphi|^2 \omega^n dt. \tag{3.9} \]

In particular, \( \mu_{\omega_0} (\varphi(t, \cdot)) \) is decreasing for \( t \in [0, \infty) \).

Proof Notice that on any compact set \( K \subset X \setminus D \),

\[ \mu_{\omega_s} (\varphi_{s, r_1, r_2}(t_1, \cdot)) - \mu_{\omega_s} (\varphi_{s, r_1, r_2}(t_2, \cdot)) = \frac{1}{V_s} \int_{t_1}^{t_2} \int_K \nabla \varphi_{s, r_1, r_2} \frac{\partial^2}{\partial s_{r_1, r_2}^2} \omega^n_{s, r_1, r_2} dt \]

\[ \geq \frac{1}{V_s} \int_{t_1}^{t_2} \int_K \nabla \varphi_{s, r_1, r_2} \frac{\partial^2}{\partial s_{r_1, r_2}^2} \omega^n_{s, r_1, r_2} dt. \]
Due to the uniform estimate of $\varphi_{s,r_1,r_2}$, by letting $s,r_1,r_2 \to 0$, we get:

$$\mu_{\omega_0}(\varphi(t_1, \cdot)) - \mu_{\omega_0}(\varphi(t_2, \cdot)) \geq \frac{1}{V_0} \int_{t_1}^{t_2} \int_K \| \nabla \dot{\varphi} \|^2 \omega^n dt.$$  

The proposition follows by letting $K \to X \setminus D$.

**Corollary 3.2** We have

$$\int_0^\infty \int_{X \setminus D} |\nabla \dot{\varphi}|^2 \omega^n dt < \infty.$$  

**Proof** The corollary is a direct consequence of Proposition 3.5 and Proposition 3.6.

**Proposition 3.7** Let $\nabla_0$ be the gradient operator with respect to the metric $\omega_0$. Then in any compact set $K \subset X \setminus D$,

$$\sup_K |\nabla_0 \dot{\varphi}(t, \cdot)|^2_{\omega_0} \to 0 \text{ as } t \to \infty.$$  

**Proof** Otherwise suppose that there exist $\delta > 0$, $z_j \in K$ and $t_j \to \infty$ such that

$$|\nabla_0 \dot{\varphi}(t_j, z_j)|^2_{\omega_0} > \delta.$$  

We can assume that $t_{j+1} - t_j > 1$ for all $j \geq 1$ by passing to a subsequence. By the uniform $C^k$ estimates for $\varphi_{s,r_1,r_2}$ in $[0, \infty) \times K'$ where $K \subset K' \subset X \setminus D$, we have the uniformly bounded geometry of $\omega_{s,r_1,r_2}$ on $K' \subset X \setminus D$ along the Monge-Ampère flow (3.1), namely $C_1 \omega_0 \geq \omega \geq \frac{1}{C_1} \omega_0$ on $K'$ for $C_1 > 0$. Therefore there exist $\epsilon > 0$, $r > 0$ such that

$$|\nabla_0 \dot{\varphi}(t, z)|^2_{\omega_0} > C_2 \delta$$

for $t \in (t_j - \epsilon, t_j + \epsilon)$, $z \in B(z_j, r) \subset K'$ and $C_2 > 0$. This leads to the contradiction against Corollary 3.2.

Now, we are in the position to prove the convergence.

**Proposition 3.8** Let $v(t, \cdot) = \varphi(t, \cdot) - \max_X \varphi(t, \cdot)$. Then $v(t, \cdot)$ converges in $L^\infty(X)$-norm to $\varphi_\infty \in PSH(X, \omega_0) \cap L^\infty(X) \cap C^\infty(X \setminus D)$, which solves the following degenerate Monge-Ampère equation

$$\begin{cases}
(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_\infty)^n = \Omega \\
\max_X \varphi_\infty = 0.
\end{cases}$$

**Proof** Notice that $v$ is still a uniformly bounded solution of the equation $(\omega_0 + \sqrt{-1} \partial \bar{\partial} v)^n = e^\varphi \Omega$.

First of all, we argue that $v$ converges to a solution of the degenerate Monge-Ampère equation (3.10) along a subsequence. By the uniform estimates (Proposition 3.3), after passing to a convergent subsequence, we can always assume

$$v(t_j, \cdot) \to \varphi_\infty \in PSH(X, \omega_0) \cap L^\infty(X) \cap C^\infty(X \setminus D).$$
Furthermore, on any connected \( K \subset X \setminus D \), \( v(t_j, \cdot) \to \varphi_\infty \) in \( C^\infty \)-topology. Then on \( K \),

\[
|\nabla_0 \varphi(t, \cdot)|_{\omega_0} = \left| \nabla_0 \log \frac{\omega^n}{\Omega} \right|_{\omega_0} \to 0 = \left| \nabla_0 \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_\infty)^n}{\Omega} \right|_{\omega_0}
\]
as \( t \to \infty \). Hence on \( K \),

\[
\log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_\infty)^n}{\Omega} = C.
\]

By letting \( K \to X \setminus D \), \( \varphi_\infty \) solves the equation (3.10) on \( X \setminus D \). \( \varphi_\infty \) extends to a bounded \( \omega_0 \)-psh function and therefore \( C = 0 \).

Secondly, we will show that \( v \) converges to \( \varphi_\infty \) in \( L^\infty(X) \) by using the stability theorem derived in [Kol2, DiZh, DP]. Notice that \( \varphi \) is uniformly bounded from above away from \( t = 0 \) by Lemma 3.5. Then for any \( \delta > 0 \), there exists a compact set \( K_\delta \subset X \setminus D \), such that for all \( t > 0 \),

\[
\int_{X \setminus K_\delta} (e^\varphi + 1) \Omega \leq \frac{\delta}{2}.
\]

Note that \( \varphi \) tends to 0 uniformly on any compact subset of \( X \setminus D \). Therefore there exists \( T_\delta > 0 \) such that for \( t > T_\delta \), \( \int_{K_\delta} |e^\varphi - 1| \Omega < \frac{\delta}{2} \).

Now for \( t > T_\delta \),

\[
\int_X |e^\varphi - 1| \Omega \leq \int_{K_\delta} |e^\varphi - 1| \Omega + \int_{X \setminus K_\delta} (e^\varphi + 1) \Omega \leq \delta.
\]

Then

\[
\|v - \varphi_\infty\|_{L^\infty(X)} \leq C \delta^{\frac{1}{\omega + \gamma}}
\]
holds for some fixed \( C > 0, \gamma > 0 \) by applying the stability theorem in [DiZh]. The proposition is thus proved by letting \( \delta \to 0 \) as \( t \to \infty \).

\[\square\]

4 Proof of theorems

4.1 Convergence of the Kähler-Ricci flow on singular Calabi-Yau varieties

Let \( X \) be a projective Calabi-Yau variety of complex dimension \( n \). Let \( \iota : X \to \mathbb{CP}^N \) be a projective embedding and \( \pi : \hat{X} \to X \) be the resolution of singularities with exceptional divisor \( E \) and \( F \). Let \( H \) be the pullback of the hyperplane divisor of \( \mathbb{CP}^N \) to \( X \) by \( \iota \), and let \( \varpi \in |H| \) be the pullback of the Fubini-Study metric by \( \iota \). Thus \( \varpi \) is a real semi-positive closed \((1, 1)\)-form. Define \( \hat{\varpi} := \pi^* \varpi \) in the semi-ample and big class \(|\pi^*H|\). Therefore, \( \hat{\varpi} \) satisfies Condition A in section 3: \( \hat{\varpi} \geq C_\varpi |S_E|^2_{h_E} \hat{\varpi} \), where \( \varpi \) is a fixed Kähler metric on \( \hat{X} \), \( C_\varpi \) is a positive constant, \( S_E \) is a defining section of an effective divisor \( E \subset \hat{X} \) and \( h_E \) is a smooth hermitian metric on the line bundle associated to \( E \). Let initial metric \( \omega_0 \) be a real smooth semi-positive closed \((1, 1)\)-form on \( X \), equivalent to \( \varpi \) and define \( \hat{\omega}_0 := \pi^* \omega_0 \). Hence \( \hat{\omega}_0 \) also satisfies Condition A.

Let \( E = \sum_i a_i E_i, F = \sum_j b_j F_j \) be the linear combination of exceptional divisors with coefficient \( a_i \geq 0 \) and \( 0 < b_j < 1 \). Let \( K_{\hat{X}/X} = K_{\hat{X}}/\pi^* K_X \) be the relative canonical line bundle and its local defining function can be written as the Jacobian of \( \pi \), i.e. locally it could be written as \( |S_E|^2_{h_E} \cdot |S_F|^2_{h_F} \) near exceptional divisors, where \( S_E, S_F \) is the canonical defining section of \( E, F \) and \( h_E, h_F \) is the smooth hermitian metric on the line bundle associated to \( E, F \) respectively. Define \( \hat{\Omega} := \pi^* \Omega \) be the pull back of the Calabi-Yau volume form \( \Omega \). Then \( \hat{\Omega} \) is a smooth,
semi-positive Calabi-Yau volume form on $\hat{X}$ with zeros along $E_i$ of order $a_i$ and poles along $F_j$ of order $b_j$. Thus one can choose $\Theta$ to be a non-vanishing smooth volume form on $\hat{X}$ such that

$$\hat{\Omega} = \frac{|S_E|_{h_E}^2}{|S_F|_{h_F}^2} \Theta$$

and so $\hat{\Omega}$ satisfies Condition B in section 3. Apparently,

$$\int_{\hat{X}} \hat{\Omega} = \int_{\hat{X}} \frac{|S_E|_{h_E}^2}{|S_F|_{h_F}^2} \Theta < \infty$$

as the order of poles is less than 2.

The weak Kähler-Ricci flow (the Monge-Ampère flow) is studied in [ST3] on the projective variety $X$ by lifting the equation to the resolution $\hat{X}$. Following the idea of [ST3], we can prove the following proposition by the study of the Monge-Ampère flow in section 3.

**Proposition 4.1** The following statements hold.

1. There exists a unique solution $\hat{\varphi} \in C^\infty((0, \infty) \times (\hat{X} \setminus \text{Exc}(\pi))) \cap L^\infty([0, \infty) \times \hat{X})$ and for all $t \in [0, \infty)$, $\hat{\varphi}(t, \cdot) \in L^\infty(\hat{X}) \cap PSH(\hat{X}, \hat{\omega}_0)$ satisfying the Monge-Ampère flow

$$\begin{cases} 
\frac{\partial}{\partial t} \hat{\varphi} = \log \left( \frac{\hat{\omega}_0 + \sqrt{-1} \partial \bar{\partial} \hat{\varphi}}{\hat{\Omega}} \right)^n & \text{on } (0, \infty) \times (\hat{X} \setminus \text{Exc}(\pi)), \\
\hat{\varphi}(0, \cdot) = 0 & \text{on } X,
\end{cases} \quad (4.1)$$

where $\text{Exc}(\pi)$ is the exceptional locus of the resolution $\pi$. Moreover, $\hat{\varphi}(t, \cdot) - \sup_\hat{X} \hat{\varphi}(t, \cdot)$ converges in $L^\infty(\hat{X})$ to a solution $\hat{\varphi}_\infty \in C^\infty(\hat{X} \setminus \text{Exc}(\pi)) \cap L^\infty(\hat{X}) \cap PSH(\hat{X}, \hat{\omega}_0)$ of the degenerate Monge-Ampère equation

$$(\hat{\omega}_0 + \sqrt{-1} \partial \bar{\partial} \hat{\varphi}_\infty)^n = \hat{\Omega}.$$ 

2. $\hat{\varphi}(t, \cdot)$ descends to $\varphi(t, \cdot) \in PSH(X, \omega_0)$. Then $\varphi(t, \cdot)$ solves (2.2). Moreover $\varphi(t, \cdot) - \sup_X \varphi(t, \cdot)$ converges in $L^\infty(X)$ to a solution $\varphi_{CY} \in C^\infty(X_{\text{reg}}) \cap L^\infty(X) \cap PSH(X, \omega_0)$ of the degenerate Monge-Ampère equation

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{CY})^n = \Omega.$$ 

**Proof** 1. Since $\hat{\omega}_0, \hat{\Omega}$ satisfies Condition A, B respectively, the existence and uniqueness follows from Proposition 3.4 and the convergence follows from Proposition 3.8.

2. Note that $\hat{\varphi}(t, \cdot)$ is constant on each connected component in the fibre of $\pi$, as $\hat{\omega}_0 \equiv 0$ restricted on each connected component. Then it is a direct consequence of part 1. □

Now we are ready to prove our main theorem.

**Proof of Theorem 1.1** The uniqueness of $\omega$ follows from Theorem 4.4 in [ST3] and the existence and the convergence follows from Proposition 4.1. Part 2. □
4.2 Scalar curvature estimates for the Kähler-Ricci flow on Calabi-Yau varieties with crepant resolutions

We are going to derive the scalar curvature estimates for the Kähler-Ricci flow on projective Calabi-Yau manifolds with degenerate initial metrics. Thus the corresponding curvature estimates on the projective Calabi-Yau variety can be obtained by lifting the Monge-Ampère flow to its crepant resolution.

Let $X$ be a projective Calabi-Yau manifold. Let $\omega_0$ be a big and real smooth semi-positive closed $(1,1)$-form satisfying Condition A and $\Omega$ be a smooth Calabi-Yau volume form on $X$.

Thus, the existence of the Kähler-Ricci flow, and of the corresponding Monge-Ampère flow on $X$ follows from the argument in the previous section. And in this situation, (3.1) will be written in the following easier form:

$$
\begin{cases}
\frac{\partial}{\partial t} \varphi_s = \log \left( \frac{(\omega_s + \sqrt{-1}\partial\bar{\partial}\varphi_s)^n}{\Omega} \right), \\
\varphi_s(0, \cdot) = 0.
\end{cases}
$$

(4.2)

We now prove the following smoothing lemma.

**Lemma 4.1** There exists $C > 0$ such that along (4.2) for $t \in (0, \infty)$, we have

$$
-C - \frac{C}{t} \leq \frac{\partial}{\partial t} \varphi_s \leq C.
$$

**Proof** The upper bound follows from the maximum principle directly, so it suffices to derive the lower bound. Let $\tilde{\varphi}_s$ be smooth solutions to the family of Monge-Ampère equations (3.4) with $r_1 = r_2 = 0$. Then we have $-C_1 \leq \tilde{\varphi}_s \leq 0$ for some constant $C_1 > 0$.

Let $H_s = t \dot{\varphi}_s + A\tilde{\varphi}_s - A\varphi_s$ for some $A > 0$ with $\bar{\varphi}_s := \varphi_{s,0,0}$ in (3.2). Hence $H_s(0) \geq 0$. Then

$$
(\frac{\partial}{\partial t} - \tilde{\Delta}_s)H_s = (A + 1)\dot{\varphi}_s - A(n + c_s) + Atr_{\tilde{\omega}}(\tilde{\omega}_s - \sqrt{-1}\partial\bar{\partial}\varphi_s + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_s)
$$

$$
\leq (A + 1)\dot{\varphi}_s + An(\frac{e^{c_s} \Omega}{\tilde{\omega}_s})^\frac{1}{n} - A(n + c_s)
$$

$$
\geq (A + 1)\dot{\varphi}_s + C_2 e^{-\frac{1}{n}\tilde{\varphi}_s} - C_3
$$

$$
\geq -C_4
$$

Hence,

$$
\frac{\partial}{\partial t} \min_X H_s \geq (\frac{\partial}{\partial t} - \tilde{\Delta}_s) \min_X H_s \geq -C_4.
$$

(4.3)

and it follows that

$$
H_s(t, \cdot) \geq \min_X H_s(0, \cdot) - C_4t \geq -C_4t.
$$

(4.4)

The lower bound of $\frac{\partial}{\partial t} \varphi_s$ is derived as $\tilde{\varphi}_s, \varphi_s$ are uniformly bounded.

We are ready to show that the scalar curvature decays at the rate of $\frac{1}{t}$ along (4.2).

**Proposition 4.2** Let $\tilde{\omega}_s := \omega_s + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_s$. For any $\delta > 0$, there exists $C > 0$ such that along (4.2) the scalar curvature satisfies:

$$
-\frac{n}{t} \leq S(\tilde{\omega}_s(t)) \leq \frac{C}{t} \quad \text{when} \quad t > \delta.
$$

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Proof In the proof, we use $\langle \cdot, \cdot \rangle_s$, $| \cdot |_s$ to denote the inner product and norm with respect to the metric $\tilde{\omega}_s$. As computed in [ST3], one would have:

$$(\frac{\partial}{\partial t} - \tilde{\Delta}_s)[tS(\tilde{\omega}_s(t))] \geq S(\tilde{\omega}_s(t)) + \frac{t}{n}S(\tilde{\omega}_s(t))^2.$$  

By the maximum principle, one can show that $S(\tilde{\omega}_s(t)) \geq -\frac{\tau}{t}$. For the upper bound, one needs to apply the gradient and Laplacian estimates as in [ST3]. The proposition is for the first time proved in [ST3] for general projective varieties. In our situation, the proof is easier and we include the proof in the following.

Let $H = \frac{[\nabla_s \hat{\varphi}_s]^2}{C - \hat{\varphi}_s}$ and $K = -\tilde{\Delta}_s \hat{\varphi}_s + BH$ for sufficiently large $B$. Then the straightforward calculation gives the evolution:

$$(\frac{\partial}{\partial t} - \tilde{\Delta}_s)H = -\frac{\langle \tilde{\nabla}_s \tilde{\nabla}_s \hat{\varphi}_s \rangle^2_s}{C - \hat{\varphi}_s} - \frac{2Re < \tilde{\nabla}_s \hat{\varphi}_s, \tilde{\nabla}_s H >_s}{C - \hat{\varphi}_s} \leq -\epsilon \frac{\langle \tilde{\nabla}_s \hat{\varphi}_s \rangle^4_s}{(C - \hat{\varphi}_s)^3} - (1 - 2\epsilon)\frac{\langle \nabla_s \nabla_s \hat{\varphi}_s \rangle^2_s}{C - \hat{\varphi}_s} - (1 - \epsilon)\frac{2Re < \tilde{\nabla}_s \hat{\varphi}_s, \tilde{\nabla}_s K >_s}{C - \hat{\varphi}_s}$$  

and

$$(\frac{\partial}{\partial t} - \tilde{\Delta}_s)K = -\frac{(B - 1)\langle \tilde{\nabla}_s \tilde{\nabla}_s \hat{\varphi}_s \rangle^2_s + B\langle \nabla_s \nabla_s \hat{\varphi}_s \rangle^2_s}{C - \hat{\varphi}_s} - \frac{2Re < \tilde{\nabla}_s \hat{\varphi}_s, \tilde{\nabla}_s(t - \delta)H >_s}{C - \hat{\varphi}_s}.$$  

Suppose that $(t - \delta)H$ achieves maximum at $(t_0, z_0)$ when $t \geq \delta$. Then it follows from the maximum principle and Lemma 4.1 that:

$$(t_0 - \delta)H(t_0, z_0) \leq C_1.$$  

Hence, when $t > \delta$, we have

$$H(t, \cdot) \leq \frac{C_1}{t - \delta}.$$  

It follows from (4.5) that:

$$(\frac{\partial}{\partial t} - \tilde{\Delta}_s)[(t - \delta)K] \leq -C_2(t - \delta)\frac{(\tilde{\Delta}_s \hat{\varphi}_s)^2}{C - \hat{\varphi}_s} - \frac{\Delta_s \hat{\varphi}_s}{C - \hat{\varphi}_s} + BH(t, \cdot) - \frac{2Re < \tilde{\nabla}_s \hat{\varphi}_s, \tilde{\nabla}_s[(t - \delta)K] >_s}{C - \hat{\varphi}_s}.$$  

Suppose that $(t - \delta)K$ achieves maximum for $t \geq \delta$ at $(t', z')$. Then it follows from the maximum principle that

$$0 \leq -C_2([t' - \delta] - \frac{\tilde{\Delta}_s \hat{\varphi}_s(t', z')}^2 + [(t' - \delta) - \frac{\tilde{\Delta}_s \hat{\varphi}_s(t', z')}^2] + B(t' - \delta)H(t', z')$$  

$$\leq -C_2([t' - \delta] - \frac{\tilde{\Delta}_s \hat{\varphi}_s(t', z')}^2 + [(t' - \delta) - \frac{\tilde{\Delta}_s \hat{\varphi}_s(t', z')}^2] + C_3.$$
Then we obtain:

\[(t' - \delta) \frac{-\tilde{\Delta}_s \phi_s (t', z')}{C - \phi_s} \leq C_4,\]

and hence

\[(t' - \delta) K(t', z') \leq C_5.\]

Therefore when \(t > \delta\),

\[-\tilde{\Delta}_s \phi_s \leq \frac{C_5}{t - \delta} (C - \phi_s) \leq \frac{C_6}{t - \delta},\]

as \(H\) is nonnegative and the upper bound of \(S(\tilde{\omega}(t))\) follows.

\[\square\]

Now Theorem 1.2 can be deduced from the above proposition.

**Proof of Theorem 1.2** Let \(\pi : \hat{X} \to X\) be the crepant resolution with \(\pi^* K_X = K_{\hat{X}}\) and so \(\hat{X}\) is a Calabi-Yau manifold. Let \(\Omega\) be the Calabi-Yau volume form on \(X\), and \(\hat{\omega}_0 = \pi^* \omega_0\), \(\hat{\Omega} = \pi^* \Omega\) be the pull back of the initial metric and the Calabi-Yau volume form respectively. Then \(\hat{\omega}\) satisfies **Condition A** and \(\hat{\Omega}\) is a smooth, strictly positive Calabi-Yau volume form in the sense that \(\text{Ric}(\hat{\Omega}) = 0\). Hence the theorem follows from Proposition 4.2 by letting \(s \to 0\).

\[\square\]

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**References**


Perelman, G., unpublished work on the Kähler-Ricci flow.


Song, J. and Weinkove, B., *Contracting exceptional divisors by the Kähler-Ricci flow*, arXiv:1003.0718


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