On local holomorphic maps preserving invariant $(p,p)$-forms from unit balls into their products

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Abstract
In this note, we study local holomorphic maps from $B^n$ into $B^{N_1} \times \cdots \times B^{N_m}$ preserving the invariant $(p,p)$-forms induced from the normalized Bergman metrics up to some conformal factors. Assume that each conformal factor is smooth Nash algebraic, we prove that the local holomorphic map extends as a Nash algebraic map. Applying holomorphic continuation and analyzing the blown-up rate of $(p,p)$-forms when approaching the boundary of $B^n$ carefully, we show that each component is either a total geodesic embedding or has rank less than $p$.

1 Introduction
Write $B^n := \{ z \in \mathbb{C}^n : |z| < 1 \}$ for the unit ball in $\mathbb{C}^n$. Let

$$\omega_n = \sum_{j,k \leq n} \frac{1}{(1 - |z|^2)^2} ((1 - |z|^2) \delta_{jk} + \bar{z}_j z_k) dz_j \wedge d\bar{z}_k$$

be the invariant Kähler form associated to the normalized Bergman metric. Let $U \subset B^n$ be a connected open subset. Consider a holomorphic map

$$F = (F_1, \ldots, F_m) : U \to B^{N_1} \times \cdots \times B^{N_m} \quad (1)$$

that preserves invariant $(p,p)$-forms up to conformal constants $\{ \lambda_1, \cdots, \lambda_m \}$ in the sense that

$$\omega^n_p = \sum_{j=1}^m \lambda_j F_j^* (\omega_{N_j}^p). \quad (2)$$

Let $p \leq n \leq N_i$ for each $1 \leq i \leq m$. More precisely, for each $j$, $F_j$ is a holomorphic map from $U$ to $B^{N_j}$ and write $F_j = (f_{j,1}, \ldots, f_{j,l}, \ldots, f_{j,N_j})$, where $f_{j,l}$ is the $l$-th component of $F_j$. In this note, we prove the following rigidity theorem:
Theorem 1.1. Suppose \( n \geq 2 \). Under the above notation and assumption, and in addition, assume that each \( F_j \) is of rank at least \( p \) for \( 1 \leq j \leq m \) at some point in \( U \). Then we have, for each \( F_j \) extends to a totally geodesic holomorphic embedding from \( (B^n, \omega_n) \) into \( (B^N_j, \omega_{N_j}) \). Moreover, we have the following identity

\[
\sum_{1 \leq j \leq m} \lambda_j = 1.
\]

Recall that a function \( h(z, \bar{z}) \) is called a Nash algebraic function over \( \mathbb{C}^n \) if there is an irreducible polynomial \( P(z, \xi, X) \) in \( (z, \xi, X) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \) with \( P(z, \bar{z}, h(z, \bar{z})) \equiv 0 \) over \( \mathbb{C}^n \).

We mention that a holomorphic map from \( B^n \) into \( B^N \) is a totally geodesic embedding with respect to the normalized Bergman metric if and only if there are a (holomorphic) automorphism \( \sigma \in Aut(B^n) \) and an automorphism \( \tau \in Aut(B^N) \) such that \( \tau \circ F \circ \sigma(z) \equiv (z, 0) \).

The study of the global extension and rigidity problem for local isometric embeddings (i.e. \( p = 1 \)) is first carried out in a paper of Calabi [Ca]. Quite a few papers have appeared after [Ca] along these lines of research (see [DL] for instance as a very recent one). In 2003, motivated by problems from arithmetic dynamical system, Clozel and Ullmo [CU] reduce the problem of characterization of modular correspondence among Hecke correspondence between Shimura varieties \( \Omega/\Gamma \) to the rigidity problem for a local isometric embedding or a local holomorphic measure-preserving map with a certain symmetry from a bounded symmetric domain \( \Omega \) into its product \( \Omega \times \cdots \Omega \). More recently, Mok carried out a systematic study of such problems in a very general setting. Many far reaching deep results have been obtained by Mok and later by Ng and Mok-Ng. (See [Mo1] [Mo3] [MN2] [Ng1] and the references therein). For the problems of local holomorphic isometries, the rigidity for local isometric embeddings is obtained in the case of \( B^n \) into \( B^n \times \cdots \times B^n \) in [Mo1], of \( B^n \) into \( B^{N_1} \times B^{N_2} \) with \( N_1, N_2 < 2n \) in [Ng2], of \( D \) into \( \Omega \) with any bounded symmetric domain \( \Omega \) and the rank of any irreducible component of \( D \) at least 2 in [Mo3]. In a joint paper with Zhang [YZ], the author proves the rigidity for local isometric embeddings is obtained in the case of \( B^n \) into \( B^{N_1} \times \cdots \times B^{N_m} \) without any restriction on the codimension between \( n \) and \( N_i \) for \( 1 \leq i \leq m \). However, a very unexpected non-rigidity result is obtained by Mok [Mo5] that there exists non-standard isometric embeddings from \( B^n \) into a certain bounded symmetric domain \( \Omega \) of high rank. Regarding the problems of local measure-preserving maps, the rigidity is obtained for local holomorphic measure-preserving maps between bounded symmetric domains \( \Omega \) and its product \( \Omega \times \cdots \times \Omega \) with \( \dim \mathbb{C} \Omega \geq 2 \) in [MN2], answering the original question posed by Clozel and Ullmo.

The question on local holomorphic maps between bounded symmetric domains preserving invariant \((p, p)\)-forms is raised by Mok in [Mo4], which generalizes the problems considered in [Mo1] [MN2]. In this note, the goal is to answer Mok’s question for the local holomorphic map from the unit ball into the product of unit balls of any codimension. We emphasis that, by the work of Mok [Mo2] [Mo5], the result in Theorem 1.1 does not hold when \( n = 1 \) even if each \( \lambda_j \) is a constant function (the only possibility here is \( p = 1 \)) and also the rigidity for local holomorphic maps from \( B^n \) with \( n \geq 2 \) into \( \Omega \) preserving invariant \((p, p)\)-forms is not
expected to be true if $\Omega$ is a general bounded symmetric domain of rank at least 2. See also many non-standard examples and related classification results in [Ng1] [Mo2] [MN1] for the local holomorphic isometric embeddings from $\mathbb{B}^1$.

The proof of the theorem is based on an algebraic extension theorem (Theorem 2.2) derived by using Huang’s theorem [Hu1] and the argument of holomorphic continuation. The total geodesy is proved by comparing the blown-up rate for the Bergman metric of $\mathbb{B}^n$ when approaching the boundary. We remark that the case $q = 1$ can not be treated using the method in this note and has been proved in [YZ].

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2 Algebraic extension

Lemma 2.1. Let the Hermitian holomorphic vector bundle $E \rightarrow X$ over a complex manifold $X$ be Griffiths negative. Then $\wedge^k E$ is also Griffiths negative.

Proof of Lemma: Firstly, we show that $\otimes^k E$ is Griffiths negative. Inductively, one suffices to show that $E \otimes E$ is Griffiths negative. Use $D, \Theta$ to denote the connection and curvature operator of a Hermitian vector bundle (following notation in [De]). It follows that

\[ \Theta(D_{E\otimes E}) = \Theta(D_E) \otimes I_E + I_E \otimes \Theta(D_E), \]

(see [De] formula (V-4.2') on p. 258) where $I_E$ is the identity matrix with rank equal to $\text{rank}(E)$. One can easily check that $E \otimes E$ is Griffiths negative by showing that for any nonzero local holomorphic section $s$, \( \left( \Theta(D_{E\otimes E})\left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \bar{x}^j}, s, \bar{s} \right) \right) \) is a strictly negative-definite \((\text{dim}(X))^2 \times (\text{dim}(X))^2\) matrix, where \( \{x_i\} \) is the local coordinate of $X$.

Secondly, as $\wedge^k E$ is a subbundle of $\otimes^k E$, $\wedge^k E$ is also Griffiths negative by Proposition (6.10) in [De] (p. 340). In fact, there is an analogue Gauss-Codazzi equation for the vector bundle,

\[ \theta_{\wedge^k E}(u, u) = \theta_{\otimes^k E}(u, u) - |\beta \cdot u|^2, \]

where $u \in TX \otimes E$ and $\beta \in \wedge^{1,0}(X, Hom(\wedge^k E, \otimes^k E/ \wedge^k E))$ is the second fundament form of $\wedge^k E$ in $\otimes^k E$. 

3
Theorem 2.2. Let $F := (F_1, \ldots, F_m) : U \rightarrow \mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}$ be the holomorphic map defined on $U \subset \mathbb{B}^n$ that preserves invariant $(p, p)$-forms up to conformal factors $\{\lambda_1(z, \bar{z}), \ldots, \lambda_m(z, \bar{z})\}$ in the sense that
\[
\omega^p_n = \sum_{j=1}^m \lambda_j(z, \bar{z}) F_j^*(\omega^p_{N_j}),
\]
where $\lambda_j(z, \bar{z})$ for each $j = 1, \ldots, m$ is assumed to be positive smooth Nash algebraic functions over $\mathbb{C}^n$. Let $p \leq n \leq N_i$ for each $1 \leq i \leq m$. Then $F$ is Nash algebraic.

Proof of Theorem: Let $K > 0$ be a large constant to be determined. Consider $S_1 \subset \wedge^p(TU)$ and $S_2 \subset U \times \wedge^p(T\mathbb{B}^{N_1}) \times \cdots \times \wedge^p(T\mathbb{B}^{N_m})$ as follows:
\[
S_1 := \{(t, \zeta) \in \wedge^p(TU) : (1 + K|t|^2)\omega^p_n(t)(\zeta, \bar{\zeta}) = 1\},
\]
\[
S_2 := \{(t, z_1, \xi_1, \ldots, z_m, \xi_m) \in U \times T\mathbb{B}^{N_1} \times \cdots \times T\mathbb{B}^{N_m} : (1 + K|t|^2)[\lambda_1(t, \bar{t})\omega^p_{N_1}(z_1)(\xi_1, \bar{\xi}_1) + \cdots + \lambda_m(t, \bar{t})\omega^p_{N_m}(z_m)(\xi_m, \bar{\xi}_m)] = 1\}. \tag{5}
\]

The defining functions $\rho_1, \rho_2$ of $S_1, S_2$ are, respectively, as follows:
\[
\rho_1 = (1 + K|t|^2)\omega^p_n(t)(\zeta, \bar{\zeta}) - 1,
\]
\[
\rho_2 = (1 + K|t|^2)[\lambda_1(t, \bar{t})\omega^p_{N_1}(z_1)(\xi_1, \bar{\xi}_1) + \cdots + \lambda_m(t, \bar{t})\omega^p_{N_m}(z_m)(\xi_m, \bar{\xi}_m)] - 1.
\]
Here $\{t\}, \{z_i\}$ are the canonical Euclidean coordinates on $\mathbb{C}^n, \mathbb{C}^{N_i}$ respectively. Then one can easily check that the map $(id, F_1, dF_1, \cdots, F_m, dF_m)$ maps $S_1$ to $S_2$ according to the equation (3). It is obvious that $S_1, S_2$ are both real algebraic CR hypersurfaces. To finish the proof the theorem, it suffices to show that $(id, F_1, dF_1, \cdots, F_m, dF_m)$ maps a strongly pseudoconvex point on $S_1$ to a strongly pseudoconvex point in $S_2$. We show the strong pseudoconvexity of $S_2$ at $Q = (0, 0, \xi_1, \ldots, 0, \xi_m)$ as follows and the strong pseudoconvexity of $S_1$ follows from the same computation.

Let $\{s_{K_j}\}$ be the basis of $\wedge^p T\mathbb{B}^{N_i}$ and write $\xi_j = \xi_{K_j} s_{K_j}$, where $K_j$ is the multi-index $(k_{j_1}, \ldots, k_{j_p})$. Denote $\Delta_j = N_j \cdots (N_j - p + 1)$ to be the rank of the vector bundle $\wedge^p T\mathbb{B}^{N_j}$. By applying $\partial \bar{\partial}$ to $\rho_2$ at $Q = (0, 0, \xi_1, \ldots, 0, \xi_m)$, we have the following Hessian matrix
\[
\mathcal{H} = \begin{bmatrix}
A & 0 & D_1 & \cdots & 0 & D_m \\
0 & B_1 & 0 & \cdots & 0 & 0 \\
D_1 & 0 & C_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & B_m & 0 \\
D_m & 0 & 0 & \cdots & 0 & C_m
\end{bmatrix}. \tag{6}
\]
where $A, B_j, C_j, D_j, j = 1, 2, \ldots, m$ are function-valued matrices with the following (in)equalities:

$$A := \left( \partial_{ij} \partial_{j} \rho_2 \right) (Q) = p! \left( K \sum_{\nu=1}^{m} \lambda_{\nu}(0) |\xi_{\nu}|^2 \delta_{ij} + \sum_{\nu=1}^{m} \partial_{ij} \lambda_{\nu}(0) |\xi_{\nu}|^2 \right) \geq \delta K (|\xi_1|^2 + \cdots + |\xi_m|^2) I_n, \tag{7}$$

$$B_j := \left( \partial_{j} \partial_{j} \rho_2 \right) (Q) = \left( -\lambda_j(0) \sum_{K_j, L_j=1}^{\Delta_j} \Theta_{\lambda \rho T \rho \rho \rho} \left( \frac{\partial}{\partial z_{jk}}, \frac{\partial}{\partial z_{jl}}, s_{K_j, L_j} \right)(0) \xi_{K_j} \xi_{L_j} \right) \geq \delta |\xi_j|^2 I_{N_j}, \tag{8}$$

$$C_j := \left( \partial_{s_{K_j}} \partial_{s_{L_j}} \rho_2 \right) (Q) = p! \left( \lambda_j(0) \delta_{K_j, L_j} \right) \geq \delta I_{\Delta_j}, \tag{9}$$

$$D_j := \left( \partial_{i} \partial_{\xi_{L_j}} \rho_2 \right) (Q) = p! \left( \partial_{i} \lambda_j(0) \xi_{K_j} \right)_{1 \leq i \leq n, K_j = (k_{j_1}, \ldots, k_{j_p})}, \tag{10}$$

at $(0, 0, \xi_1, \ldots, 0, \xi_m)$ for some $\delta > 0$. Here $B_j$ is positive definite (the inequality in (8) holds) because $\wedge^p T \rho \rho$ is Griffiths negative by applying Lemma 2.1. By the similar argument as in Lemma 4.1 of [YZ], one can show that $\mathcal{H}$ is positive definite for sufficiently large $K > 0$. This implies that $Q \in S_2$ is a strongly pseudoconvex point.

Without loss of generality, one can assume that $F(0) = 0$ by composing elements from $Aut(\mathbb{B}^n)$ and $Aut(\mathbb{B}^N_1) \times \cdots \times Aut(\mathbb{B}^N_m)$. Furthermore, since $F_1, \ldots, F_m$ are of rank at least $p$ at 0, we can assume that $dF_j|_0 \neq 0, \ldots, dF_m|_0 \neq 0$. Therefore, there exists $0 \neq \zeta \in \wedge^p T_0 \mathbb{B}^n$, such that $dF_j(\zeta) \neq 0$ for all $j$. After scaling, we assume that $(0, \zeta) \in S_1$. Notice that both the fiber of $S_1$ over $0 \in U$ and the fiber of $S_2$ over $(0, 0, \ldots, 0) \in U \times \mathbb{B}^N_1 \times \cdots \times \mathbb{B}^N_m$ are independent of the choice of $K$. Now the theorem follows by applying the algebraicity theorem of Huang [Hu1] to the map $(id, F_1, dF_1, \ldots, F_m, dF_m)$ from $S_1$ into $S_2$. □

3 Total geodesy

**Lemma 3.1.** Let $F : \mathbb{B}^n \to \mathbb{B}^N$ be a rational, proper holomorphic map. Let $\Xi = \omega_n - F^* \omega_N$. Then $\Xi$ is a non-negative $(1, 1)$-form in $\mathbb{B}^n$ that is real analytic on an open neighborhood of $\overline{\mathbb{B}}^n$. Moreover, $F$ is a totally geodesic embedding if and only if either $\Xi \equiv 0$ in $\mathbb{B}^n$ or $\Xi \equiv 0$ on an open piece $V$ of $\partial \mathbb{B}^n$ of real dimension $2n - 1$.

**Proof:** It follows from the Schwarz Lemma that $\Xi$ is non-negative and $\Xi \equiv 0$ if and only if $F$ is a totally geodesic embedding. $\Xi$ is real analytic on an open neighborhood of $\overline{\mathbb{B}}^n$ is proved in Corollary 2.3 in [YZ]. Moreover, $\Xi$ is closely related to the first fundamental form of the CR map between unit spheres. By Proposition 2.5 in [YZ] and Theorem 4.2 in [Hu2] to the second normalization $F_p^* = \sigma \circ F \circ \tau_p^0$ with $\tau_p^0 \in Aut(\mathbb{B}^n), \sigma \in Aut(\mathbb{B}^N)$ and $\tau_p^0(0) = p, \sigma(F(p)) = 0$ for each $p \in V$, it follows that $a_{ij} = 0$ and hence $F$ is totally geodesic. □
Lemma 3.2. Let $V \subset \mathbb{C}^{N_1}$ be a connected open set and $F = (f_1, \cdots, f_{N_2}) : V \to \mathbb{C}^{N_2}$ be a holomorphic map. Let $\{w_i\}_{i=1}^{N_1}$ and $\{z_k\}_{k=1}^{N_2}$ be the coordinates of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. Assume $N_1 \geq p, N_2 \geq p$. Then

$$F^*(dz_{k_1} \wedge \cdots \wedge dz_{k_p} \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_p) = \sum_{i_1 < \cdots < i_p, j_1 < \cdots < j_p} \det \left( \frac{\partial (f_{k_1}, \cdots, f_{k_p})}{\partial (w_{i_1}, \cdots, w_{i_p})} \right) \det \left( \frac{\partial (f_{k_1}, \cdots, f_{k_p})}{\partial (w_{j_1}, \cdots, w_{j_p})} \right) dw_{i_1} \wedge \cdots \wedge dw_{i_p} \wedge d\overline{w}_{j_1} \wedge \cdots \wedge d\overline{w}_{j_p}. \quad (11)$$

Proof of Theorem 1.1: Let $X$ be the union of the branch varieties of $F_j$ for $1 \leq j \leq m$. Since $\dim_{\mathbb{C}} X \leq n - 1$, for any $Q_0 \in \mathbb{C}^n \setminus X$, there is a real curve $\gamma$ connecting $Q_0$ and $U$ such that any branch of $F$ is holomorphically continued along $\gamma$ to the germ of holomorphic map at $Q_0$, still denoted by $F$. Define $E = \bigcup_{j=1}^m \{ z \in \mathbb{B}^n \setminus X | |F_j(z)| = 1 \}$ and $\dim_{\mathbb{R}} E \leq 2n - 1$.

At the first step, we are going to show $\dim_{\mathbb{R}} E \leq 2n - 2$. Suppose not. Then there is a curve $\gamma$ connecting $U$ and a point $Q_0 \in E$ such that $\dim_{\mathbb{R}} O = 2n - 1$, where $O \subset E$ is an open neighborhood of $Q_0$, and moreover, assume $\{Q_0\} = \gamma \cap E$. Without loss of generality, assume $Q_0 \in O \subset \{ z \in E | |F_1(z)| = 1 \}$. Since equation (3) holds in a small open neighborhood of $\gamma$ for any branch of $F$ by the holomorphic continuation, one has the following equation as points $Q_s$ on $\gamma$ approaches $Q_0$:

$$\omega^p_n(Q_s) = \sum_{j=1}^m \lambda_j(Q_s, Q_s) F_j^*(\omega^p_{N_j}(Q_s)).$$

Denote the coordinates of $\mathbb{B}^n$ and $\mathbb{B}^{N_1}$ by $\{z_i\}_{i=1}^n$ and $\{w_k\}_{k=1}^{N_1}$ respectively and $F_1 = (f_1, \cdots, f_{N_1})$. It follows that

$$\omega^p_n(Q_s) \geq \lambda_1(Q_s, Q_s) F_1^*(\omega^p_{N_1}(Q_s)) \geq \lambda_1(Q_s, Q_s) F_1^* \left( \left( \sum_{k=1}^{N_1} dw_k \wedge d\overline{w}_k \right)^p / \left( 1 - |w|^2 \right)^p \right) (Q_s)$$

$$= \frac{\lambda_1(Q_s, Q_s)}{(1 - |F_1(Q_s)|^2)^p} \sum_{k_1 < \cdots < k_p} C_{k_1, \cdots, k_p} \Theta_{k_1, \cdots, k_p} (Q_s), \quad (12)$$

where $C_{k_1, \cdots, k_p}$ is the constant coefficient of $dw_{k_1} \wedge \cdots \wedge dw_{k_p} \wedge d\overline{w}_{k_1} \wedge \cdots \wedge d\overline{w}_{k_p}$ in $(\sum_{k=1}^{N_1} dw_k \wedge d\overline{w}_k)^p$ and

$$\Theta_{k_1, \cdots, k_p} = \sum_{i_1 < \cdots < i_p, j_1 < \cdots < j_p} \frac{\partial (f_{k_1}, \cdots, f_{k_p})}{\partial (z_{i_1}, \cdots, z_{i_p})} \frac{\partial (f_{k_1}, \cdots, f_{k_p})}{\partial (z_{j_1}, \cdots, z_{j_p})} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_p}.$$
By the uniqueness of holomorphic functions, it follows that $\omega^p_n(Q_s)$ is a curve, still denoted by $U$ on that for any $k_1 \leq \cdots \leq k_p, i_1 \leq \cdots \leq i_p$ and any $Q_0 \in O$,\[
\frac{\partial (f_{k_1}, \ldots, f_{k_p})}{\partial (z_{i_1}, \ldots, z_{i_p})}(Q_0) = 0.
\]
By the uniqueness of holomorphic functions, it follows that \[
\frac{\partial (f_{k_1}, \ldots, f_{k_p})}{\partial (z_{i_1}, \ldots, z_{i_p})} \equiv 0
\]
on $U$. This contradicts to the assumption that $F_1$ is of rank at least $p$.

Given any point, still denoted by $Q_0 \in \partial B^n \setminus X$, it follows from the previous step that there is a curve, still denoted by $\gamma$, connecting $U$ and $Q_0$, such that $\gamma \cap E = \emptyset$. Since the equation (3) holds on a small open neighborhood $O'$ of $\gamma$ and as $Q_s \in O'$ approaches $Q_0$, the coefficients of $\omega^p_n(Q_s)$ go to $+\infty$, then the coefficients of $F_j^*(\omega^p_{N_j})(Q_s)$ go to $+\infty$ for certain $j$. This implies $|F_j(Q_0)| = 1$, i.e. $F_j(\partial B^n) \subset \partial B^{N_j}$. Assume that $F_j$ maps an open subset of $\partial B^n$ into $\partial B^{N_j}$ exactly for $1 \leq j \leq m_0$ after reordering $\{1, \ldots, j\}$. By the theorems of Forstneric [Fo] and Cima-Suffridge [CS], $F_j$ extends to the unique proper holomorphic map between $B^n$ and $B^{N_j}$, which is rational, for $1 \leq j \leq m_0$. Therefore, there exists an open subset of $\partial B^n$, still denoted by $O$ such that $F_j(O) \subset \partial B^{N_j}$ for $1 \leq j \leq m_0$ and $F_j(O) \subset B^{N_j}$ for $m_0 + 1 \leq j \leq N_j$. Rewrite the equation (3):

\[
\sum_{j=1}^{m_0} \lambda_j(z, \bar{z})(\omega^p_n - (F_j^*\omega_{N_j})^p) + (1 - \sum_{j=1}^{m_0} \lambda_j(z, \bar{z}))\omega^p_n = \sum_{j=m_0+1}^{m} \lambda_j(z, \bar{z})F_j^*(\omega^p_{N_j}) \quad (13)
\]
Letting $\Xi_j = \omega_n - F_j^*\omega_{N_j}$ and applying difference formula, it follows that

\[
\sum_{j=1}^{m_0} \lambda_j(z, \bar{z})\Xi_j \wedge \left(\sum_{t=0}^{p-1} \omega^t_n \wedge (F_j^*\omega_{N_j})^{p-1-t}\right) + (1 - \sum_{j=1}^{m_0} \lambda_j(z, \bar{z}))\omega^p_n = \sum_{j=m_0+1}^{m} \lambda_j(z, \bar{z})F_j^*(\omega^p_{N_j}) \quad (14)
\]
For each $j$, it follows from $\Xi_j \geq 0$ that

\[
\Xi_j \wedge \omega^{p-1}_n \leq \Xi_j \wedge \left(\sum_{t=0}^{p-1} \omega^t_n \wedge (F_j^*\omega_{N_j})^{p-1-t}\right) \leq p\Xi_j \wedge \omega^{p-1}_n.
\]
Now we rewrite equation (17) in the coordinate of the Siegel upper half space $\mathbb{H}^n = \{(Z, W) \in \mathbb{C}^{n-1} \times \mathbb{C} : \Re W - |Z|^2 > 0\}$. Recall the following Cayley transformation

\[
\rho_n(Z, W) = \left(\frac{2Z}{1-iW}, \frac{1+iW}{1-iW}\right).
\]

7
Then $\rho_n$ biholomorphically maps $\mathbb{H}^n$ to $\mathbb{B}^n$, and biholomorphically maps $\partial \mathbb{H}^n$, the Heisenberg hypersurface, to $\partial \mathbb{B}^n \backslash \{(0, 1)\}$. Applying the Cayley transformation, one can compute the normalized Bergman metric on $\mathbb{H}^n$, still denoted by $\omega_n$, by pulling back the normalized Bergman metricon $\mathbb{B}^n$, as follows:

$$
\omega_n = \sum_{j,k<n} \frac{\delta_{jk}(3W - |Z|^2)}{(3W - |Z|^2)^2} dZ_j d\bar{Z}_k + \frac{dW \wedge d\bar{W}}{4(3W - |Z|^2)^2} + \sum_{j<n} \frac{Z_j dZ_j \wedge d\bar{W}}{2i(3W - |Z|^2)^2} - \sum_{j<n} \frac{Z_j dW \wedge d\bar{Z}_j}{2i(3W - |Z|^2)^2}.
$$

(16)

Note that $\omega_n$ is also an invariant metric under the action of the holomorphic automorphism group of $\mathbb{H}^n$. Still denote $\rho^{-1}(Q_0)$ by $Q_0$. Without loss of generality, one may assume that $Z(Q_0) = 0$ by composing the holomorphic automorphism of $\mathbb{H}^n$. One chooses $Q_s$ such that $Z(Q_s) = 0$. Hence one has:

$$
\omega_n(Z(Q_s), W(Q_s)) = \sum_{k<n} \frac{1}{3W(Q_s)} dZ_k d\bar{Z}_k + \frac{1}{4(3W(Q_s))^2} dW \wedge d\bar{W}.
$$

The right hand side of equation (17) is bounded, implying $1 - \sum_{j=0}^{m_0} \lambda_j = 0$, because the blown-up rate for $\Xi_j \wedge \left( \sum_{t=0}^{p-1} \omega_n^t \wedge (Q_s)(F_j^* \omega_{N_j})^{p-1-t} \right) \left( \frac{\partial}{\partial Z_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial Z_{i_p}} \wedge \frac{\partial}{\partial W} \wedge \frac{\partial}{\partial \bar{W}} \wedge \cdots \wedge \frac{\partial}{\partial Z_{i_p}} \wedge \frac{\partial}{\partial \bar{W}} \right)$ and $\omega^p(Q_s) \left( \frac{\partial}{\partial Z_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial Z_{i_p}} \wedge \frac{\partial}{\partial W} \wedge \frac{\partial}{\partial \bar{W}} \wedge \cdots \wedge \frac{\partial}{\partial Z_{i_p}} \wedge \frac{\partial}{\partial \bar{W}} \right)$ is $\frac{1}{(3W(Q_s))^p}$ and that of $\omega^p(Q_s) \left( \frac{\partial}{\partial Z_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial Z_{i_p}} \wedge \frac{\partial}{\partial W} \wedge \frac{\partial}{\partial \bar{W}} \wedge \cdots \wedge \frac{\partial}{\partial Z_{i_p}} \wedge \frac{\partial}{\partial \bar{W}} \right)$ is $\frac{1}{(3W(Q_s))^p+1}$, which is higher. Therefore, it follows that

$$
\sum_{j=m_0+1}^{m} \lambda_j F_j^* \omega_{N_j}^p = \sum_{j=1}^{m_0} \lambda_j \Xi_j \left( \sum_{t=0}^{p-1} \omega_n^t \wedge (F_j^* \omega_{N_j})^{p-1-t} \right) \geq \sum_{j=1}^{m_0} \lambda_j \Xi_j \wedge \omega_n^{p-1}.
$$

(17)

We only need consider the case $p \geq 2$ (the case $p = 1$ is solved in [YZ]). By the similar argument as above, it follows that $\Xi_j \equiv 0$ on an open piece of $\partial \mathbb{B}^n$ of real dimension $2n - 1$ for each $1 \leq j \leq m_0$. Hence $F_j$ is a totally geodesic embedding for each $j$ by Lemma 3.1, and therefore $\Xi_j \equiv 0$ on $\mathbb{B}^n$. Now the equation (17) reads

$$
\sum_{j=m_0+1}^{m} \lambda_j F_j^* \omega_{N_j}^p \equiv 0
$$

(18)

on $U$, implying $F_j^* \omega_{N_j}^p \equiv 0$ on $U$ for $m_0 + 1 \leq j \leq m$. It follows from Lemma 3.2 that for any $1 \leq k_1 < \cdots < k_p \leq N_j$, $i_1 < \cdots < i_p$,

$$
\frac{\partial(f_{j k_1}, \ldots, f_{j k_p})}{\partial(z_{i_1}, \ldots, z_{i_p})} \equiv 0,
$$

where $F_j = (f_{j1}, \ldots, f_{jN_j})$ for each $m_0 + 1 \leq j \leq m$. This contradicts the assumption that each $F_j$ is of rank at least $p$. ■
References


[YZ] Yuan, Y. and Zhang, Y.: *Rigidity for local holomorphic isometric embeddings from $B^n$ into $B^{N_1} \times \ldots \times B^{N_m}$ up to conformal factors*, J. Differential Geom. 90 (2012), no. 2, 329-349.

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