The isoperimetric inequality and $Q$-curvature

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ABSTRACT

A well-known question in differential geometry is to control the constant in isoperimetric inequality by intrinsic curvature conditions. In dimension 2, the constant can be controlled by the integral of the positive part of the Gaussian curvature. In this paper, we showed that on simply connected conformally flat manifolds of higher dimensions, the role of the Gaussian curvature can be replaced by the Branson’s $Q$-curvature. We achieve this by exploring the relationship between $A_p$ weights and integrals of the $Q$-curvature.

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1. Introduction

The classical isoperimetric inequality on $\mathbb{R}^2$ states that for any bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary,

$$\text{vol}(\Omega) \leq \frac{1}{4\pi} \text{Area}(\partial \Omega)^2.$$
On a complete noncompact simply connected surface \( M^2 \), the well-known Fiala–Huber’s \([16,18]\) isoperimetric inequality is

\[
\text{vol}(\Omega) \leq \frac{1}{2(2\pi - \int_{M^2} K_g^+ \, dv_g)} \text{Area}(\partial \Omega)^2, \tag{1.1}
\]

where \( K_g^+ \) is the positive part of the Gaussian curvature \( K_g \).

\[
\int_{M^2} K_g^+ \, dv_g < 2\pi \tag{1.2}
\]

is the sharp bound for the isoperimetric inequality to hold.

In this paper, we aim to study if one can prove Fiala–Huber’s result in higher dimensions. We want to see if some correct curvature quantities could play a similar role as the Gaussian curvature does on surfaces, such that their integrals control the isoperimetric property of the manifold. More precisely, we want to answer the following questions:

1) Can we derive isoperimetric inequalities of Fiala–Huber’s type in higher dimensions, using the integral of some curvature quantity to control the isoperimetric constant? There are many results of isoperimetric inequalities in higher dimensions using pointwise curvature assumptions. For example, the works of \([1,6,25,12]\) proved the isoperimetric inequality with pointwise sectional curvature or Ricci curvature bound. However, pointwise assumptions on the curvature are not the most natural ones. One can see it from the example of a rotationally symmetric cone. The curvature at the vertex of the cone (in weak sense) is infinity. However, the cone does satisfy the isoperimetric inequality.

2) What is the suitable substitute of the Gaussian curvature in higher dimension to control the isoperimetric behavior? In dimension 2, the Gaussian curvature controls the geometry of the manifolds. However, in higher dimensions, there are many choices. For example, the sectional curvature, the Ricci curvature, the isotropic curvature, the scalar curvature, etc. So it is not clear at all what curvature quantity/quantities play the role in higher dimensions.

In this paper, we answer the above two questions. It turns out that in the setting of conformal geometry, one only needs to impose integral curvature assumptions as in Fiala–Huber’s result, and the Branson’s \( Q \)-curvature is the correct curvature quantity to look at on higher dimensional manifolds. To prove the result, we find out that the conformal structure is the key structure to allow this generalization of Fiala–Huber’s isoperimetric inequality to higher dimensions.

The \( Q \)-curvature, introduced by Branson \([5]\), is an important notion in conformal geometry. In dimension 2, \( Q_g = K_g / 2 \), and in dimension 4, \( Q_g = \frac{1}{12} (-\Delta R_g + \frac{1}{4} R_g^2 - 3|E_g|^2) \), where \( R_g \) denotes the scalar curvature and \( E_g \) denotes the traceless part of the Ricci tensor. In general case, \( Q \)-curvature remains a mysterious quantity that it is defined via analytic continuation in dimensions. (See the definition in Section 2.)
conformal geometry, there has been much progress made on its analytic properties based on higher order elliptic PDEs. For a incomplete list of works in this direction, see the work of Fefferman–Graham [15] which studies the $Q$-curvature and ambient metrics; that of Chang–Yang [10], Chang–Gursky–Yang [7] on the existence and regularity of the PDE that a constant $Q$-curvature metric satisfies, etc.

Let us begin with the statement of the main result.

**Theorem 1.1.** Let $(M^n, g) = (\mathbb{R}^n, g = e^{2u}|dx|^2)$ be a complete noncompact even dimensional manifold. Let $Q^+$ and $Q^-$ denote the positive and negative part of $Q_g$ respectively, and $dv_g$ denote the volume form of $M$. Suppose $g = e^{2u}|dx|^2$ is a “normal” metric, i.e.

$$u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x - y|} Q_g(y) dv_g(y) + C;$$

(1.3)

for some constant $C$. Then if the $Q$-curvature satisfies

$$\alpha := \int_{M^n} Q^+ dv_g < c_n$$

(1.4)

where $c_n = 2^{n-2}(\frac{n-2}{2})!\pi^{\frac{n}{2}}$, and

$$\beta := \int_{M^n} Q^- dv_g < \infty,$$

(1.5)

then $(M^n, g)$ satisfies the isoperimetric inequality with isoperimetric constant depending only on $n$, $\alpha$ and $\beta$. Namely, for any bounded domain $\Omega \subset M^n$ with smooth boundary,

$$|\Omega|^{\frac{n-1}{n}} \leq C(n, \alpha, \beta)|\partial \Omega|_g.$$

(1.6)

It is important to notice that the constant $c_n$ in the assumption (1.4) is sharp. In fact, $c_2 = 2\pi$ which is sharp bound in (1.2) of Fiala–Huber’s inequality. $c_n$ is equal to the integral of the $Q$-curvature on a half cylinder (a cylinder with a round cap attached to one of its two ends); but obviously a half cylinder fails to satisfy the isoperimetric inequality.

The assumption that $g = e^{2u}|dx|^2$ is a “normal” metric is a natural and necessary assumption in higher dimensions. There are well-known counterexamples to the isoperimetric inequality in Theorem 1.1 if we remove this assumption. In addition, this analytic assumption has geometric meaning. In dimension 4, by a maximal principle argument, one can see that if the scalar curvature satisfies $\lim \inf_{|x| \to \infty} R_g(x) \geq 0$, then the metric is a normal metric. See for example [8, Theorem 4.1] for the proof. We will give more explanations about normal metric in Remark 5.4 in Section 5.
Let us return to the statement of the main result. It is nice to notice that the influence of the positive part of the $Q$-curvature to the validity of the isoperimetric inequality is much more essential than that of the negative part of the $Q$-curvature. This can be easily seen from the assumptions (1.4) and (1.5). This agrees with the general intuition of the isoperimetric inequality and the negative curvature. A well-known conjecture in differential geometry asserts that the Euclidean isoperimetric inequality holds on complete simply connected manifolds with nonpositive sectional curvature. This conjecture was proved in dimension 2 by Weil [27], in dimension 3 by Kleiner [20], and in dimension 4 by Croke [13]; but it is still open for higher dimensions.

In the works of Chang, Qing and Yang [8,9], the authors explored the relationship between the $Q$-curvature and Cohn-Vossen inequality. More precisely, it was proved that if the metric is “normal” (as defined by (1.3)) and $\int_{\mathbb{R}^n} |Q_g| dv_g \leq \infty$, then the isoperimetric profile for very big balls is captured by the integral of the $Q$-curvature. The relation is that

$$\chi(\mathbb{R}^4) - \frac{1}{4\pi^2} \int_{\mathbb{R}^4} Q_g dv_g = \lim_{r \to \infty} \frac{\text{Area}_g(\partial B(r))^{4/3}}{4(2\pi^2)^{1/3} \text{Vol}_g(B(r))^{1/3}},$$

where $B(r)$ is the Euclidean ball with radius $r$. This generalizes the works of Cohn-Vossen [11] and Huber [18] for surfaces. This also gives us a hint that $Q$-curvature may be a good substitute of the Gaussian curvature in higher dimensions to control the isoperimetric profile, not just at the end as well.

Previous work on the $Q$-curvature and its relationship to the geometry of the complete manifolds was done by Bonk, Heinonen and Saksman [3]. They showed that if the metric is “normal”, and if in addition $\int_{\mathbb{R}^n} |Q_g| dv_g \leq \epsilon_0$ for some small $\epsilon_0 << 1$, then the manifold is bi-Lipschitz to the Euclidean space, which in particular implies the isoperimetric inequality. In my previous work [26], we proved the isoperimetric inequality with weaker assumptions, but the isoperimetric constant is not uniformly controlled. It is finally in this paper, that we derive the isoperimetric inequality with sharp integral assumption on the $Q$-curvature, and the isoperimetric constant only depends on $\alpha = \int_{M^n} Q^+ dv_g$, $\beta := \int_{M^n} Q^- dv_g$ and the dimension $n$.

The methods applied by Bonk, Heinonen and Saksman [3] is quasi-conformal flow. Smallness assumption $\int_{\mathbb{R}^n} |Q_g| dv_g \leq \epsilon_0$ when dimension $n \geq 3$ is used as the small energy condition to prove the existence of the flow. When $n = 2$, the existence of quasi-conformal maps is proved by Bonk and Lang [4]. In order to overcome this major difficulty, we adopt a very different method in this paper. The main proof relates conformal invariants and geometric behavior of manifolds to $A_p$ and strong $A_\infty$ weights. Inspired by Peter Jones’ result [19] on the decomposition of $A_\infty$ weights, in particular, the idea of dyadic decomposition of BMO functions, we notice that proper decomposition of the weight $e^{nu}$ characterizes different roles of the positive and negative parts of the $Q$-curvature. Conceptually, the observation is that there is a parallel structure between the geometric obstruction of having isoperimetric inequality with the analytic obstruction of being in
suitable classes of $A_p$ and strong $A_\infty$ weights. We conclude that the volume form $e^{nu}$ is a strong $A_\infty$ weight, and thus by a classical result of Guy David and Stephen Semmes [14] (see Theorem 2.1 below), this implies the isoperimetric inequality is valid.

The paper is organized as follows. In Section 2, we present preliminaries on the $Q$-curvature and weights. We then decompose the volume form $e^{nu}$ into two parts: $e^{nu+}$ and $e^{nu-}$ (see Definition 3.1), and discuss their different behaviors in Section 3 and Section 4 respectively. In Section 5 we put these pieces together, and show that $e^{nu}$ is a strong $A_\infty$ weight and finish the proof.

2. Preliminaries

2.1. $Q$-curvature in conformal geometry

In past decades, there are many works focusing on the study of the $Q$-curvature equation and the associated conformal covariant operators, both from PDE point of view and from the geometry point of view. We now discuss some background of it in conformal geometry. Consider a 4-manifold $(M^4, g)$, the Branson’s $Q$-curvature of $g$ is defined as

$$Q_g := \frac{1}{12} \left\{ -\Delta R_g + \frac{1}{4} R_g^2 - 3 |E|^2 \right\},$$

where $R_g$ is the scalar curvature, $E_g$ is the traceless part of $Ric_g$, and $| \cdot |$ is taken with respect to the metric $g$. It is well known that the $Q$-curvature is an integral conformal invariant associated with the fourth order Paneitz operator $P_g$

$$P_g := \Delta^2 + \delta \left( \frac{2}{3} R_g g - 2 Ric_g \right) d.$$

Under the conformal change $g_u = e^{2u} g_0$, $P_{g_u} = e^{-4u} P_{g_0}$, $Q_{g_u}$ satisfies the fourth order differential equation,

$$P_{g_0} u + 2Q_{g_0} = 2Q_{g_u} e^{4u}. \quad (2.1)$$

This is analogous to the Gaussian curvature equation on surfaces

$$-\Delta_{g_0} u + K_{g_0} = K_{g_u} e^{2u}.$$

One particular situation is when the background metric $g_0 = |dx|^2$. In this case, the equation (2.1) reduces to

$$(-\Delta)^2 u = 2Q_{g_u} e^{4u},$$

where $\Delta$ is the Laplacian operator of the flat metric $g_0$. 

Another analogy between the $Q$-curvature and the Gaussian curvature is the invariance of the integral of the $Q$-curvature, due to the Chern–Gauss–Bonnet formula for closed manifold $M^4$:

$$\chi(M^4) = \frac{1}{4\pi^2} \int_{M^4} \left( \frac{|W_g|^2}{8} + Q_g \right) dv_M,$$

where $W_g$ denotes the Weyl tensor.

For higher dimensions, the $Q$-curvature is defined via the analytic continuation in the dimension and the formula is not explicit in general. However when the background metric is flat, it satisfies, under the conformal change of metric $g_u = e^{2u} |dx|^2$, the $n$-th order differential equation

$$(-\Delta)^\frac{n}{2} u = 2Q_{g_u} e^{nu},$$

where $\Delta$ is the Laplacian operator of $|dx|^2$. We will only use this property of $Q$-curvature in this paper.

2.2. $A_p$ weights and strong $A_\infty$ weights

In this subsection, we are going to present the definitions and the properties of $A_p$ weights and strong $A_\infty$ weights.

In harmonic analysis, $A_p$ weights ($p \geq 1$) are introduced to characterize when a function $\omega$ could be a weight such that the associated measure $\omega(x) dx$ has the property that the maximal function $M$ of an $L^1$ function is weakly $L^1$, and that the maximal function of an $L^p$ function is $L^p$ if $p > 1$.

For a nonnegative locally integrable function $\omega$, we call it an $A_p$ weight $p > 1$, if

$$\frac{1}{|B|} \int_B \omega(x) dx \cdot \left( \frac{1}{|B|} \int_B \omega(x)^{-p'/p} dx \right)^{p/p'} \leq C < \infty,$$

for all balls $B$ in $\mathbb{R}^n$. Here $p'$ is conjugate to $p$: $\frac{1}{p'} + \frac{1}{p} = 1$. The constant $C$ is uniform for all $B$ and we call the smallest such constant $C$ the $A_p$ bound of $\omega$. The definition of $A_1$ weight is given by taking limit of $p \to 1$ in (2.2), which gives

$$\frac{1}{|B|} \int_B \omega \leq C \omega(x),$$

for almost all $x \in B$. Thus it is equivalent to say the maximal function of the weight is bounded by the weight itself:

$$M \omega(x) \leq C' \omega(x),$$
for a uniform constant $C'$. Another extreme case is the $A_\infty$ weight. $\omega$ is called an $A_\infty$ weight if it is an $A_p$ weight for some $p > 1$. It is not difficult to see $A_1 \subseteq A_p \subseteq A'_p \subseteq A_\infty$ when $1 \leq p \leq p' \leq \infty$.

One of the most fundamental properties of $A_p$ weight is the reverse Hölder inequality: if $\omega$ is $A_p$ weight for some $p \geq 1$, then there exist an $r > 1$ and a $C > 0$, such that

$$\left( \frac{1}{|B|} \int_B \omega^r \, dx \right)^{1/r} \leq C \frac{1}{|B|} \int_B \omega \, dx,$$

(2.3)

for all balls $B$. This would imply that any $A_p$ weight $\omega$ satisfies the doubling property: there is a $C > 0$ (it might be different from the constant $C$ in (2.3)), such that

$$\int_{B(x_0, 2r)} \omega(x) \, dx \leq C \int_{B(x_0, r)} \omega(x) \, dx$$

for all balls $B(x_0, r) \subset \mathbb{R}^n$.

Suppose $\omega_1$ and $\omega_2$ are $A_1$ weights, and let $t$ be any positive real number. Then it is not hard to show $\omega_1 \omega_2^{-t}$ is an $A_\infty$ weight. Conversely, the factorization theorem of $A_\infty$ weight proved by Peter Jones [19]: if $\omega$ is an $A_\infty$ weight, then there exist $\omega_1$ and $\omega_2$ which are both $A_1$ weights, and $t > 1$ such that $\omega = \omega_1 \omega_2^{-t}$. Later, in the proof of the main theorem, we will decompose the volume form $e^{nu}$ into two pieces. The idea to decompose $e^{nu}$ is inspired by Peter Jones’ factorization theorem. In our case, we give an explicit decomposition of the weight $e^{nu}$, and by analyzing each part in the decomposition we finally prove that $e^{nu}$ is a strong $A_\infty$ weight, a class of weights much stronger than $A_\infty$ that we will introduce in the following.

The notion of strong $A_\infty$ weight was first proposed by David and Semmes in [14]. Given a positive continuous weight $\omega$, we define $\delta_\omega(x, y)$ to be:

$$\delta_\omega(x, y) := \left( \int_{B_{xy}} \omega(z) \, dz \right)^{1/n},$$

(2.4)

where $B_{xy}$ is the ball with diameter $|x - y|$ that contains $x$ and $y$. One can prove that $\delta_\omega$ is only a quasi-distance in the sense that it satisfies the quasi-triangle inequality

$$\delta_\omega(x, y) \leq C(\delta_\omega(x, z) + \delta_\omega(z, y)).$$

On the other hand, for a continuous function $\omega$, by taking infimum over all rectifiable arc $\gamma \subset B_{xy}$ connecting $x$ and $y$, one can define the $\omega$-distance to be

$$d_\omega(x, y) := \inf_\gamma \int_\gamma \omega^{1/n} \, ds.$$  

(2.5)
If $\omega$ is an $A_\infty$ weight, then it is easy to prove (see for example Proposition 3.12 in [22])

$$d_\omega(x, y) \leq C\delta_\omega(x, y) \tag{2.6}$$

for all $x, y \in \mathbb{R}^n$. If in addition to the above inequality, $\omega$ also satisfies the reverse inequality, i.e.

$$\delta_\omega(x, y) \leq Cd_\omega(x, y), \tag{2.7}$$

for all $x, y \in \mathbb{R}^n$, then we say $\omega$ is a strong $A_\infty$ weight, and $C$ is the bound of this strong $A_\infty$ weight.

Every $A_1$ weight is a strong $A_\infty$ weight, but for any $p > 1$ there is an $A_p$ weight which is not strong $A_\infty$. Conversely, for any $p > 1$ there is a strong $A_\infty$ weight which is not $A_p$. It is easy to verify by definition the function $|x|^\alpha$ is $A_1$ thus strong $A_\infty$ if $-n < \alpha \leq 0$; it is not $A_1$ but still strong $A_\infty$ if $\alpha > 0$. And $|x|^\alpha$ is not strong $A_\infty$ for any $\alpha > 0$ as one can choose a curve $\gamma$ contained in the $x_2$-axis.

The notion of strong $A_\infty$ weight was initially introduced in order to study weights that are comparable to the Jacobian of quasi-conformal maps. It was proved by Gehring that the Jacobian of a quasi-conformal map on $\mathbb{R}^n$ is always a strong $A_\infty$ weight, and it was conjectured that the converse was assertive: every strong $A_\infty$ weight is comparable to the Jacobian of a quasi-conformal map. Later, however, counterexamples were found by Semmes [23] in dimension $n \geq 3$, and by Laakso [21] in dimension 2. Nevertheless, it was proved by David and Semmes that a strong $A_\infty$ weight satisfies the Sobolev inequality:

**Theorem 2.1.** *(See [14].)* Let $\omega$ be a strong $A_\infty$ weight. Then for $f \in C_0^\infty(\mathbb{R}^n)$,

$$\left( \int_{\mathbb{R}^n} |f(x)|^{p^*} \omega(x) dx \right)^{1/p^*} \leq C \left( \int_{\mathbb{R}^n} (\omega^{-\frac{1}{n}}(x)|\nabla f(x)|)^p \omega(x) dx \right)^{1/p}, \tag{2.8}$$

where $1 \leq p < n$, $p^* = \frac{np}{n-p}$. Take $p = 1$, it is the standard isoperimetric inequality. The constant $C$ in the inequality only depends on the strong $A_\infty$ bound of $\omega$ and $n$.

By taking $f$ to be a smooth approximation of the indicator function of domain $\Omega$, this implies the validity of the isoperimetric inequality with respect to the weight $\omega$. In this paper, we will take $\omega = e^{nu}$, the volume form of $(\mathbb{R}^n, e^{2u}|dx|^2)$. We aim to show $e^{nu}$ is a strong $A_\infty$ weight. By **Theorem 2.1**, this implies the isoperimetric inequality on $(\mathbb{R}^n, e^{2u}|dx|^2)$:

$$\left( \int_\Omega e^{nu(x)} dx \right)^{\frac{n-1}{n}} \leq C \int_{\partial\Omega} e^{(n-1)u(x)} d\sigma_x,$$
or equivalently, for $g = e^{2u}|dx|^2$,

$$|\Omega| g_{\alpha \beta}^{n-1} \leq C|\partial\Omega|_g.$$ 

A good reference for $A_p$ weights is Chapter 5 in [24]. For more details on strong $A_\infty$ weight, we refer the readers to [14], where the concept was initially proposed.

3. Analysis on the negative part of the $Q$-curvature

We first remark that since $Q_g(y)e^{nu(y)}$ is integrable, $\log \frac{|y|}{|x-y|} Q_g(y)e^{nu(y)}$ is also integrable in $y$ for each fixed $x \in \mathbb{R}^n$. In fact, for a fixed $x$, the integral over the domain $|y| >> |x|$ is finite because $\log \frac{|y|}{|x-y|}$ is bounded and $Q_g(y)e^{nu(y)}$ is absolutely integrable by assumption (1.4) and (1.5); on the other hand, since the $Q$-curvature is smooth, and thus locally bounded, the integral over $B(x,1)$ is finite as well. Later in the paper, we will replace $Q_g(y)$ by either the positive or the negative part of it. The function $\log \frac{|y|}{|x-y|} Q^\pm(y)e^{nu(y)}$ is still integrable for each fixed $x$. We will not repeat this point in the following sections.

To begin with, let us decompose $u = u_+ + u_-$, where $u_+$ and $u_-$ are defined in the following.

**Definition 3.1.**

$$u_-(x) := -\frac{1}{cn} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q^-(y)e^{nu(y)} dy,$$ 

and

$$u_+(x) := \frac{1}{cn} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q^+(y)e^{nu(y)} dy.$$ 

Note that $u_-(x)$ and $u_+(x)$ are not the negative and positive part of the function $u(x)$. They form a decomposition of $u(x)$ using the negative and positive part of the $Q$-curvature.

In this section, we consider the analytic property of $e^{nu_-(x)}$. For simplicity, we denote it by $\omega^2(x)$. By (1.5), $\beta := \int_{\mathbb{R}^n} Q^-(y)e^{nu(y)} dy < \infty$. We recall the definitions (2.4) and (2.5) for a nonnegative continuous function $\omega(x)$,

$$d_\omega(x, y) := (\int_{B_{xy}} \omega(z)dz)^{\frac{1}{n}},$$

$$\delta_\omega(x, y) := \inf_{\gamma} \int \omega^{\frac{1}{2}}(\gamma(s))ds,$$
where $B_{xy}$ is the ball with diameter $|x - y|$ that contains $x$ and $y$, the infimum is taken over all curves $\gamma \subset B_{xy}$ connecting $x$ and $y$, and $ds$ is the arc length.

**Theorem 3.2.** $\omega_2(x) := e^{nu_-(x)}$ is a strong $A_\infty$ weight, i.e. there exists a constant $C = C(n, \beta)$ such that

$$\frac{1}{C(n, \beta)}d_{\omega_2}(x, y) \leq \delta_{\omega_2}(x, y) \leq C(n, \beta) d_{\omega_2}(x, y).$$

(3.3)

We first observe that without generality we can assume $|x - y| = 2$. This is because we can dilate $u$ by a factor $\lambda > 0$, $u_\lambda(x) = u(\lambda x) = -\frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|\lambda x - y|} Q^{-}(y) e^{nu(y)} dy. \quad (3.4)$

By the change of variable, this is equal to

$$-\frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x - y|} Q^{-}(\lambda y) e^{nu(\lambda y)} \lambda^n dy.$$

Notice $Q^{-}(\lambda y) e^{nu(\lambda y)} \lambda^n$ is still an integrable function on $\mathbb{R}^n$, with integral equal to $\beta$. Thus by choosing $\lambda = \frac{2}{|x - y|}$, the problem reduces to proving inequality (3.3) for $u_\lambda$ and $|x - y| = 2$.

Let us denote the midpoint of $x$ and $y$ by $p_0$. And from now on, we adopt the notation $\lambda B := B(p_0, \lambda)$. Since $|x - y| = 2$, we have $B_{xy} = B(p_0, 1) = B$. We also define

$$u_1(x) := -\frac{1}{c_n} \int_{10B} \log \frac{|y|}{|x - y|} Q^{-}(y) e^{nu(y)} dy, \quad (3.5)$$

and

$$u_2(x) := -\frac{1}{c_n} \int_{\mathbb{R}^n \setminus 10B} \log \frac{|y|}{|x - y|} Q^{-}(y) e^{nu(y)} dy. \quad (3.6)$$

In the following lemma, we prove that when $z$ is close to $p_0$, the difference between $u_2(z)$ and $u_2(p_0)$ is controlled by $\beta$.

**Lemma 3.3.**

$$|u_2(z) - u_2(p_0)| \leq \frac{\beta}{4c_n} \quad (3.7)$$

for $z \in 2B$. 
Proof.

\[ |u_2(z) - u_2(p_0)| \]
\[ = \frac{1}{c_n} \left| \int_{\mathbb{R}^n \setminus 10B} -\log \frac{|y|}{|z - y|} Q^-(y) e^{nu(y)} dy + \int_{\mathbb{R}^n \setminus 10B} \log \frac{|y|}{|p_0 - y|} Q^-(y) e^{nu(y)} dy \right| 
\]
\[ = \frac{1}{c_n} \left| \int_{\mathbb{R}^n \setminus 10B} \log \frac{|z - y|}{|p_0 - y|} Q^-(y) e^{nu(y)} dy \right| 
\]
\[ \leq \frac{|z - p_0|}{c_n} \cdot \int_{\mathbb{R}^n \setminus 10B} \frac{1}{|(1 - t^*)(p_0 - y) + t^*(z - y)|} Q^-(y) e^{nu(y)} dy, \tag{3.8} \]

for some \( t^* \in [0, 1] \). Since \( y \in \mathbb{R}^n \setminus 10B \) and \( z, p_0 \in 2B \),
\[ \frac{1}{|(1 - t^*)(p_0 - y) + t^*(z - y)|} \leq \frac{1}{8}, \]
\[ |u_2(z) - u_2(p_0)| \] is bounded by
\[ \frac{|z - p_0|}{8c_n} \cdot \int_{\mathbb{R}^n \setminus 10B} Q^-(y) e^{nu(y)} dy. \tag{3.9} \]

Note that for \( z \in 2B \), \( |z - p_0| \leq 2 \). From this, (3.7) follows. \( \square \)

Now we adopt some techniques used in [2] for potentials to deal with the \( \epsilon \)-singular set \( E_\epsilon \).

Lemma 3.4 (Cartan’s lemma). For the Radon measure \( Q^-(y) e^{nu(y)} dy \), given \( \epsilon > 0 \), there exists a set \( E_\epsilon \subseteq \mathbb{R}^n \), such that
\[ \mathcal{H}^1(E_\epsilon) := \inf_{E_\epsilon \subseteq \cup B_i} \{ \sum_i \text{diam } B_i \} < 10\epsilon \]

and for all \( x \notin E_\epsilon \) and \( r > 0 \),
\[ \int_{B(x,r)} Q^-(y) e^{nu(y)} dy \leq \frac{r^\beta}{\epsilon}. \]

The proof of Lemma 3.4 follows from standard measure theory argument. Thus we omit it here.
Proposition 3.5. Given $\epsilon > 0$,

$$
\mathcal{H}^1 \left( \left\{ x \in 10B : \left| \frac{-1}{c_n} \int_{10B} \log \frac{1}{|x-y|} Q^{-}(y)e^{nu(y)}dy \right| > \frac{C_0\beta}{\epsilon} \right\} \right) < 10\epsilon,
$$

for some $C_0$ depending only on $n$.

Proof. Fix $\epsilon > 0$. By Lemma 3.4, there exists a set $E_\epsilon \subseteq \mathbb{R}^n$, s.t. $\mathcal{H}^1(E_\epsilon) < 10\epsilon$ and for $x \notin E_\epsilon$ and $r > 0$

$$
\int_{B(x,r)} Q^{-}(y)e^{nu(y)}dy \leq \frac{r\beta}{\epsilon}.
$$

(3.10)

If we can show for some $C_0 = C_0(n)$

$$
10B \setminus E_\epsilon \subseteq \left\{ x \in 10B : \left| \frac{-1}{c_n} \int_{10B} \log \frac{1}{|x-y|} Q^{-}(y)e^{nu(y)}dy \right| \leq \frac{C_0}{\epsilon} \beta \right\},
$$

(3.11)

then

$$
\mathcal{H}^1 \left( \left\{ x \in 10B : \left| \frac{-1}{c_n} \int_{10B} \log \frac{1}{|x-y|} Q^{-}(y)e^{nu(y)}dy \right| > \frac{C_0\beta}{\epsilon} \right\} \right) \leq \mathcal{H}^1(E_\epsilon) < 10\epsilon.
$$

To prove (3.11), we notice for $x \in 10B \setminus E_\epsilon$, $r = 2^{-j} \cdot 10$, (3.10) implies

$$
\left| \frac{-1}{c_n} \int_{10B} \log \frac{1}{|x-y|} Q^{-}(y)e^{nu(y)}dy \right| \\
\leq \frac{1}{c_n} \sum_{j=-1}^{\infty} \left| \int_{B(x,2^{-j} \cdot 10) \setminus B(x,2^{-(j+1)} \cdot 10)} \log \frac{1}{|x-y|} Q^{-}(y)e^{nu(y)}dy \right| \\
\leq \frac{1}{c_n} \sum_{j=-1}^{\infty} \left( \max \{ |\log 2^{-j}|, |\log 2^{-(j+1)}| \} + \log 10 \right) \\
\cdot \int_{B(x,2^{-j} \cdot 10) \setminus B(x,2^{-(j+1)} \cdot 10)} Q^{-}(y)e^{nu(y)}dy \\
\leq \frac{1}{c_n} \sum_{j=-1}^{\infty} \left( \max \{ |\log 2^{-j}|, |\log 2^{-(j+1)}| \} + \log 10 \right) \cdot \frac{2^{-j} \cdot 10\beta}{\epsilon} \\
\leq \frac{C_0\beta}{\epsilon},
$$

(3.12)
where
\[ C_0 = \frac{10 \sum_{j=-1}^{\infty} \left( \max \{ |\log 2^{-j}|, |\log 2^{-(j+1)}| \} + \log 10 \right) \cdot 2^{-j}}{c_n} < \infty, \]
depending only on \( n \). This completes the proof of the proposition. \( \square \)

We next estimate the integral of \( e^{nu_-(z)} \) over \( 2B \).

**Proposition 3.6.** Let \( \tilde{c} := -\frac{1}{c_n} \int_{10B} \log |y| Q^{-}(y) e^{nu(y)} dy \), \( \tilde{c} < \infty \), since \( Q^{-}(y) e^{nu(y)} \) is continuous thus bounded near the origin. Then
\[
\int_{2B} e^{nu_-(z)} \, dz \leq C_1(n, \beta) e^{nu_2(p_0)} e^{n\tilde{c}}, \tag{3.13}
\]
for \( C_1(n, \beta) = e^{\frac{n\beta}{n \cdot \omega_{n-1}}} 12\frac{n^{2n}}{n} \omega_{n-1} ^{-2n} \), where \( \omega_{n-1} \) denotes the area of the \((n-1)\)-dimensional unit sphere in \( \mathbb{R}^n \) and \( \beta_{10} := \int_{10B} Q^{-}(y) e^{nu(y)} dy \leq \beta < \infty \).

**Proof.** Recall
\[
u_1(x) := -\frac{1}{c_n} \int_{10B} \log \frac{|y|}{|x-y|} Q^{-}(y) e^{nu(y)} dy, \tag{3.14}
\]
and
\[
u_2(x) := -\frac{1}{c_n} \int_{\mathbb{R}^n \setminus 10B} \log \frac{|y|}{|x-y|} Q^{-}(y) e^{nu(y)} dy. \tag{3.15}
\]

By Lemma 3.3,
\[
\int_{2B} e^{nu_-(z)} \, dz = \int_{2B} e^{nu_1(z)} e^{nu_2(z)} \, dz \leq e^{\frac{n\beta}{n \cdot \omega_{n-1}}} e^{nu_2(p_0)} \int_{2B} e^{nu_1(z)} \, dz. \tag{3.16}
\]

To estimate \( u_1 \), by definition \( \beta_{10} := \int_{10B} Q^{-}(y) e^{nu(y)} dy \leq \beta < \infty \). If \( \beta_{10} = 0 \), then \( u_1(z) = 0 \) and \( \tilde{c} := -\frac{1}{c_n} \int_{10B} \log |y| Q^{-}(y) e^{nu(y)} dy = 0 \). So (3.13) follows immediately. If \( \beta_{10} \neq 0 \), \( \frac{Q^{-}(y) e^{nu(y)}}{\beta_{10}} \) is a nonnegative probability measure on \( 10B \). Hence by Jensen’s inequality
\[
\int_{2B} e^{nu_1(z)} \, dz = e^{n\tilde{c}} \cdot \int_{2B} e^{\frac{n}{c_n} \int_{10B} \log |z-y| Q^{-}(y) e^{nu(y)} dy \, dz} \leq e^{n\tilde{c}} \cdot \int_{2B} \int_{10B} |z-y| \frac{n\beta_{10}}{c_n} Q^{-}(y) e^{nu(y)} dydz. \tag{3.17}
\]
Since \( z \in 2B \) and \( y \in 10B \),
\[
\int_{2B} |z - y|^{\frac{n\beta_{10}}{c_n}} \, dz \leq 12 \frac{n\beta_{10}}{c_n} \frac{\omega_n - 1}{n} 2^n.
\] (3.18)

From this, we get
\[
\int_{2B} e^{nu_1(z)} \, dz \leq e^{n\bar{c}} 12 \frac{n\beta_{10}}{c_n} \frac{\omega_n - 1}{n} 2^n \int_{10B} \frac{Q^-(y)e^{nu(y)}}{\beta_{10}} \, dy \leq e^{n\bar{c}} 12 \frac{n\beta_{10}}{c_n} \frac{\omega_n - 1}{n} 2^n.
\] (3.19)

Plugging it to (3.16), we finish the proof of the proposition. \( \Box \)

Now we are ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** Let us assume \( \omega_2 := e^{nu_-} \) is an \( A_p \) weight for some large \( p \), with bounds depending only on \( n \) and \( \beta \). We will prove this statement, in fact for a more general setting, in Proposition 5.1. So by the reverse Hölder’s inequality for \( A_p \) weights, it is easy to prove (see for example Proposition 3.12 in [22]),
\[
\delta_{\omega_2}(x, y) \leq C_2(n, \beta) d_{\omega_2}(x, y).
\]

Hence we only need to prove the other side of the inequality:
\[
\delta_{\omega_2}(x, y) \geq C_3(n, \beta) d_{\omega_2}(x, y),
\] (3.20)

for some constant \( C_3(n, \beta) \). By Proposition 3.5, for a given \( \epsilon > 0 \), there exists a Borel set \( E_\epsilon \subseteq \mathbb{R}^n \), such that
\[
\mathcal{H}^1(E_\epsilon) \leq 10\epsilon,
\] (3.21)

and for \( z \in 10B \setminus E_\epsilon \), according to (3.11)
\[
|\hat{u}_1(z)| \leq \frac{C_0}{\epsilon} \beta.
\] (3.22)

Here
\[
\hat{u}_1(z) := \frac{-1}{c_n} \int_{10B} \log \frac{1}{|x - y|} Q^-(y)e^{nu(y)} \, dy.
\]

With this, we claim the following estimate.
Claim. Suppose $\mathcal{H}^1(E_\epsilon) < 10\epsilon$ with $\epsilon \leq \frac{1}{20}$. Then the Euclidean length of $\gamma \setminus E_\epsilon$

$$\text{length}(\gamma \setminus E_\epsilon) > \frac{3}{2}, \quad (3.23)$$

where $\gamma \subset B_{xy}$ is a curve connecting $x$ and $y$.

Proof of Claim. Let $P$ be the projection map from points in $B_{xy}$ to the line segment $I_{xy}$ between $x$ and $y$. Since the Jacobian of the projection map is less or equal to 1,

$$\text{length}(\gamma \setminus E_\epsilon) \geq \text{length}(P(\gamma \setminus E_\epsilon)) = m(P(\gamma \setminus E_\epsilon)), \quad (3.24)$$

where $m$ is the Lebesgue measure on the line segment $I_{xy}$. Notice $P(\gamma) = I_{xy}$, and $P(\gamma) \setminus P(E_\epsilon)$ is a subset of $P(\gamma \setminus E_\epsilon)$. Therefore

$$m(P(\gamma \setminus E_\epsilon)) \geq m(P(\gamma)) - m(P(E_\epsilon)) = 2 - m(P(E_\epsilon)). \quad (3.25)$$

Now by assumption, $\mathcal{H}^1(E_\epsilon) < 10\epsilon$, so $\mathcal{H}^1(\gamma \cap E_\epsilon) < 10\epsilon$. Hence there is a covering $\cup_i B_i$ of $\gamma \cap E_\epsilon$, so that

$$\sum_i \text{diam } B_i < 10\epsilon.$$ 

This implies that $\cup_i P(B_i)$ is a covering of the set $P(\gamma \cap E_\epsilon)$ and

$$\sum_i \text{diam } P(B_i) = \sum_i \text{diam } B_i \leq 10\epsilon.$$ 

Thus $m(P(E_\epsilon)) = \mathcal{H}^1(P(E_\epsilon)) < 10\epsilon < \frac{1}{2}$, by choosing $\epsilon \leq \frac{1}{20}$. Plug it to (3.25), and then to (3.24). This completes the proof of the claim. □

We now continue the proof of Theorem 3.2. Since $\gamma \subset B$, then by Lemma 3.3,

$$\int_{\gamma} e^{u_1 - (\gamma(s))} ds = \int_{\gamma} e^{(u_1 + u_2)(\gamma(s))} ds \geq e \frac{\bar{c}}{m} \int_{\gamma} e^{\tilde{u}_1(\gamma(s))} ds. \quad (3.26)$$

Here $\bar{c}$ is the constant defined in Proposition 3.6. Let $\epsilon = \frac{1}{20}$. By (3.22),

$$|\tilde{u}_1(z)| \leq 20C_0\beta$$

for $z \in 10B \setminus E_\epsilon$. Thus

$$\int_{\gamma} e^{\tilde{u}_1(\gamma(s))} ds \geq e^{-20C_0\beta} \text{length}(\gamma \setminus E_\epsilon). \quad (3.27)$$

By (3.23), it is bigger than
\[
\frac{3}{2} e^{-20C_0 \beta}.
\]
Therefore
\[
\int_{\gamma} e^{u_-(\gamma(s))} ds \geq \frac{3}{2} e^{-20C_0 \beta} e^{u_2(p_0)} e^c = C_4(n, \beta) e^{u_2(p_0)} e^c
\]
for \(C_4(n, \beta) = \frac{3}{2} e^{-20C_0 \beta}\). By inequality (3.28) and Proposition 3.6, we conclude for any curve \(\gamma \subset B_{xy}\) connecting \(x\) and \(y\), there is a \(C_3 = C_3(n, \beta)\) such that
\[
\int_{\gamma} e^{u_-(\gamma(s))} ds \geq C_3(n, \beta) \left( \int_{B_{xy}} e^{nu_-(z)} dz \right)^{\frac{1}{n}}.
\]
This implies inequality (3.20) and thus completes the proof of Theorem 3.2. \(\square\)

4. On the positive part of the \(Q\)-curvature

In this section, we consider the positive measure \(\frac{1}{cn} Q^+(x) e^{nu(x)} dx\). We recall the assumption (1.4), \(\alpha := \int_{\mathbb{R}^n} Q^+(x) e^{nu(x)} dx < c_n\). Recall Definition 3.1,
\[
u_+(x) := \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q^+(y) e^{nu(y)} dy.
\]

\textbf{Theorem 4.1.} Suppose \(\alpha := \int_{\mathbb{R}^n} Q^+(x) e^{nu(x)} dx < c_n\). Then \(\omega_1(x) := e^{nu_+(x)}\) is an \(A_1\) weight, i.e.
\[
M(\omega_1)(x) \leq C(n, \alpha) \omega_1(x) \quad \text{a.e.} \quad x \in \mathbb{R}^n,
\]
where \(M(\cdot)\) denotes the maximal function
\[
M(f)(x) := \sup_{r>0} \int_{B(x,r)} |f(y)| dy.
\]
\textbf{Proof.} Note that
\[
\frac{M(\omega_1)(x)}{\omega_1(x)} = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \exp \left( \frac{n}{c_n} \int_{\mathbb{R}^n} \log \frac{|z|}{|y-z|} Q^+(z) e^{nu(z)} dz \right) dy
\]
\[
= \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \exp \left( \frac{n}{c_n} \int_{\mathbb{R}^n} \log \frac{|x-z|}{|y-z|} Q^+(z) e^{nu(z)} dz \right) dy
\]
If $\alpha = 0$, then (4.2) is obviously true. So let us assume $\alpha \neq 0$ and define the nonnegative probability measure $\nu_+(z) := Q^+(z)e^{n\nu(z)}dz$. By Jensen’s inequality, we get for any $r > 0$,

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} \exp \left( \frac{n}{c_n} \int_{\mathbb{R}^n} \log \frac{|x-z|}{|y-z|} Q^+(z)e^{n\nu(z)}dz \right) dy \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} \left( \frac{|x-z|}{|y-z|} \right)^{\frac{\alpha n}{c_n}} d\nu_+(z) dy \leq \int_{\mathbb{R}^n} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left( \frac{|x-z|}{|y-z|} \right)^{\frac{\alpha n}{c_n}} dy d\nu_+(z).$$

(4.4)

As discussed in Section 2, $\frac{1}{|x|^\frac{\alpha n}{c_n}}$ is an $A_1$ weight on $\mathbb{R}^n$ with $A_1$ bound depending on $n$ and $\alpha$ when $\alpha < c_n$. Hence for any $x \in \mathbb{R}^n$, $r > 0$,

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} \left( \frac{1}{|y|^\frac{\alpha n}{c_n}} \right) dy \leq C(n, \alpha).$$

(4.5)

Obviously if we shift the function $\frac{1}{|x|^\frac{\alpha n}{c_n}}$ by any point $z \in \mathbb{R}^n$, the inequality is still valid with the same constant $C(n, \alpha)$, i.e.

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} \left( \frac{1}{|y-z|^\frac{\alpha n}{c_n}} \right) dy \leq C(n, \alpha).$$

(4.6)

Applying it to (4.4), we obtain

$$\int_{\mathbb{R}^n} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left( \frac{|x-z|}{|y-z|} \right)^{\frac{\alpha n}{c_n}} dy d\nu_+(z) \leq \int_{\mathbb{R}^n} C(n, \alpha) d\nu_+(z) = C(n, \alpha),$$

(4.7)

for any $r > 0$ and $x \in \mathbb{R}^n$. Thus (4.2) follows. This finishes the proof of the theorem.  

5. Proof of Theorem 1.1

We begin this section by showing that $e^{n\nu}$ is an $A_p$ weight for large $p$. 

Proposition 5.1. For
\[ u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q(y) e^{nu(y)} dy \]  
(5.1)

with assumptions (1.4) and (1.5), \( e^{nu(x)} \) is an \( A_p \) weight for some large \( p \). Its \( A_p \) bound depends only on \( n, \alpha \) and \( \beta \).

As a direct corollary, this proposition implies that \( e^{nu_\epsilon} \) is an \( A_p \) weight for large \( p \) when (1.5) is assumed. Such a conclusion has been used in the proof of Theorem 3.2 in Section 3.

Proof of Proposition 5.1. By Theorem 4.1, \( e^{nu_\epsilon} \) is an \( A_1 \) weight, so there is a uniform constant \( C = C(n, \alpha) \), so that for all \( x_0 \in \mathbb{R}^n \) and \( r > 0 \)
\[ \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} e^{nu_\epsilon(x)} dx \leq C(n, \alpha) e^{nu_\epsilon(x_0)}. \]  
(5.2)

So for all \( y \in B(x_0, r) \)
\[ \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} e^{nu_\epsilon(x)} dx \leq \frac{1}{|B(y, 2r)|} \int_{B(y, 2r)} e^{nu_\epsilon(x)} dx \]
\[ = \frac{2^n}{|B(y, 2r)|} \int_{B(y, 2r)} e^{nu_\epsilon(x)} dx \]  
(5.3)
\[ \leq 2^n C(n, \alpha) e^{nu_\epsilon(y)}. \]

Namely, for all ball \( B \) in \( \mathbb{R}^n \) and \( y \in B \),
\[ \frac{1}{|B|} \int_{B} e^{nu_\epsilon(x)} dx \leq 2^n C(n, \alpha) e^{nu_\epsilon(y)}. \]  
(5.4)

We observe that \( e^{-\epsilon nu_{-\epsilon}(x)} \) is also an \( A_1 \) weight for \( \epsilon = \epsilon(n, \beta) << 1 \). In fact,
\[ e^{-\epsilon nu_{-\epsilon}} = e^{nu_{-\epsilon}} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} e^{-\epsilon y} dy. \]  
(5.5)
\[ Q^{-}(y)e^{nu(y)} \geq 0 \] and \( \int_{\mathbb{R}^n} \epsilon Q^{-}(y)e^{nu(y)} dy < c_n \) if \( \epsilon \) is small enough. Thus by Theorem 4.1, \( e^{-\epsilon nu_{-\epsilon}(x)} \) is an \( A_1 \) weight. Using (5.4), we get for all ball \( B \) in \( \mathbb{R}^n \) and all \( y \in B \),
\[ \frac{1}{|B|} \int_{B} e^{-\epsilon nu_{-\epsilon}(x)} dx \leq 2^n C(n, \beta) e^{-\epsilon nu_{-\epsilon}(y)}. \]  
(5.6)
Choose $1 < p < \infty$ such that $\epsilon = \frac{p'}{p}$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Using $e^{nu} = e^{nu_+} \cdot e^{nu_-}$, we get

\[
\left( \int_{B} e^{nu(x)} \, dx \right) \left( \int_{B} \frac{e^{nu(x)} - \frac{p'}{p}}{dx} \right) \left( \int_{B} \frac{e^{nu_+}}{dx} \right) \left( \int_{B} \frac{e^{nu_-}}{dx} \right)^{\frac{1}{p'}} = \left( \int_{B} e^{nu_+} \cdot (e^{-\epsilon nu_-})^{-\frac{1}{p}} \, dx \right) \left( \int_{B} \frac{e^{nu_+}}{dx} \right) \left( \int_{B} \frac{e^{-\epsilon nu_-}}{dx} \right)^{\frac{1}{p'}} \tag{5.7}
\]

By (5.6), if $p$ is large enough and thus $\epsilon$ is small enough, then for all ball $B$ and all $y \in B$,

\[
(e^{-\epsilon nu_-}(y))^{-\frac{1}{p}} \leq \frac{1}{2^n C(n, \beta) |B|} \int_{B} e^{-\epsilon nu_-} \, dx \tag{5.8}
\]

So

\[
\left( \int_{B} e^{nu_+} \cdot (e^{-\epsilon nu_-})^{-\frac{1}{p}} \, dx \right) \left( \int_{B} \frac{e^{nu_+}}{dx} \right) \left( \int_{B} \frac{e^{-\epsilon nu_-}}{dx} \right)^{-\frac{1}{p'}} \leq \left( \int_{B} \frac{e^{nu_+}}{dx} \right) \left( \int_{B} \frac{e^{-\epsilon nu_-}}{dx} \right)^{-\frac{1}{p'}} \tag{5.9}
\]

Similarly, by (5.4) for all ball $B$ and all $y \in B$

\[
(e^{nu_+}(y))^{-\frac{p'}{p}} \leq \frac{1}{2^n C(n, \alpha) |B|} \int_{B} e^{nu_+} \, dx \tag{5.10}
\]

So

\[
\left( \int_{B} \frac{e^{nu_+}}{dx} \right)^{-\frac{p'}{p}} \cdot e^{-\epsilon nu_-} \, dx \right) \leq \left( \int_{B} \frac{e^{nu_+}}{dx} \right)^{-\frac{1}{p}} \left( \int_{B} \frac{e^{-\epsilon nu_-}}{dx} \right)^{\frac{1}{p'}} \tag{5.11}
\]

Applying (5.8) to (5.9) in (5.7), we have
\[
\left( \int_B e^{nu(x)} dx \right)^{\frac{1}{p}} \left( \int_B (e^{nu(x)})^{-\frac{\nu}{p}} dx \right)^{\frac{1}{q'}} \leq \left( \frac{1}{C(n, \alpha, \beta)|B|} \right)^{\frac{1}{p'} - \frac{1}{q'}} = C(n, \alpha, \beta)|B|
\]

(5.10)

for \( p >> 1 \). This shows that \( e^{nu(x)} \) is an \( A_p \) weight for \( p >> 1 \). The bound \( C \) depends only on \( n, \alpha \) and \( \beta \). \( \square \)

Now we recall a lemma [22, Lemma 3.17]. The set-up of [22] is in slightly different but equivalent form of ours; the proof of the lemma is straightforward following the definitions of strong \( A_\infty \) weight and \( A_1 \) weight. Thus we omit it here.

**Lemma 5.2.** (See [22, Lemma 3.17].) Assume that \( \omega_1 \) is an \( A_1 \) weight, \( \omega_2 \) is a strong \( A_\infty \) weight, and that \( r \) is a positive real number. If \( \omega_1^r \omega_2 \) is \( A_\infty \), then \( \omega_1^r \omega_2 \) is strong \( A_\infty \). The bound of strong \( A_\infty \) weight \( \omega_1^r \omega_2 \) depends only on the bounds of \( A_1 \) weight \( \omega_1 \), strong \( A_\infty \) weight \( \omega_2 \), \( A_\infty \) weight \( \omega_1^r \omega_2 \) and \( r \).

**Proof of Theorem 1.1.** In Theorem 4.1 and Theorem 3.2 we have proved that \( \omega_1(x) := e^{nu(x)} \) is an \( A_1 \) weight with bound depending only on \( n \) and \( \alpha \); and \( \omega_2(x) := e^{nu(x)} \) is a strong \( A_\infty \) weight with bound depending only on \( n \) and \( \beta \). Also by Proposition 5.1, \( e^{nu} = e^{nu} \cdot e^{nu} \) is an \( A_p \) weight for \( p >> 1 \), with bound depending only on \( n, \alpha \) and \( \beta \). Therefore \( e^{nu(x)} \) is an \( A_\infty \) weight. Applying Lemma 5.2 (with \( r = 1 \)), we obtain \( e^{nu} \) is a strong \( A_\infty \) weight with bound depending only on \( n \), \( \alpha \) and \( \beta \). Therefore according to Theorem 2.1, the isoperimetric inequality is valid with constant depending only on \( n, \alpha \) and \( \beta \). This completes the proof of Theorem 1.1. \( \square \)

**Remark 5.3.** As we pointed out in the introduction, the assumption (1.4) is sharp. In fact, \( c_n \) is equal to the integral of the \( Q \)-curvature of the standard sphere metric on a unit hemisphere, and the \( Q \)-curvature is equal to 0 on a flat cylinder. Thus a cylinder with a hemisphere attached to one of its ends (one can slightly perturb the metric in order to glue smoothly) has \( \alpha = c_n \) and \( \beta = 0 \); and it is conformal equivalent to \((\mathbb{R}^n, |dx|^2)\). But such a manifold certainly fails to satisfy the isoperimetric inequality.

**Remark 5.4.** The definition of “Normal metric” was given in 2-dimension by Finn [17], and generalized in higher dimensions by Chang, Qing and Yang [8]. This is a necessary assumption when dimension is higher than 2, due to the nature of the problem. On one hand, if this assumption is removed, there are examples of manifolds with non-uniform isoperimetric constant; on the other hand, no assumption on “normal metric” is needed when \( n = 2 \). Because by Huber’s result [18], every complete non-compact metric with integrable Gaussian curvature is “normal”. So the assumption is implicit when \( n = 2 \).

**Remark 5.5.** In fact, by a similar argument, one can even show \( e^{nu} \) is a stronger \( A_\infty \) weight (see [22, Definition 5.1] for the definition), which is a stronger conclusion than
being a strong $A_\infty$ weight. For the purpose of the present paper, there is no need to get into the details of this point.

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