Morera’s Theorem

Let \( f \) be a continuous function on a simply connected domain \( D \), then \( f \) is analytic if and only if \( \int_\Gamma f(z)\,dz = 0 \) for every closed contour in \( D \).

Proof:

- The ”only if” part is Cauchy’s Integral Theorem. We focus on the ”if” part.

- Suppose \( \int_\Gamma f(z)\,dz = 0 \) for every closed contour in \( D \), then by the ”independence of path” theorem, \( f \) has an antiderivative \( F \) on \( D \). As \( F' = f \), \( F \) is analytic. So \( F^{(n+1)} = f^{(n)} \) exists by Cauchy’s Differentiation Formula for every \( n \), and thus \( f \) is analytic.
A bounded entire function must be constant.

**Proof:** Suppose $f$ is an entire function satisfying $|f| \leq M$.

For any $a$, $f'(a) = \frac{1}{2\pi i} \int_{|z-a|=R} \frac{f(z)}{(z-a)^2} \, dz$ for all $R > 0$, with the circle $|z - a| = R$ oriented positively.

On the circle, $\left| \frac{f(z)}{(z-a)^2} \right| \leq \frac{M}{R^2}$. And the length of the circle is $2\pi R$. Therefore, $|f'(a)| \leq \left| \frac{1}{2\pi i} \frac{M}{R^2} \cdot 2\pi R \right| = \frac{M}{R}$.

As $R$ can be arbitrarily large, $f'(a) = 0$, and thus $f$ must be constant. \qed
Example 1:

Show that: If an entire function \( f(z) \) satisfies that \( f^{(k)} \) is bounded for some given \( k \). Then \( f(z) \) is a polynomial of degree at most \( k \).

**Proof:**

\( f^{(k)} \) is also an entire function by Cauchy’s differentiation formula. So by Liouville Theorem, \( f^{(k)} \) is constant, denote it by \( c_k \).

This implies \((f^{(k-1)}(z) - c_k z)\)' = \( f^{(k)}(z) - c_k \) = 0 for all \( z \). So \( f^{(k-1)}(z) - c_k z \) is constant, denote it by \( c_{k-1} \). Then \( f^{(k-1)}(z) = c_k z + c_{k-1} \).

Repeating this, we get \( f(z) = c_k z^k + c_{k-1} z^{k-1} + \cdots + c_0 \). \( \square \)
Example 2

Show that: If an entire function $f(z)$ satisfies $|f(z)| \leq C|z|^k$ for some given $k$, then $f(z)$ is a polynomial of degree at most $k$.

Proof: We know $f^{(k+1)}(a) = \frac{(k+1)!}{2\pi i} \int_{|z-a|=R} \frac{f(z)}{|z-a|^{k+2}} dz$.

$\left| \frac{f(z)}{|z-a|^{k+1}} \right| \leq \frac{C(R+|a|)^k}{R^{k+2}}$ on the circle $|z-a| = R$.

This is at most $\frac{C(2R)^k}{R^{k+2}} \leq \frac{C2^k}{R^2}$ when $R > |a|$.

$\left| \frac{f(z)}{|z-a|^{k+1}} \right|$ The circle has length $2\pi R$.}

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Example 2, continued

So \(|f^{(k+1)}(a)|\) is bounded by \(\frac{(k+1)!}{2\pi i} \cdot \frac{C2^k}{R^2} \cdot 2\pi R \leq \frac{C2^k(k+1)!}{R}\)
for all \(R > |a|\).

Let \(R \to \infty\), we known \(f^{(k+1)}(a) = 0\) for all \(a \in \mathbb{C}\). Hence \(f^{(k)}(z)\) is constant. By previous example, \(f(z)\) is a polynomial of degree at most \(k\).