Theorem: Let \( f_j : U \to \mathbb{C}, j = 1, 2, 3, \ldots \) be a sequence of holomorphic functions on an open set \( U \) in \( \mathbb{C} \). Suppose that there is a function \( f : U \to \mathbb{C} \) such that for each compact subset \( E \) of \( U \), the sequence \( f_j|_E \) converges uniformly to \( f|_E \). Then \( f \) is holomorphic on \( U \). In particular, \( f \in C^\infty(U) \).

(See Theorem 3.5.1 on Page 88.)
The Zeros of a Holomorphic Function

**Theorem:** Let $U \subset \mathbb{C}$ be a connected open set and let $f : U \rightarrow \mathbb{C}$ be holomorphic. Let $Z := \{z \in U, f(z) = 0\}$. If there exists a $z_0 \in Z$ and $\lim z_i = z_0$, $z_i \neq z_0 \in Z$ (accumulating point), then $f \equiv 0$.

(See Theorem 3.6.1 on Page 90.)
Proof of Fundamental Theorem of Algebra

Any polynomial \( p(z) \) of degree \( n \) has \( n \) roots (counted with multiplicity).

(See Theorem 3.4.5, Corollary 3.4.6 on Page 87.)

Proof:

- It suffices to show for \( n \geq 1 \) that \( p(z) \) has at least one root. This is because:
  - If \( z_1 \) is a root of \( p \), then \( p(z) = (z - z_1)p_1(z) \)
  - \( p_1(z) \): polynomial of degree \( n - 1 \).
  - We can assume \( p_1(z) \) have \( n - 1 \) roots by induction. So \( p(z) \) has \( n \) roots.
Proof of Fundamental Theorem of Algebra, continued

Suppose $p(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0$ and has no root.

- $\frac{1}{p(z)}$ has no pole and is an entire function;
- Take $R > \min(1, 2 \cdot \frac{|a_{n-1}| + \cdots + |a_0|}{|a_n|})$, then for any $|z| > R$,

  $$|p(z)| \geq |a_n z^n| - |a_{n-1} z^n + \cdots + a_0|$$
  $$\geq |a_n| R^n - (\sum_{j=0}^{n-1} |a_j|) R^{n-1}$$
  $$\geq |a_n| R^n - \frac{|a_n| R}{2} \cdot R^{n-1}$$
  $$= \frac{|a_n|}{2} R^n.$$

- So $\frac{1}{p(z)}$ is bounded on $|z| > R$.  

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Bounds on holomorphic functions
Proof of Fundamental Theorem of Algebra, continued

- $\frac{1}{p(z)}$ is bounded on $|z| \leq R$, too.
- Therefore, $\frac{1}{p(z)}$ is a bounded entire function, hence constant.
- So $p(z)$ is also constant, this contradicts that $n \geq 1$. □
► first proof of Fundamental Theorem of Algebra: Gauss 1799
► first time taught in a textbook:
**Figure:** Augustin-Louis Cauchy (1789-1857)
Cauchy’s Integral Formula, circle case

Suppose a disk $|z - a| \leq r$, boundary included, is inside the domain of analyticity of $f$. Then Cauchy’s Integral Formula says:

$$f(a) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{z-a} \, dz$$

$$= \frac{1}{2\pi i} \int_{t=0}^{2\pi} \frac{f(a+re^{it})}{re^{it}} \, d(a + re^{it})$$

$$= \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{f(a+re^{it})}{re^{it}} \cdot rie^{it} \, dt$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} f(a + re^{it}) \, dt.$$  

That is, $f(a)$ is the average value of $f$ on the circle $|z - a| = r$. 

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Corollary

In the setting above, \( f(a) = \frac{1}{\pi r^2} \int \int_{|z-a| \leq r} f(z) dx dy \)

"Proof: " Take average on every circle centered at \( a \) with radius \( \leq r \), then take weighted average over these circles.
Maximum Principle

**Lemma:** For a holomorphic function $f$, if $|f|$ achieves its maximum value in the disk $D = \{|z - a| \leq r\}$ at $a$, then $f$ is a constant function in $D$.

**Proof of Lemma**

- Write $w = f(a)$, then $1 = \frac{f(a)}{w}$ equals the average value $\int\int_D \frac{f(z)}{w} \, dx \, dy$ in the disk.
- For each $z$, $|f(z)| \leq |w|$, $\left| \frac{f(z)}{w} \right| \leq 1$ and $\Re \frac{f(z)}{w} \leq 1$. So $\Re \int\int_D \frac{f(z)}{w} \, dx \, dy = \int\int_D \Re \frac{f(z)}{w} \, dx \, dy \leq 1$.
- As equality is achieved, we must have $\Re \frac{f(z)}{w} = 1$ for every $z$. This implies $\frac{f(z)}{w} = 1$ for every $z$ in the disk. □
Maximum Principle

Theorem

If \( f \) is holomorphic in a domain \( D \), continuous on \( \partial D \), and \( |f(z)| \) achieves its maximum at a point \( a \) inside \( D \), then \( f \) is a constant.

(See Theorem 5.4.2 on Page 170.)

In other words, a non-constant analytic function must achieve its maximal absolute value on the boundary of the domain.
Proof of Maximum Principle

Suppose \( f(b) \neq f(a) \) for some \( b \in D \). As \( D \) is connected, we can find a piecewise smooth path \( \gamma \) in \( D \) from \( a \) to \( b \).

Let \( c \) be the furthest point from \( a \) on the path (not by Euclidean distance, but by the travel distance along the path from \( a \)), such that \( f(c) = f(a) \).
Proof of Maximum Principle

Fix a tiny disk $|z - c| \leq r$ in $D$ centered at $c$. Then the disk contains points $c' \in \gamma$ with $f(c') \neq f(a)$. But $|f|$ achieves at $c$ its maximum in the disk.

This is a contradiction to the previous lemma! □
Homeworks

Reading material: Chapter 3.2 Power series convergence and its convergence tests
MATH311: Complex Analysis

Chapter 5: Series

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What is a series?

A series is:

- a formal sum \( \sum_{j=0}^{\infty} c_j \);
- or equivalently: \( c_0 + c_1 + c_2 + \cdots \)
  the terms \( c_j \) are complex numbers.

The index \( j \) is a dummy variable. For example, \( \sum_{n=0}^{\infty} c_n \) and \( \sum_{k=0}^{\infty} c_k \) are the same series.

- One may also let the series start at a different index other than 0. For example, \( \sum_{n=5}^{\infty} \frac{1}{n^2} = \frac{1}{5^2} + \frac{1}{6^2} + \cdots \).
Convergence of a series

■ When does it make sense, i.e. have a value?

Definition A series $\sum_{j=0}^{\infty} c_j$ converges if the sequence

$$S_n = \sum_{j=0}^{n} c_j$$

converges. And in this case the value of the series is $\lim_{n \to \infty} S_n$. Otherwise, we say the series diverges.

■ Changing or omitting finitely many terms at the beginning of the series would not affect convergence/divergence, e.g.

$\sum_{j=0}^{\infty} c_j$ converges if and only if $\sum_{j=100}^{\infty} c_j$ does.
Let \( \{a_n : n = 0, 1, \cdots \} \) be a sequence of complex values.

\( a_n \) converges to a limit \( L \) if for any \( \epsilon > 0 \), there is \( n_0 \), such that
\[
|a_n - L| < \epsilon \quad \text{whenever} \quad n \geq n_0.
\]

Example

- The sequence \( 1, 0, 1, 0, 1, 0, \cdots \) doesn’t converge.

Hence \( \sum_{j=0}^{\infty} (-1)^j \) doesn’t converge.

(Notice \( S_n = 1 \) if \( n \) is even and \( S_n = 0 \) if \( n \) is odd).
Review: convergence of sequences

Let \( \{a_n : n = 0, 1, \cdots \} \) be a sequence of complex values.

\[ a_n \text{ converges if and only if it is a Cauchy sequence}, \text{ i.e:} \]

For all \( \epsilon > 0 \), there is \( n_0 \) such that \( |a_m - a_n| < \epsilon \) whenever \( m, n \geq n_0 \).

For sequences, this can be formulated as:

\[ \sum_{j=0}^\infty c_j \text{ converges if and only if: for all } \epsilon > 0, \text{ there is } n_0 \text{ such that } |c_n + \cdots + c_m| < \epsilon \text{ whenever } m \geq n \geq n_0. \]

**Corollary:** In particular, if \( \sum_{j=0}^\infty c_j \) converges, then \( \lim_{j \to \infty} c_j = 0 \).
Convergence tests: 1. Comparison Test

Theorem

If: $|c_j| \leq M_j$; and $\sum_{j=1}^{\infty} M_j$ converges, then $\sum_{j=1}^{\infty} c_j$ converges.

Definition: $\sum_{j=1}^{\infty} c_j$ absolutely converges if $\sum_{j=1}^{\infty} |c_j|$ converges.

Corollary: In particular, $\sum_{j=1}^{\infty} c_j$ converges if it absolutely converges.
Examples

Given the fact that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges:

- $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.
- $\sum_{n=1}^{\infty} \frac{e^{3in^5}}{n^2+1}$ absolutely converges, and therefore converges.
Convergence tests: 2. Ratio Test

**Theorem**

Suppose \( \lim_{j \to \infty} \left| \frac{c_{j+1}}{c_j} \right| \) exists and equals \( L \).

- If \( L < 1 \), then \( \sum_{j=1}^{\infty} c_j \) converges;
- If \( L > 1 \), then \( \sum_{j=1}^{\infty} c_j \) diverges.

When \( L = 1 \), the known information is not sufficient to guarantee either convergence or divergence. A more detailed study of the series is required.
**Corollary:** The geometric series $\sum_{j=0}^{\infty} c^j$ converges if $|c| < 1$ and diverges if $|c| > 1$.

**Question:** What if $|c| = 1$?

No, because $c^j \not\to 0$ as $j \to \infty$. 

Geometric series
If $|c| < 1$, then $\sum_{j=0}^{\infty} c^j = \frac{1}{1-c}$.

Proof: Let $S_n = \sum_{j=0}^{n} c^j = 1 + c + \cdots + c^n$. Then

$$c \cdot S_n = c + \cdots + c^n + c^{n+1} = S_n - 1 + c^{n+1},$$

so $S_n = \frac{1-c^{n+1}}{1-c}$.

$$\sum_{j=0}^{\infty} c^j = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1-c^{n+1}}{1-c} = \frac{1}{1-c}. \qed$$
Example 1

Does \( \sum_{j=0}^{\infty} \left( \frac{8j \sin j}{2j^2 - 1} \right) j \) converge?

If we can show the series absolutely converges, that is,

\[
\sum_{j=0}^{\infty} \left| \left( \frac{8j \sin j}{2j^2 - 1} \right) j \right| \leq \sum_{j=0}^{\infty} \left( \frac{8j}{|2j^2 - 1|} \right) j
\]

converges, then the series converges.

\( \triangleright \) When \( j \geq 1 \), \( j^2 \geq 1 \) and \( 2j^2 - 1 \geq j^2 \geq 0 \), so

\[
\frac{8j}{|2j^2 - 1|} \leq \frac{8j}{j^2} \leq \frac{8}{j};
\]

\( \triangleright \) When \( j \geq 9 \), \( \frac{8j}{|2j^2 - 1|} \leq \frac{8}{9} \);
Example 1, continued

We compare \( \sum_{j=0}^{\infty} \left( \frac{8j}{|2j^2 - 1|} \right)^j \) to \( \sum_{j=0}^{\infty} \left( \frac{8}{9} \right)^j \).

The first finitely many terms can be ignored. And starting from \( j = 9 \), the terms in the first series are smaller.

\[ \sum_{j=0}^{\infty} \left( \frac{8}{9} \right)^j \] converges, because it is a geometric series with \( c < 1 \).

So \( \sum_{j=0}^{\infty} \left( \frac{8j}{|2j^2 - 1|} \right)^j \) converges.

Therefore, \( \sum_{j=0}^{\infty} \left( \frac{8j \sin j}{2j^2 - 1} \right)^j \) converges.
Example 2

Recall that \( j! = 1 \cdot 2 \cdots j \) and \( 0! = 1 \). Show that for any complex number \( a \), \( \sum_{j=0}^{\infty} \frac{a^j}{j!} \) converges.

**Proof:** Use ratio test, the quantity to be tested is

\[
\lim_{j \to \infty} \left| \frac{a^{j+1}}{(j+1)!} / \frac{a^j}{j!} \right| = \lim_{j \to \infty} \left| \frac{a^{j+1} / a^j}{(j+1)! / j!} \right| = \lim_{j \to \infty} \frac{|a|}{j+1} = 0.
\]

Since \( 0 < 1 \), this shows the original series converges. \( \Box \)

We will see later that the series is equal to \( e^a \).
Now we let $c_j = f_j(z)$ depend on $z$ in a series.

**Definition:** $\sum_{j=0}^{\infty} f_j(z)$ uniformly converges to $F(z)$ on a set $D \subset \mathbb{C}$ if:

for any $\epsilon > 0$, there is $n_0$, such that $|\sum_{j=0}^{n} f_j(z) - F(z)| < \epsilon$

whenever $n \geq n_0$ and $z \in D$.

The uniform convergence of $\sum_{j=0}^{\infty} f_j(z)$ on $D$ is equivalent to being **uniformly Cauchy**, i.e.

for any $\epsilon > 0$, there is $n_0$, such that $|\sum_{j=n}^{m} f_j(z)| < \epsilon$ whenever $m \geq n \geq n_0$ and $z \in D$. 
Example 3

Fix $a \in \mathbb{C}$.

a. For which $z$ does the series $\sum_{j=0}^{\infty} \frac{z^j}{a^j}$ converge?

Answer: $\sum_{j=0}^{\infty} \frac{z^j}{a^j} = \sum_{j=0}^{\infty} \left(\frac{z}{a}\right)^j$ converges $\iff |\frac{z}{a}| < 1 \iff |z| < |a|$.

b. What is the limit function?

Answer: $\frac{1}{1-z} = \frac{a}{a-z}$.

c. Does this series uniformly converge to $\frac{a}{a-z}$ on $|z| < |a|$?

Answer: No. The sequence is not uniformly Cauchy, since for $\epsilon = \frac{1}{2}$, for any $n_0$, there are $n \geq n_0$ and $z$ with $|z| < |a|$ such that $\left|\left(\frac{z}{a}\right)^n\right| > \frac{1}{2}$. In fact, we can take any $n \geq n_0$, and $z$ with $\frac{|z|}{a}$ sufficiently close to 1, depending on $n$. 
Example 3, continued

d. For $r < |a|$ show that $\sum_{j=0}^{\infty} \frac{z^j}{a^j}$ uniformly converges on $|z| \leq r$.

Proof: We know that $\sum_{j=0}^{\infty} \frac{r^j}{|a|^j}$ converges. So for any $\epsilon$, there is $n_0$ such that for all $n \geq n_0$, $\sum_{j=n+1}^{\infty} \frac{r^j}{|a|^j} < \epsilon$. Then $\sum_{j=n+1}^{\infty} \left| \frac{z^j}{a^j} \right| < \epsilon$ for all $z$ with $|z| \leq r$. This shows $\left| \sum_{j=n+1}^{\infty} \frac{z^j}{a^j} \right| < \epsilon$, or equivalently there exists a limit function $F(z)$, such that $\left| \sum_{j=n}^{\infty} \frac{z^j}{a^j} - F(z) \right| < \epsilon$ for all $n \geq n_0$ and $|z| \leq r$. \qed
Definition

- If \( f(z) \) has derivatives of all orders at \( a \), then the **Taylor series** around \( a \) is

\[
\sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!} (z - a)^j = f(a) + f'(a)(z - a) + \frac{f''(a)}{2}(z - a)^2 + \cdots
\]

- Taylor series around 0 is also called **Maclaurin series**

- In calculus of a real variable, there are bad functions \( f \) such that, all derivatives are defined at a point \( x_0 \) but the Taylor series never converges to \( f(x) \) except at \( x = x_0 \).

- Key philosophy of this chapter: this cannot happen for complex analytic functions.
Main Theorem

If $f$ is an analytic function on $|z - a| < R$, then its Taylor series around $a$ converges to $f(z)$ at every $z$ with $|z - a| < R$. Moreover, for any $r < R$, the convergence is uniform on $|z - a| \leq r$. 

![Diagram of circle centered at a with radius r and R]