From previous example, we learned:

$$z^n = (|z| e^{i \arg z})^n = |z|^n e^{in \arg z}.$$ 

What is $n$-th root of $z$?

Suppose $w^n = z$, then $|w|^n e^{in \arg w} = |z| e^{i \arg z}$.

- $|w| = |z|^{\frac{1}{n}}$
- $\arg w = \frac{\arg z}{n} = \ldots, \frac{\arg z - 2\pi}{n}, \frac{\arg z + 2\pi}{n}, \ldots$
  
  $\quad = \ldots, \frac{\Arg z}{n} - \frac{2\pi}{n}, \frac{\Arg z}{n}, \frac{\Arg z}{n} + \frac{2\pi}{n}, \ldots$

- $\Arg w$ can take $n$ different values

So $z^{\frac{1}{n}}$ is not unique:

- If $w$ is an $n$-th root of $z$, then so is $w e^{i \frac{2k\pi}{n}}$ for $k = 0, \ldots, n - 1$
Examples

- The $n$th roots of 1 are called the $n$-th roots of unity.
  - There are $n$ of them:
    \[ e^{i \frac{2k\pi}{n}} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} = \omega_n^k, \quad k = 0, \ldots, n - 1 \]
    where $\omega_n = e^{i \frac{2\pi}{n}} = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$

- 3rd (cubic) roots of unity: $e^{i \cdot 0} = 1$, $e^{i \frac{2\pi}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $e^{i \frac{4\pi}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

- 4th roots of unity: 1, $i$, $-1$, $-i$. 
Examples

■ Solve \((1 + i)^{\frac{1}{3}}\)

- \(|1 + i| = \sqrt{2}, \ \text{Arg}(1 + i) = \frac{\pi}{4}\)
- One root: \(\sqrt[6]{2}e^{\frac{\pi}{12}}i\)
- All cubic roots:
  \[
  \sqrt[6]{2}e^{i\frac{\pi}{12}} \\
  \sqrt[6]{2}e^{i\left(\frac{\pi}{12} + \frac{2\pi}{3}\right)} = \sqrt[6]{2}e^{i\frac{3\pi}{4}} \\
  \sqrt[6]{2}e^{i\left(\frac{\pi}{12} + \frac{4\pi}{3}\right)} = \sqrt[6]{2}e^{i\frac{17\pi}{12}}
  \]
Planar sets

- Smooth arc
- Smooth closed curve
- Open set
- Closed set
Planar sets

Definition: We say a set $S$ is open if for every point $p \in S$, there is a ball $B(p, r)$, centered at $p$ with radius $r$, such that $B(p, r) \subset S$.

Definition: Suppose for every sequence $\{z_i\}_{i=1}^{\infty}$ that satisfies (1) $z_i \in S$ for all $i = 1, 2, 3, \cdots$, and (2) $\lim_{i \to \infty} z_i = z_0$ exists, we have $z_0 \in S$. Then we say $S$ is a closed set. In other words, a set that contains all its limit points is a closed set.
Planar sets

- Connected set
- Disconnected set
Domain

- **domain** = open connected set

**Example of domains**

- Bounded, simply connected
- Bounded, not simply connected
- Unbounded, simply connected
- Unbounded, not simply connected
Domain

Theorem. Suppose $u(x, y)$ is a real-valued function defined on a domain $D$. If \( \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \) at all points of $D$, then $u$ is a constant on $D$. 
Functions of a complex variable

- $f(z) = w$, where $z, w \in \mathbb{C}$
  - Domain of definition: values of $z$ for which $f$ is defined
  - Range: values of $w$ that can be achieved by $f$

- $z = x + iy$, $w = u + iv$
  - $f$ is determined by $u = u(x, y)$ and $v = v(x, y)$:
    
    $$f(x + iy) = u(x, y) + iv(x, y)$$
Examples

- \( f(z) = e^z \)
  - \( e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y \)
  - \( u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y \)
  - for \( f(z) = e^z \) defined on \( \mathbb{C} \)
    - Domain of definition: \( \mathbb{C} \); Range: \( w \neq 0 \)
  - for \( f(z) = e^z \) defined on the right half plane \( \text{Re} z > 0 \)
    - Domain of definition: \( \text{Re} z > 0 \); Range: \( |w| > 1 \),
      because \( |e^z| = e^{\text{Re} z} > e^0 = 1 \)

- \( f(z) = \frac{1}{z} \)
  - Domain of definition: \( z \neq 0 \); Range: \( w \neq 0 \)
Examples

- \( f(z) = \frac{1}{1-z} \)

  - Domain of definition: \( z \neq 1; \) Range: \( w \neq 0 \)

  - \( f(z) = \frac{1}{1-x-iy} = \frac{1-x-iy}{|1-x-iy|^2} = \frac{1-x+iy}{(1-x)^2+y^2} \)

  - \( u(x, y) = \frac{1-x}{(1-x)^2+y^2} = \frac{1-x}{1-2x+x^2+y^2}, \)

  - \( v(x, y) = \frac{y}{(1-x)^2+y^2} = \frac{y}{1-2x+x^2+y^2} \)

- Show that \( f(z) \) maps the circle \(|z| = 1\) to \( \text{Re} \ w = \frac{1}{2}. \)

  Proof: \( x^2 + y^2 = |z|^2 = 1 \Rightarrow u(x, y) = \frac{1-x}{2-2x} = \frac{1}{2}. \)