Zeros of \(\cos z\) and \(\sin z\)

Recall \(\cos z\), \(\sin z\) are entire, i.e. no poles

- **The zeros of \(\sin z\) are \(k\pi, k \in \mathbb{Z}\)**

  **Proof:**

  \[
  \sin z = 0 \iff \frac{e^{iz} - e^{-iz}}{2i} = 0 \iff e^{iz} = e^{-iz} \iff 2iz = 2k\pi i \iff z = k\pi
  \]

- **The zeros of \(\cos z\) are \((k + \frac{1}{2})\pi, k \in \mathbb{Z}\)**

  **Proof:** \(\cos z = \cos z \cdot 1 + \sin z \cdot 0 = \sin(z + \frac{\pi}{2})\), so

  \[
  \cos z = 0 \iff z + \frac{\pi}{2} = k\pi \iff z = (k - \frac{1}{2})\pi, k \in \mathbb{Z}.
  \]

That’s the same as \(z = (k + \frac{1}{2})\pi, k \in \mathbb{Z}\).
Examples

\[ f(z) = \cos(3z) - \sin \frac{\pi i}{z} \]

Find \( f(2i) \).

\[
\begin{align*}
 f(2i) &= \cos(6i) - \sin \frac{\pi}{2} = \frac{e^{i \cdot 6i} + e^{-i \cdot 6i}}{2} - \sin \frac{\pi}{2} \\
 &= \frac{e^{-6} + e^{6}}{2} - 1
\end{align*}
\]

Find \( f'(2i) \)

\[
\begin{align*}
 f'(z) &= -\sin(3z) \cdot (3z)' - \cos \frac{\pi i}{z} \cdot \left( \frac{\pi i}{z} \right)' = -3 \sin(3z) - \cos \frac{\pi i}{z} \cdot \left( -\frac{\pi i}{z^2} \right) \\
 f'(2i) &= -3 \sin(6i) - \cos \frac{\pi}{2} \cdot \left( -\frac{\pi i}{1} \right) = -3 \cdot \frac{e^{-6} - e^{6}}{2i} - 0 \\
 &= \frac{3(e^{-6} - e^{6})}{2} i
\end{align*}
\]
Example

How many solutions does \( \cos z = 100 \) have in the region \(-\pi \leq \text{Re } z < 3\pi\)?

\[
\begin{align*}
\text{\( \cos z = 100 \) } & \iff \frac{e^{iz} + e^{-iz}}{2} = 100 \iff e^{iz} - 200 + e^{-iz} = 0 \\
& \iff e^{2iz} - 200e^{iz} + 1 = 0, \text{ i.e. } e^{iz} \text{ is a zero of } w^2 - 200w + 1 \\
& \iff w^2 - 200w + 1 \text{ has two distinct zeros } w_1, w_2, \text{ and } w_1, w_2 \neq 0. \\
& \iff e^z = w_1 \text{ has exactly one solution in the fundamental region } -\pi \leq \text{Im } z < \pi. \\
& \iff \text{So } e^{iz} = w_1 \text{ has exactly one solution in } -\pi \leq \text{Re } z < \pi. \text{ And so does } e^{iz} = w_2. \\
& \iff \text{Therefore, } \cos z = 100 \text{ has two solutions in } -\pi \leq \text{Re } z < \pi, \text{ as } \cos \text{ has period } 2\pi, \text{ there are two solutions in } \pi \leq \text{Re } z < 3\pi. \\
\text{Answer: } 4 \text{ solutions in } -\pi \leq \text{Re } z < 3\pi.
\end{align*}
\]
cosh $z$ and sinh $z$

$$\cosh z := \frac{e^z + e^{-z}}{2}, \quad \sinh z := \frac{e^z - e^{-z}}{2}$$

**Basic identities**

- $\cosh z = \cos(iz), \quad \sinh z = \frac{\sin(iz)}{i}$, both are analytic, have period $2\pi i$
- $\cosh' z = \sinh z, \quad \sinh' z = \cosh z$
- $\cosh^2 z - \sinh^2 z = 1$
**log and Log**

- \( w = \log z \) if \( e^w = z \).

  Equivalently, \( \log = \text{Log} |z| + i \arg z \). (Here, \( \text{Log} |z| \) stands for the usual logarithm of a positive real number.)

- Is \( \log z \) unique?

  No: \( w + 2k\pi i \) are also solutions, because \( e^w \) is periodic with period \( 2\pi i \).
log and Log

What can we do?

Choose $w$ in a fundamental region, where the solution is unique.

$w = \log z$ if $e^w = z$ and $-\pi < \text{Im } w \leq \pi$.

Equivalently, $\log z = \log |z| + i \arg z$

Example: $\log 2i = \log 2 + i \frac{\pi}{2} \quad$ 

$\log 2i = \log 2 + i\left(\frac{\pi}{2} + 2k\pi\right), \; k \in \mathbb{Z}$. 
Basic facts about Log

- Domain = \{z \neq 0\}, Range = \{-\pi < \text{Im } w \leq \pi\}

- Continuous except along Arg \( z = \pi \)

  At \( z_0 \) with Arg \( z_0 = \pi \): limit is \( \log |z| + i\pi \) when approaching from above, but \( \log |z| - i\pi \) from below.

- \( \log zw = \log z + \log w \), \( \log \frac{z}{w} = \log z - \log w \).

  But the same equalties are not always true for Log. (Both sides may differ by \( 2\pi i \)).
Basic facts about Log

- Log z is analytic except along Arg z = π

  **Proof:** If f has derivative f'(z_0) at z_0, then f^{-1}(z) has derivative \( \frac{1}{f'(z_0)} \) at f(z_0). Apply this to f(z) = e^z defined on \( \{-\pi < \text{Im } w < \pi\} \).

- (Log z)' = \frac{1}{(e^w)'} = \frac{1}{e^w} = \frac{1}{z}.
Definition: $F$ is a branch of a multiple valued function $f$ on a domain $D$ if $F$ is continuous on $D$, and $F(z)$ equals one of the values of $f(z)$ for all $z \in D$.

Example

Log $z$ is a branch of log $z$ on \( \{z \neq 0, \text{Arg } z \neq \pi\} \)
More branches of log $z$

- $\log z + 4\pi i$ is a branch on $\{z \neq 0, \text{Arg } z \neq \pi\}$ (Domain A)

- There is a unique branch $\text{Arg}_{-3\frac{3}{4}\pi} z$ of arg $z$ on $\{z \neq 0, \text{Arg } z \neq -3\frac{3}{4}\pi\}$ (Domain B) s.t. $-\frac{3}{4}\pi < \text{Arg}_{-3\frac{3}{4}\pi} z < \frac{5}{4}\pi$.
  
  - $\log |z| + i \text{Arg}_{-3\frac{3}{4}\pi} z$ is a branch of log $z$ on Domain B.
More branches of log $z$

- There is a unique branch $\text{Arg}_C z$ of $\text{arg } z$ on Domain $C$, s.t. $\text{Arg}_C 1 = 0$ and $\text{Arg}_C z$ is continuous on Domain $C$.

- $\log|z| + i \text{Arg}_C z$ is a branch of $\log z$ on Domain $C$. 

![Graphs A, B, C showing branch cuts and continuity]

Chapter 3: Elementary Functions,   Section 3.3: Log Function
Fact: For any branch $L(z)$ of log $z$ on any domain $D$, $L$ is analytic and $L'(z) = \frac{1}{z}$ on $D$.

Proof: same as for Log $z$.

Example: Find a branch $g(z)$ of log($z^2 - 4$) that is analytic at $z = 1$, find $g(1)$ and $g'(1)$.

$1^2 - 4 = -3$ is in Domain B above, so

$g(z) = \text{Log} |z^2 - 4| + i \text{Arg}_{-\frac{3}{4}\pi}(z^2 - 4)$ is analytic at $z = 1$.

$g(1) = \text{Log} 3 + i \text{Arg}_{-\frac{3}{4}\pi}(-3) = \text{Log} 3 + i\pi$;

$g'(1) = \frac{1}{z^2 - 4} \cdot (z^2 - 4)'|_{z=1} = \frac{2z}{z^2 - 4}|_{z=1} = -\frac{2}{3}$.
A contour $\gamma$ is a piecewise smooth oriented curve consisting of finitely many oriented smooth arcs $\gamma_k$.
Parametrization of a contour

Each oriented smooth arc $\gamma_i$ is parametrized by

$$\{ z(t) = x(t) + iy(t) : a_i \leq t \leq b_i \}$$

with the following conditions:

- $z'(t) = x'(t) + iy'(t)$ exists and is continuous and never vanishes on $[a, b]$
- $z(t)$ is one-to-one on $[a, b]$

In order to be a contour, we also need:

- $b_k = a_{k+1}$
Length of a smooth arc $z(t): a \leq t \leq b$ is $\int_{a}^{b} |z'(t)| \, dt$.

Length of a contour is the sum of lengths of its arc components.
Example

Draw the contour parametrized by

\[
\begin{align*}
\gamma_1 : z(t) &= (1 - i) + ti, \quad 0 \leq t \leq 1; \\
\gamma_2 : z(t) &= \frac{t + i}{-t + i}, \quad 0 \leq t \leq 1;
\end{align*}
\]

and calculate its length.

For \( \gamma_2 \): note as \( t \) is real, \( |t + i| = |-t + i| = \sqrt{t^2 + 1} \), hence \( z(t) = \frac{t+i}{-t+i} \) is on the unit circle.
Length of \( \gamma_1 : z(t) = (1 - i) + ti, \quad 0 \leq t \leq 1; \)
\[\gamma_2 : z(t) = \frac{t + i}{-t + i}, \quad 0 \leq t \leq 1; \]

For \( \gamma_1 : z'(t) = i, \)
\[L_1 = \int_0^1 |i| dt = \int_0^1 1 dt = 1\]

For \( \gamma_2 : z'(t) = \frac{1 \cdot (-t+i) - (t+i) \cdot (-1)}{(-t+i)^2} = \frac{2i}{(-t+i)^2}\]
\[L_2 = \int_0^1 \frac{2i}{(-t+i)^2} |dt = \int_0^1 \frac{|2i|}{|-t+i|^2} = \int_0^1 \frac{2}{t^2 + 1} dt\]
\[= 2 \arctan t \Big|_0^1 = 2 \cdot \frac{\pi}{4} - 2 \cdot 0 = \frac{\pi}{2}.\]

Thus total length of contour is \( L_1 + L_2 = 1 + \frac{\pi}{2}. \)