Small Excess and the Topology of Open Manifolds

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Abstract: In this paper, we study the relations between the excess of open manifolds and their topology by using the methods of comparison geometry. We prove that for an open manifold with curvature bounded from below, it has finite topological type or it is diffeomorphic to R^n when its excess is bounded by some function of its critical radius.

Key words: Excess function; Open manifold; Finite topological type; Critical point; Comparison theorem

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1 Introduction and main results.

Let $(M, g)$ be a complete noncompact $n$-manifold with curvature (sectional curvature or Ricci curvature) bounded from below. The topological properties of $(M, g)$ have received much attention recently. Many new notions and tools have been established, see, for example [1], [2], [4], [8] etc. In this paper, we get some relations between the excess of open manifolds and their topology.

Let $M$ be a complete $n$-manifold. In their pioneering paper [1], Abresch and Gromoll defined the following excess function. For any $p, q \in M$, $d$ denotes the distance function on $M$ induced from the metric, the excess function $e_{pq}(x)$ is defined by

$$e_{pq}(x) = d(x, p) + d(x, q) - d(p, q), \forall x \in M.$$ 

Now take $M$ to be a complete noncompact $n$-manifold, $p \in M$. The excess of $M$ at $p$ is defined to be ([9])

$$e(p) = \sup_{x \in M} (d(x, p) - b_p(x)).$$

By the triangle inequality, this is nonnegative. Define two notion of excess of $M$ by

$$e(M) = \sup_{p \in M} e(p), E(M) = \inf_{p \in M} e(p).$$

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1
Here, $b_p(x) = \limsup_{q \to \infty}(d(p, q) - d(q, x)) = \lim_{t \to \infty}(t - d(x, S(p, t)))$ is the generalized busemann function with respect to $p \in M$.

Given $p \in M$, we say that $K^\text{min}_p \geq c$ if for any minimal geodesic $\gamma$ issuing from $p$, all sectional curvature of the planes which are tangent to $\gamma$ are greater than or equal to $c$.

Notice that the distance function $d_p$ is not a smooth function (on the cut locus of $p$), hence the critical points of $d_p$ are not defined in a usual sense. The notion of critical points of $d_p$ is introduced by Grove-Shiohama [4]. A point $q \in M$ is called a critical point of $d_p$ if for any unit vector $v \in T_qM$, there is a minimizing geodesic $\sigma$ from $p$ to $q$ such that $\angle(\sigma(t), v) \leq \frac{\pi}{2}$. The critical radius $r_p$ at $p$ is the supremum of $r$ such that $B(p, r)$ has no critical point of $p$. We also define critical radius of $M$ as $r(M) = \inf_{p \in M} r_p$.

A manifold $M$ is said to have finite topological type if there is a compact domain $\Omega \subset M$ whose boundary $\partial \Omega$ is a topological manifold such that $M \setminus \Omega$ is homeomorphic to $\partial \Omega \times [0, +\infty)$. It is well known that $M$ is diffeomorphic to $\mathbb{R}^n$ if there is a point $p \in M$ such that $p$ has no critical point other than $p$. We also know (see [11]) that if all the critical points of $p$ are in some compact set in $M$, then $M$ has finite topological type.

In Shiohama’s paper [9], he proved that if $K_M \geq 0$ and $E(M) = 0$, then $M$ is diffeomorphic to $\mathbb{R}^n$. He also proved that if $K_M \geq -1$ and $\epsilon(M) < \epsilon(n)$, then $M$ homeomorphic to $S \times \mathbb{R}^k$, where $S$ is a compact manifold. In this paper, we will prove the following:

**Theorem 1.1** Let $M$ be a complete noncompact Riemannian $n$-manifold. If there is a point $p \in M$ with $K^\text{min}_p \geq -1$ and $\epsilon(p) < \ln 2$, then $M$ has finite topological type.

**Theorem 1.2** Let $M$ be a complete noncompact Riemannian $n$-manifold. If there is a point $p \in M$ with $K^\text{min}_p \geq -1$ and $\epsilon(p) < \ln \frac{2}{1 + e^{2}}$, then $M$ is diffeomorphic to $\mathbb{R}^n$.

**Theorem 1.3** Let $M$ be a complete noncompact Riemannian $n$-manifold with $\text{Ric}_M \geq -(n - 1)$. If there is a point $p \in M$ with

$$\epsilon(p) < \frac{1}{4} \min\left\{\frac{1}{4}\rho_c(p), \left(\frac{1}{C_0(n, \rho_c(p))} \ln \frac{9}{8}\right)^2\right\},$$

where $\rho_c(p)$ and $C_0(n, \rho_c(p))$ are as in Lemma 1.6, then $M$ has finite topological type.

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where $\rho_c(p)$ and $C_0(n, \rho_c(p))$ are as in Lemma 1.6, then $M$ is diffeomorphic to $\mathbb{R}^n$.

The following two Toponogov-type comparison theorems are to be used in controlling the existence of critical points.
Lemma 1.5 ([5]) Let $M$ be a complete Riemannian $n$-manifold, $p \in M$ and $K^\text{min}_p \geq c$. Suppose $\gamma_i : [0, l_i] \to M, i = 0, 1, 2$ be minimal geodesics with $\gamma_1(0) = \gamma_2(l_2) = p, \gamma_0(0) = \gamma_1(l_1), \gamma_0(l_0) = \gamma_2(0)$. Then, there exist minimal geodesics $\tilde{\gamma}_i : [0, l_i] \to M^2(c), i = 0, 1, 2$ with $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(l_2), \tilde{\gamma}_0(0) = \tilde{\gamma}_1(l_1), \tilde{\gamma}_0(l_0) = \tilde{\gamma}_2(0)$ such that

$$\angle(-\gamma'_1(l_1), \gamma'_0(0)) \geq \angle(-\tilde{\gamma}'_1(l_1), \tilde{\gamma}'_0(0)),$$

$$\angle(-\gamma'_0(l_0), \gamma'_2(0)) \geq \angle(-\tilde{\gamma}'_0(l_0), \tilde{\gamma}'_2(0)),$$

where $M^2(c)$ denotes the complete simple connected surface of constant curvature $c$.

Before the next lemma, we will first introduce the following modified version of the conjugate radius function $\rho_c(p)$:

$$\rho_c(p) = \sup\{\rho > 0 | \text{con}(q) \geq \rho, \forall q \in B_\rho(p)\}.$$

Lemma 1.6 ([3],[8]) Let $M$ be a complete $n$-manifold with $\text{Ric}_M \geq -(n - 1)$. Then for every $p \in M$, there is a constant $C_0 = C_0(n, \rho_c(p)) > 0$ such that if $\sigma_i : [0, r_i] \to M$ are minimizing geodesics from $p$, $\rho = \max\{r_1, r_2\} \leq \frac{1}{4} \rho_c(p)$, then

$$d(\sigma_1(r_1), \sigma_2(r_2)) \leq e^{C_0^2/4} |r_1 v_1 - r_2 v_2|,$$

where $v_i = \frac{d}{dt}(0), i = 1, 2$. Moreover, if $\text{con}_M \geq c_0 > 0$, then in the same conditions, there is a constant $C_0 = C_0(n, c_0) > 0$ such that the above conclusion holds.

2 The excess function of open manifolds.

Now we will study some properties of excess function of open manifolds, which will be used in the proof of our theorems.

Proposition 2.1 Let $\gamma_1, \gamma_2$ be minimal geodesics from $x$ to $p, q$ respectively, $p^*$ is a point in $\gamma_1$ and $q^*$ is a point in $\gamma_2$. Then $e_{p^*, q^*}(x) \leq e_{pq}(x)$.

Proof. From the triangle inequality, we know that

$$e_{pq}(x) - e_{p^*, q^*}(x) = d(p, p^*) + d(p^*, q^*) + d(q^*, q) - d(p, q) \geq d(p, q) \geq 0.$$

Proposition 2.2 If $q, x \in M$ satisfied $d(q, x) = d(x, S(p, d(p, q)))$ and $d(p, x) \leq d(p, q)$, then $e_{pq}(x) \leq e_{pq}(x)$.

Proof. From the triangle inequality, we know that

$$e_{pq}(x) - e_{p^*, q^*}(x) = d(p, p^*) + d(p^*, q^*) + d(q^*, q) - d(p, q) \geq d(p, q) \geq 0.$$
Proof. First, we will show that when \( t \geq d(x, p) \), function \( t \mapsto t - d(x, S(p, t)) \) is monotone decreasing. In fact, suppose \( t \geq d(x, p) \) and \( \varepsilon > 0 \), take \( a \in S(p, t + \varepsilon) \) with \( d(x, a) = d(x, S(p, t + \varepsilon)) \). \( \gamma \) is a minimal geodesic from \( a \) to \( x \) which cross \( S(p, t) \) at point \( b \), then \( d(x, S(p, t + \varepsilon)) = d(x, a) \geq d(x, b) + d(a, b) \geq d(x, S(p, t)) + \varepsilon \). Thus we have

\[
[t + \varepsilon - d(x, S(p, t + \varepsilon))] - [t - d(x, S(p, t))] = \varepsilon + d(x, S(p, t)) - d(x, S(p, t + \varepsilon)) \leq 0.
\]

By this, we can get

\[
e(p) = \sup_{y \in M} (d(p, y) - b_p(y)) \geq d(p, x) - \lim_{t \to \infty} [t - d(x, S(p, t))] \geq d(p, x) - d(p, q) + d(x, S(p, d(p, q))) = e_{pq}(x).
\]

The following two lemmas tell us that excess function can control the existence of critical points far away.

**Lemma 2.3** Let \( M \) be a complete noncompact Riemannian \( n \)-manifold, \( p \in M \) with \( K_p^{\min} \geq -1 \). There exist a function \( \epsilon_1(a, b) > 0 \), such that for \( \forall a, b > 0 \), if \( e_{pq}(x) < \epsilon_1(a, b) \), where \( a = d(x, p) \) and \( b = d(x, q) \), then \( x \) is not a critical point of \( p \). Moreover, the function \( \epsilon_1(a, b) \) satisfies \( \lim_{a, b \to \infty} \epsilon_1(a, b) = \ln \frac{2}{1 + e^{-2}} \).

**Proof.** Suppose that, on the contrary, \( x \) is a critical point of \( p \). Take \( \gamma_1 \) to be a minimal geodesics from \( x \) to \( q \), then by the definition of critical point, we know that there exists a minimal geodesic \( \gamma_2 \) from \( x \) to \( p \), such that \( \theta = \angle(\gamma_1'(0), \gamma_2'(0)) \leq \frac{\pi}{2} \). Write \( t = d(p, q) \), by Lemma 1.5 and Cosine Theorem in \( M^2(-1) \), we get

\[
cosh(t) \leq \cosh(a) \cosh(b) - \cos(\theta) \sinh(a) \sinh(b) \leq \cosh(a) \cosh(b),
\]
in other words,

\[
e^t + e^{-t} \leq \frac{1}{2} (e^a + e^{-a})(e^b + e^{-b}).
\]

Take \( f(a, b) \) to be the positive root of the equation

\[
0 = F(x) = e^x + e^{-x} - \frac{1}{2} (e^a + e^{-a})(e^b + e^{-b}),
\]

then \( t \leq f(a, b) \). By \( F(0) < 0 \) and \( F(a + b) > 0 \) we know \( 0 < f(a, b) < a + b \). Define \( \epsilon_1(a, b) = a + b - f(a, b) > 0 \), then

\[
e_{pq}(x) = a + b - t \geq a + b - f(a, b) = \epsilon_1(a, b) > 0,
\]

which is a contradiction. In addition, by computing directly, we can get

\[
f(a, b) = \ln \left[ \frac{1}{4} (e^a + e^{-a})(e^b + e^{-b}) + \frac{1}{4} \sqrt{(e^a + e^{-a})^2(e^b + e^{-b})^2 - 16} \right]
\]
\[ \epsilon_1(a, b) = a + b - f(a, b) = \ln \frac{2}{1 + e^{-2a}} + \ln \frac{1}{1 + e^{-2b}} + \ln \left( 1 + \sqrt{1 - \frac{2}{(c_0 + c_0)(2c_0 + c_0)}} \right). \]

Then the last conclusion follows from the above equation immediately.

**Lemma 2.4** Let \( M \) be a complete noncompact Riemannian \( n \)-manifold with \( \text{Ric}_M \geq -(n - 1), p \in M \) is a fixed point. If \( x, q \in M \) satisfy \( d(p, x) \geq \rho, d(q, x) \geq \rho, \epsilon_{pq}(x) < \frac{1}{4}\rho \), then \( x \) is not a critical point of \( p \). Here \( \rho \) satisfies \( 0 < \rho \leq \min\\{ \frac{1}{4}\rho_0(p), (\frac{1}{c_0(n, \rho_0(p))} \ln \frac{9}{8})^2 \} \), and \( C_0(n, \rho_0(p)) \) is the same as in Lemma 1.6. Here, the constant \( \rho_0(p) \) can be changed to be \( c_0 \) if \( \text{ Conj}_M \geq c_0 > 0 \).

**Proof.** Let \( \sigma_1, \sigma_2 \) be minimal geodesics from \( x \) to \( p, q \), \( p^* = \sigma_1(\rho), q^* = \sigma_2(\rho) \). By Proposition 2.1 we know that \( \epsilon_{p^*, q^*}(x) \leq \epsilon_{pq}(x) < \frac{1}{4}\rho \). Let \( \theta = \angle(\sigma_1(0), \sigma_2(0)) \), by Lemma 1.6 we know

\[
d[p^*, q^*] \leq e^{C_0\rho^2}(2\rho)[1 - \sin^2\left( \frac{\pi - \theta}{2} \right)] < e^{C_0\rho^2}(2\rho)[1 - \frac{1}{2}\sin^2\left( \frac{\pi - \theta}{2} \right)],
\]

thus

\[
\sin^2\left( \frac{\pi - \theta}{2} \right) \leq e^{C_0\rho^2} \sin^2\left( \frac{\pi - \theta}{2} \right) \leq 2[e^{C_0\rho^2} - 1] = 2\left( e^{C_0\rho^2} - 1 + \frac{\epsilon_{p^*, q^*}(x)}{2\rho} \right).
\]

By \( \rho \leq \left( \frac{1}{c_0} \ln \frac{9}{8} \right)^2 \) and \( \epsilon_{p^*, q^*}(x) < \frac{1}{4}\rho \) we get \( e^{C_0\rho^2} - 1 < \frac{1}{8} \) and \( \frac{\epsilon_{p^*, q^*}(x)}{2\rho} < \frac{1}{8} \). So \( \sin^2\left( \frac{\pi - \theta}{2} \right) < \frac{1}{8} \) and \( \theta > \frac{\pi}{2} \). Thus \( x \) is not a critical point of \( p \).

The last conclusion is obvious by the same discussing and the second part of Lemma 1.6.

3 The proof of theorems and some corollaries.

**Proof of Theorem 1.1:**

From \( e(p) < \ln 2 \) and \( \lim_{a \to \infty} \lim_{b \to \infty} \epsilon_1(a, b) = \ln 2 \) we know that there exist \( a, b \) large enough, \( b > a \), such that \( e(p) < \epsilon_1(a, b) \), here \( \epsilon_1(a, b) \) is the same as in Lemma 2.3. Now we will prove that for all \( x \) satisfies \( d(x, p) \geq a \), \( x \) is not a critical point of \( p \), thus all critical points of \( p \) lies in the closed ball \( B(p, a) \) and \( M \) has finite topological type.

In fact, write \( r_0 = d(x, p) \geq a \). Take \( q \in S(p, 2r_0 + b) \) with \( d(q, x) = d(x, S(p, 2r_0 + b)) \). By Proposition 2.2, we know \( \epsilon_{pq}(x) \leq e(p) < \epsilon_1(a, b) \), so by Lemma 2.3, \( x \) is not a critical point of \( p \).
From Theorem 1.1 we immediately get

**Corollary 3.1** Let $M$ be a complete noncompact Riemannian $n$-manifold with $K_M \geq -1$, $E(M) < \ln 2$, then $M$ has finite topological type.

**Proof of Theorem 1.2:**
Given $x \in M$, denote $r_0 = d(p, x)$. If $r_0 < r_p$, by the definition of $r_p$ we know that $x$ is not a critical point of $p$. If $r_0 \geq r_p$, take $b > 0$ with $e_1(r_p, b) > e(p)$, then take $q \in S(p, 2r_0 + b)$ with $d(q, x) = d(x, S(p, 2r_0 + b))$. By Proposition 2.2 we know $e_{pq}(x) \leq e(p) < e_1(r_p, b)$, so by Lemma 2.3, $x$ is not a critical point of $p$. Thus $q$ has no critical point in $M$ and $M$ is diffeomorphic to $\mathbb{R}^n$.

From Theorem 1.2 we immediately get

**Corollary 3.2** Let $M$ be a complete noncompact Riemannian $n$-manifold with $K_M \geq -1$, $r(M) > 0$, and $E(M) < \ln \frac{2}{1 + e^{-2r(M)}}$, then $M$ is diffeomorphic to $\mathbb{R}^n$.

**Proof of Theorem 1.3:**
Given $x \in M$ with $r_0 = d(p, x) > \min\{\frac{1}{4}\rho_\epsilon(p), \frac{1}{C_0(n, \rho_\epsilon(p))} \ln \frac{g}{8}\}$, take $q \in S(p, 2r_0)$ satisfies $d(x, q) = d(x, S(p, 2r_0))$. By Proposition 2.2 we know

$$e_{pq}(x) \leq e(p) < \frac{1}{4} \min\{\frac{1}{4}\rho_\epsilon(p), \frac{1}{C_0(n, \rho_\epsilon(p))} \ln \frac{g}{8}\}.$$

So by Lemma 2.4, $x$ is not a critical point of $p$, which implies that $M$ has finite topological type.

From the proof of Theorem 1.3 we immediately get

**Corollary 3.3** Let $M$ be a complete noncompact Riemannian $n$-manifold with $\text{Ric}_M \geq -(n - 1)$, $\text{conj}_M \geq c_0$ and $E(M) < \frac{1}{4} \min\{\frac{1}{4}\rho_\epsilon(p), \frac{1}{C_0(n, \rho_\epsilon(p))} \ln \frac{g}{8}\}$, then $M$ has finite topological type.

**Proof of Theorem 1.4:**
We only need to prove that $p$ has no critical point. Suppose $x$ is a critical point of $p$. Set $\rho = \min\{\frac{1}{4}\rho_\epsilon(p), \frac{1}{C_0(n, \rho_\epsilon(p))} \ln \frac{g}{8}\}$, then in $B(p, \rho)$ there is no critical point of $p$ and $r_0 = d(p, x) \geq \rho$. Take $q \in S(p, 2r_0)$ with $d(q, x) = d(x, S(p, 2r_0))$, by Proposition 2.2 we know $e_{pq}(x) \leq e(p) < \frac{1}{4} \rho$, together with Lemma 2.4, $x$ is not a critical point of $p$, which is a contradiction.
The following corollary are obvious.

**Corollary 3.4** Let $M$ be a complete noncompact Riemannian $n$-manifold with $\text{Ric}_M \geq -(n-1)$, $\text{con}_M \geq c_0 > 0$, $r(M) > 0$ and

$$E(M) < \frac{1}{4} \min \{ \frac{1}{4} c_0, \left( \frac{1}{C_0(n, c_0)} \ln \frac{9}{8} \right)^2, r(M) \},$$

then $M$ is diffeomorphic to $\mathbb{R}^n$.

**Corollary 3.5** Let $M$ be a complete noncompact Riemannian $n$-manifold with $\text{inj}_M \geq i_0 > 0$. If one of the following two conditions hold, then $M$ is diffeomorphic to $\mathbb{R}^n$:

1. $\text{Ric}_M \geq -(n-1)$ and $E(M) < \frac{1}{4} \min \{ \frac{1}{4} i_0, \left( \frac{1}{C_0(n, i_0)} \ln \frac{9}{8} \right)^2, i_0 \}$;
2. $K_M \geq -1$, and $E(M) < \ln \frac{2}{1+e^{-2i_0}}$.

**References**


小 **Excess** 与开流形的拓扑

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摘要

本文中，我们应用比较几何的方法研究开流形的 **Excess** 与其拓扑之间的关系。我们证明了对于一个曲率下有界的开流形，当它的 **Excess** 被其低界半径的某个函数所诱导时，它就有有限拓扑型或散分同胚于 $n$ 维欧氏空间。

**关键词**：**Excess** 函数；开流形；有限拓扑型；临界点；比较定理