On the fundamental group of open manifolds with nonnegative Ricci curvature *

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Abstract

In this paper we establish some uniform estimates for the distance to halfway points of minimal geodesics by terms of the distance to end points on some types of Riemannian manifolds, and then prove some theorems about the finite generation of fundamental group of Riemannian manifold with nonnegative Ricci curvature, which support the famous Milnor conjecture.

Key words Excess function, Finitely generated fundamental group, Ray density, Ricci curvature

2000 MR Subject Classification 53C20
Chinese Library Classification O186.12 Document Code A

§1. Introduction

Let \((M, g)\) be a complete noncompact \(n\)-manifold with nonnegative Ricci curvature or even Ricci curvature bounded from below. The topological properties of \((M, g)\) have received much attention recently. Numerous works had been done under the same title “curvature and topology”. We can see, for example, [1], [7],[8].

In this paper, we study the fundamental group of complete Riemannian manifolds \(M\) with nonnegative Ricci curvature \(\text{Ric}_M \geq 0\).

Recall that if \(M\) is compact (without boundary), then its fundamental group \(\pi_1(M)\) is finitely generated, and if \(M\) is compact with \(\text{Ric}_M > 0\), then by Myers Theorem \(\pi_1(M)\) is finite.

If \(M\) is noncompact, then the most basic question is whether \(\pi_1(M)\) is finitely generated. In 1968, Milnor [6] conjectured that every complete noncompact manifold \(M^n\) with nonnegative Ricci curvature must has finitely generated fundamental group. He

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*This research was supported by National Natural Science Foundation of China (Grant No. 19971081)
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proved that any finitely generated subgroup of $\pi_1(M)$ is of polynomial growth of order at most $n = \dim N$.

Later in 1971, Cheeger and Gromoll [4] proved the famous Soul theorem, which indicate that each complete noncompact manifold with nonnegative sectional curvature must diffeomorphic to the normal bundle over a compact totally geodesic submanifold called a soul. Thus Milnor conjecture was true for nonnegative sectional curvature. While, even for positive Ricci curvature, we don’t know whether the conjecture is true.

We can note that this question are open only for $\dim M \geq 4$. In dimension 2, it is obvious that $Ric_M = \text{Sec} M$. In dimension 3, Schoen and Yau [11] have proven that a complete manifold with $Ric_M > 0$ is diffeomorphic to Euclidean space and thus has a trivial fundamental group. Anderson [3] proved that $\pi_1(M)$ is either $\{e\}$, $\mathbb{Z}$ or has $\mathbb{Z} \oplus \mathbb{Z}$ as a subgroup of finite index.

In 1990 Anderson [2] proved that if a manifold with nonnegative Ricci curvature has Euclidean volume growth, then the fundamental group is actually finite and its order is less than $1/\alpha_M$, where $\alpha_M$ is the volume growth constant. Abresch, Gromoll, Shen and many other person proved that with many different extra conditions such as small diameter growth, sectional curvature bounded etc, such manifold has finite topological type, and thus has finitely generated fundamental group.

Recently, in her paper [9], C. Sormani studied the relation between fundamental group $\pi_1(M)$ and the diameter growth of a complete Riemannian manifold $M$ with nonnegative Ricci curvature $\text{Ric}_M \geq 0$. She proved that a complete Riemannian manifold with small linear diameter growth and nonnegative Ricci curvature has a finitely generated fundamental group, and she also proved a more general theorem which states that if a complete Riemannian manifold has no tangent cone at infinity which is not polar, its fundamental group is also finitely generated. It is a fairly strong result because currently it is not known whether a manifold with nonnegative Ricci curvature can have a tangent cone at infinity which is not polar. So it support Milnor conjecture strongly.

The proves of Sormani’s theorems are based on two lemmas, Halfway Lemma and Uniform Cut Lemma, and her constant $\tilde{S}_n$ comes from the Uniform Cut Lemma. In her Uniform Cut Lemma, she proved that for some special cut points $\gamma(D/2)$ which are the halfway points of some particular geodesic loops $\gamma$, and for $x \in \partial B_{\epsilon_0}(RD)$ where $R > 1/2$, the inequality

$$d_M(x, \gamma(D/2)) > (R - 1/2)D$$

can be improved to stronger one, that is

$$d_M(x, \gamma(D/2)) \geq (R - 1/2)D + 2\tilde{S}_n D,$$

where $\tilde{S}_n = \frac{n - 1}{n - 2} \left(\frac{n - 2}{n - 1}\right)^{n-1}$.

In this paper, we will show that the universal constant above can be improved greatly. In fact, we show that we can get a better uniform constant for any point whose distance to the halfway point is $\varepsilon$, here $\varepsilon$ is any small positive constant. As a corollary, we improves the constant in Sormani’s theorem to $S_n = \frac{1}{n-1} \left(\frac{n-1}{n} \right)^{n-1}$. Obviously, our constant is more bigger than Sormani’s constant, especially when $n$ is large.
We also get a $k$th-Ricci version of the estimate and give a similar theorem. Furthermore, we show that nonnegative Ricci curvature is not a necessary condition for such an estimate. For example, we have a similar estimate for complete Riemannian manifold with Ricci curvature bounded below by a negative constant, which improve Sormani’s correspondent lemma in [10].

\section{Basic Estimates}

In this section we will establish some uniform estimates for the distance to halfway points by terms of the distance to end points on some types of Riemannian manifolds.

First let us recall the concept of the excess function. The excess function $e_{pq}(x)$ is defined by

$$e_{pq}(x) = d(x,p) + d(x,q) - d(p,q), \quad \forall x \in M,$$

where $p,q \in M$ and $d$ denotes the distance function on $M$ induced from the metric. In their pioneering work [1], Abresch and Gromoll give an important excess estimate for “thin” triangular, which was used by them and many others to prove many topological finite results on many types of Riemannian manifolds. To be precise, they proved

\begin{lemma} \cite{1,9,10} \label{lem1}

Let $M$ be a complete Riemannian $n$-manifold with Ricci curvature $\text{Ric}_M \geq -(n-1)k$, $n \geq 3$ and $k \geq 0$. Denote $r_0 = d(x, \gamma(0))$ and $r_1 = d(x, \gamma(D))$ where $\gamma$ is a minimal geodesic with length $L(\gamma) = D$. Suppose $l = d(x, \gamma) \leq \min\{r_0, r_1\}$, then

$$e_{\gamma(0), \gamma(D)}(x) = r_0 + r_1 - D \leq 2 \left( \frac{n-1}{n-2} \left( \frac{1}{2} C_3 \right)^{n-1} \right)^{1/(n-1)},$$

where

$$C_3 = \left\{ \begin{array}{ll}
\frac{n-1}{n} \left( \frac{n-1}{n-1} + \frac{1}{n-1} \right), & k = 0, \\
\frac{n-1}{n} \left( \frac{n-1}{n-1} \frac{\sinh \sqrt{k}}{\sqrt{k}} \right)^{n-1} \sqrt{k} \left[ \coth \sqrt{k} (r_0 - l) + \coth \sqrt{k} (r_1 - l) \right], & k > 0.
\end{array} \right.$$ 

\end{lemma}

Now we can give our main estimate.

\begin{lemma} \label{lem2}

Let $M^n$ be a complete Riemannian $n$-manifold with Ricci curvature $\text{Ric}_M \geq -(n-1)k$, $n \geq 3$. Let $\gamma$ be a minimal geodesic from $\gamma(0)$ to $\gamma(D)$ with length $L(\gamma) = D$. If $x \in M$ is a point with $d(x, \gamma(0)) \geq \left( \frac{1}{2} + \varepsilon_1 \right) D$ and $d(x, \gamma(D)) \geq \left( \frac{1}{2} + \varepsilon_2 \right) D$, then

$$d(x, \gamma(D/2)) \geq \alpha(\varepsilon_1, \varepsilon_2) D,$$

where

$$\alpha(\varepsilon_1, \varepsilon_2) = \left\{ \begin{array}{ll}
\min \left\{ \frac{1}{2}, \left( \frac{1 + \varepsilon_2}{2} \right)^{\frac{n-1}{n}} \left( \frac{n}{n-1} \left( \frac{n-2}{n-1} \right)^{n-1} \right)^{\frac{1}{2}} \right\}, & k = 0, \\
\min \left\{ \frac{1}{2}, \left( \frac{1 + \varepsilon_2}{2} \right)^{\frac{n-1}{n}} \left( \frac{n-2}{n-1} \frac{n-1}{2} \right)^{\frac{n-1}{2}} \left( \frac{n}{n-1} \coth \sqrt{k} / \sqrt{k} \right)^{n-1} \right\}, & k > 0, D \leq 1.
\end{array} \right.$$ 

\end{lemma}
Proof: Suppose on the contrary, \( d(x, \gamma(D/2)) \leq \alpha(\varepsilon_1, \varepsilon_2)D \).
First, by definition, we can get

\[
e_{\gamma(0), \gamma(D)}(x) = d(x, \gamma(0)) + d(x, \gamma(D)) - D \geq (\varepsilon_1 + \varepsilon_2)D.
\]

On the other hand, we have \( l = d(x, \gamma) \leq d(x, \gamma(D/2)) \leq \alpha(\varepsilon_1, \varepsilon_2)D \leq \min\{r_0, r_1\} \).
Thus by lemma 1, we have

\[
e_{\gamma(0), \gamma(D)}(x) \leq 2\left(\frac{n-1}{n-2}\right)\left(\frac{1}{2}C_3 m^{1/n-1}\right).
\]

Note that \( r_0 - l \geq D/4 \).
If \( \text{Ric}_M \geq 0 \), together with \( \alpha(\varepsilon_1, \varepsilon_2) \leq \frac{1}{2} \) and lemma 1 we have

\[
C_3 \leq \frac{n-1}{n} \left(\frac{1}{(\frac{1}{2} + \varepsilon_1 - \alpha(\varepsilon_1, \varepsilon_2))D} + \frac{1}{(\frac{1}{2} + \varepsilon_1 - \alpha(\varepsilon_1, \varepsilon_2))D}\right) \leq \frac{n-1}{n} \frac{8}{D}.
\]

Thus by lemma 1,

\[
(\varepsilon_1 + \varepsilon_2)D < 2\left(\frac{n-1}{n-2}\right)\left(\frac{1}{2} \frac{n-1}{D} \frac{8}{n} \alpha(\varepsilon_1, \varepsilon_2)^n D^n \right)^{\frac{1}{n-1}} \leq 2\left(\frac{n-1}{n-2}\right)\left(\frac{1}{2} \frac{n-1}{D} \frac{8}{n} \alpha(\varepsilon_1, \varepsilon_2)^n D^n \right)^{\frac{1}{n-1}}.
\]

So we get

\[
\alpha(\varepsilon_1, \varepsilon_2) > \left(\frac{\varepsilon_1 + \varepsilon_2 n - 2}{2} \frac{n}{n-1} \frac{8}{n} \alpha(\varepsilon_1, \varepsilon_2)^n D^n \right)^{\frac{1}{n-1}} \leq \left(\frac{\varepsilon_1 + \varepsilon_2 n - 2}{2} \frac{n}{n-1} \frac{8}{n} \alpha(\varepsilon_1, \varepsilon_2)^n D^n \right)^{\frac{1}{n-1}}.
\]

which is contradicted to the definition of \( \alpha(\varepsilon_1, \varepsilon_2) \).
Now suppose \( \text{Ric}_M \geq -(n-1)k \), note that \( \frac{\sinh l}{ l} \) and \( \coth (r - l) \) are both increasing functions of \( l \) and that \( D < 1 \), from lemma 1 we know

\[
C_3 = \frac{n-1}{n} \left(\frac{\sinh \sqrt{k}}{\sqrt{k}}\right)^{n-1} \sqrt{k} \left[\coth \sqrt{k}(r_0 - l) + \coth \sqrt{k}(r_1 - l)\right] \leq \frac{n-1}{n} \left(\frac{\sinh \sqrt{k}}{\sqrt{k}}\right)^{n-1} \sqrt{k} \left(2 \coth \frac{\sqrt{k}}{4}\right).
\]

Thus

\[
(\varepsilon_1 + \varepsilon_2)D < 2\left(\frac{n-1}{n-2}\right)\left(\frac{1}{2} \frac{n-1}{n} \left(\frac{\sinh \sqrt{k}}{\sqrt{k}}\right)^{n-1} \sqrt{k} \left(2 \coth \frac{\sqrt{k}}{4}\right) \alpha(\varepsilon_1, \varepsilon_2)^n D^n \right)^{\frac{1}{n-1}} \leq 2\left(\frac{n-1}{n-2}\right)\left(\frac{1}{2} \frac{n-1}{n} \left(\frac{\sinh \sqrt{k}}{\sqrt{k}}\right)^{n-1} \sqrt{k} \left(2 \coth \frac{\sqrt{k}}{4}\right) \alpha(\varepsilon_1, \varepsilon_2)^n D^n \right)^{\frac{1}{n-1}}.
\]

4
\( \alpha(\varepsilon_1, \varepsilon_2) > \left( \frac{\varepsilon_1 + \varepsilon_2}{2} \right)^{\frac{n-1}{n}} \left( \frac{n-2}{n-1} \right)^{\frac{n-1}{n}} \left( \frac{\sqrt{k}}{n-1} \coth \sqrt{k/4} \frac{\sqrt{k/4}}{\sinh \sqrt{k/4}} \right)^{n-1} \frac{1}{n}, \)

which is a contradiction.

We can see that the excess estimate is crucial to our lemma. Note that Chen [8] gives a kth-Ricci version excess estimate, thus by the same way, we have a similar estimate for complete Riemannian n-manifold with k-th Ricci curvature \( \text{Ric}^k_M \geq 0. \)

**Lemma 3** Let \( M^n \) be a complete Riemannian n-manifold with k-th Ricci curvature \( \text{Ric}^k_M \geq 0. \) Let \( \gamma \) be a minimal geodesic from \( \gamma(0) \) to \( \gamma(D) \) with length \( L(\gamma) = D. \) If \( x \in M \) is a point with \( d(x, \gamma(0)) \geq (\frac{1}{2} + \varepsilon)D \) and \( d(x, \gamma(D)) \geq (\frac{1}{2} + \varepsilon)D, \) then

\[
d(x, \gamma(D/2)) \geq \alpha_k(\varepsilon)D,
\]

where

\[
\alpha_k(\varepsilon) = \frac{1}{4} \varepsilon^{\frac{2}{k+1}}.
\]

**Proof:** Suppose on the contrary, \( d(x, \gamma(D/2)) < \alpha_k(\varepsilon)D. \)

Obviously we have

\[
e_{\gamma(0), \gamma(D)}(x) \geq 2\varepsilon D.
\]

On the other hand, \( s(x) = \min\{d(x, \gamma(0)), d(x, \gamma(D))\} \geq (\frac{1}{2} + \varepsilon)D \) and \( h(x) = d(x, \gamma) < \alpha_k(\varepsilon)D. \) Thus by the excess estimate for Riemannian manifold with \( \text{Ric}^k_M \geq 0 \) (cf [8]) we can get

\[
2\varepsilon D \leq e_{\gamma(0), \gamma(D)}(x) < 8\alpha_k(\varepsilon)D(\frac{\alpha_k(\varepsilon)D}{(\frac{1}{2} + \varepsilon)D})^{1/k},
\]

and

\[
\alpha_k(\varepsilon) > (\frac{1}{2} + \varepsilon)^{\frac{2}{k+1}} \varepsilon^{\frac{2}{k+1}} 4^{1/k} > \frac{1}{4} \varepsilon^{\frac{2}{k+1}},
\]

which is a contradiction.

**§3. Fundamental group and nonnegative Ricci curvature**

Now we can give a constant better than Sormani’s constant in [9], and thus support Milnor conjecture more strongly. First we need a lemma.

**Lemma 4** ([9],[5]) Let \( M \) be a complete Riemannian n-manifold with fundamental group \( \pi_1(M, x_0), \) where \( x_0 \in M. \) Then there exists an ordered set of independent generators \( \{g_1, g_2, g_3, \cdots \} \) of \( \pi_1(M, x_0) \) with minimal representative geodesic loops \( \gamma_k \) of length \( d_k \) such that

\[
d_M(\gamma_k(0), \gamma_k(d_k/2)) = d_k/2.
\]
Now we give a uniform cut lemma which is superior than Sormani’s corepsondent one.

**Lemma 5** Let $M^n$ be a complete manifold with nonnegative Ricci curvature, $n \geq 3$ and $\gamma$ be a be a non-contractible geodesic loop based at a point $x_0 \in M^n$ with length $L(\gamma) = D$, such that the following conditions hold:
1) If $\sigma$ based at $x_0$ is a loop homotopic to $\gamma$, then $L(\sigma) \geq D$,
2) The loop $\gamma$ is minimal on $[0, D/2]$ and is also minimal on $[D/2, D]$.

Then for any $\varepsilon > 0$, there is a universal constant $\alpha(\varepsilon) = \alpha(\varepsilon, \varepsilon)$, such that if $x \in \partial B_{x_0}(RD)$, where $R \geq (1/2 + \varepsilon)$, then

$$d_M(x, \gamma(D/2)) \geq (R - \frac{1}{2})D + (\alpha(\varepsilon) - \varepsilon)D.$$ 

Similarly, if $M$ has nonnegative $k$th-Ricci curvature $\text{Ric}_M^k \geq 0$, then

$$d_M(x, \gamma(D/2)) \geq (R - \frac{1}{2})D + (\alpha_k(\varepsilon) - \varepsilon)D.$$ 

**Proof:** Note that by lemma 4, such loop $\gamma$ exist.

First we suppose that $R = (1/2 + \varepsilon)$.

Let $\tilde{M}$ be the universal cover of $M$, $\tilde{x}_0 \in \tilde{M}$ be a lift of $x_0$, and $\tilde{g} \in \pi_1(M, x_0)$ be the element represented by the given loop $\gamma$. By conditions of $\gamma$, its lift $\tilde{\gamma}$ is a minimal geodesic running from $\tilde{x}_0$ to $g\tilde{x}_0$. Thus $d_{\tilde{M}}(\tilde{x}_0, g\tilde{x}_0) = D$. Obviously we have

$$r_0 = d_{\tilde{M}}(\tilde{x}, \tilde{x}_0) \geq d_M(x, x_0) = (\varepsilon + 1/2)D,$$

$$r_1 = d_{\tilde{M}}(g\tilde{x}_0, \tilde{x}) \geq d_M(x, x_0) = (\varepsilon + 1/2)D.$$ 

Now we lift the minimal geodesic $C : [0, H] \to M^n$ from $\gamma(D/2)$ to $x$ to a curve $\tilde{C}$ in the universal cover, which runs from $\tilde{\gamma}(D/2)$ to a point $\tilde{x} \in \tilde{M}$. Note that $L(C) = L(\tilde{C}) = H$. Thus by applying lemma 2 to $\tilde{x}$ and $\tilde{\gamma}$ in $\tilde{M}$, we have

$$d(x, \gamma(D/2)) = L(\tilde{C}) = L(C) = d(\tilde{x}, \tilde{\gamma}(D/2)) \geq \alpha(\varepsilon)D.$$ 

For $R \geq 1/2 + \varepsilon$, take $x \in \partial B_{x_0}(RD)$ and suppose $y \in \partial B_{x_0}((1/2 + \varepsilon)D)$ lies in the minimal geodesic from $x$ to $\gamma(D/2)$. Now we have

$$d_M(x, \gamma(D/2)) = d_M(x, y) + d_M(y, \gamma(D/2))$$

$$\geq (RD - \frac{1}{2} + \varepsilon)D + \alpha(\varepsilon)D$$

$$= (R - \frac{1}{2})D + (\alpha(\varepsilon) - \varepsilon)D$$

The last assert can be proved the same way by using lemma 3 and we omit the proof here.

$\blacksquare$
Recall that the ray density function \( D(r) \) is defined as
\[
D(r) = \sup_{x \in \partial B(x_0(r))} \inf_{\gamma, \gamma(0) = x_0} d(x, \gamma(r)).
\]

Now we can give our main theorems which improved Sormani’s theorems and support Milnor conjecture strongly.

**Theorem 1** There exists a universal constant,
\[
S_n = \frac{1}{4} \frac{1}{n-1} \left( \frac{n-2}{n} \right)^{n-1},
\]
such that if \( M^n \) is complete and noncompact with nonnegative Ricci curvature and has small ray density growth,
\[
\lim_{r \to \infty} \frac{D(r)}{r} < 2S_n,
\]
then its fundamental group is finitely generated.

**Proof:** First let \( f(\varepsilon) = \alpha(\varepsilon) - \varepsilon \), then by elementary calculus we can know that the maximum point of \( f \) is \( \varepsilon_0 = \left( \frac{n-2}{n} \right)^{n-1} \) and the maximum value of \( f \) is \( f(\varepsilon) = S_n = \frac{1}{4} \frac{1}{n-1} \left( \frac{n-2}{n} \right)^{n-1} \).

Now we assume that \( M^n \) has infinitely generated fundamental group, thus by lemma 4, there is a sequence of generators \( g_k \), whose minimal representative geodesic loops \( \gamma_k \) based at \( x_0 \) satisfied the hypothesis of lemma 5. Let \( d_k = L(\gamma_k) \). Note that \( d_k \) diverges to infinity.

Now for any \( \gamma_k \) we choose a ray \( \gamma \) such that
\[
d(\gamma_k(d_k/2), \gamma(d_k/2)) = \inf_{\text{rays, } \gamma(0) = x_0} d(\gamma_k(d_k/2), \gamma(d_k/2)).
\]
Take \( x_k = \gamma((1/2 + \varepsilon_0)d_k) \) and \( y_k = \gamma((1/2)d_k) \), then we have
\[
d(y_k, \gamma_k(d_k/2)) \geq d(x_k, \gamma_k(d_k/2)) - d(x_k, y_k) \geq (\alpha(\varepsilon_0) - \varepsilon_0) d_k = S_n d_k.
\]
So
\[
\lim_{r \to \infty} \frac{D(r)}{r} \geq \lim_{k \to \infty} \frac{d(y_k, \gamma_k(d_k/2))}{d_k/2} \geq \lim_{k \to \infty} \frac{S_n d_k}{d_k/2} = 2S_n,
\]
which is a contradiction. \( \square \)

Similarly, for \( \text{Ric}^k_M \geq 0 \) we can get

**Theorem 2** There exists a universal constant,
\[
S_k = \frac{1}{4(k+1)} \left( \frac{k}{4(k+1)} \right)^k,
\]
such that if \( M^n \) is complete and noncompact manifold with nonnegative \( k \)-th Ricci curvature \( \text{Ric}^k_M \geq 0 \) and has small ray density growth,
\[
\lim_{r \to \infty} \frac{D(r)}{r} < 2S_k,
\]
then its fundamental group is finitely generated.
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