Open Manifolds with Nonnegative Ricci Curvature and Large Volume Growth

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Abstract: In this paper, we prove that if $M$ is an open manifold with nonnegative Ricci curvature and large volume growth, positive critical radius, then $\sup_{p \in M} C_p = \infty$. As an application, we give a theorem which supports strongly Petersen's conjecture.

Key words: Open manifold, Nonnegative Ricci curvature, Critical radius, Volume Growth

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§1 Introduction

Let $M$ be an $n$-dimensional complete and noncompact Riemannian manifold with $Ric_M \geq 0$. By Bishop volume comparison theorem, $\frac{Vol[B(p, r)]}{\omega_n r^n}$ is a non-increasing function of $r$, where $B(p, r)$ denotes the ball with radius $r$ around $p$ in $M$, $Vol[B(p, r)]$ denotes the volume of $B(p, r)$, $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$. Let

$$\alpha_M = \lim_{r \to \infty} \frac{Vol[B(p, r)]}{\omega_n r^n}.$$ 

Obviously, $0 \leq \alpha_M \leq 1$. If $\alpha_M = 1$, by volume comparison theorem, $M$ is isometric to $\mathbb{R}^n$. We say $M$ has large volume growth if $\alpha_M > 0$.

Notice that the distance function $r_p(x) = d(p, x)$ is not a smooth function (on the cut locus of $p$). Hence the critical point of $r_p$ are not defined in a usual sense. The notion of critical points of $r_p$ is introduced by Grove-Shiohama[1].

A point $q \in M$ is called a critical point of $r_p$ if for any unit vector $v \in T_q M$, there is a minimizing geodesic $\sigma$ from $q$ to $p$ such that $\angle(\sigma'(0), v) \leq \frac{\pi}{2}$.

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For any fixed point $p$ in $M$, let
\[ C_p = \sup \{ r > 0 \mid \text{there is no critical point of } p \in B(p, r) \}, \]
and define the critical radius of $M$ to be $c(M) = \inf_{p \in M} C_p$. It is well known that if there is some point $p \in M$ such that $C_p = \infty$, then $M$ is diffeomorphic to $\mathbb{R}^n$. In this paper we shall prove the following theorem.

**Theorem 1** Let $M$ be a complete and noncompact Riemannian manifold, $\text{Ric}_M \geq 0$, $\alpha_M > 0$, $\text{conj}_M \geq i_0 > 0$, $c(M) > 0$, then $\sup_{p \in M} C_p = \infty$.

**Theorem 2** Let $M$ be a complete and noncompact Riemannian manifold, $\text{Ric}_M \geq 0$, $\alpha_M > 0$, $K_M \geq -k^2$, $c(M) > 0$, then $\sup_{p \in M} C_p = \infty$.

Let $M$ be a complete open Riemannian manifold. The generalized Busemann function $b_p : M \to \mathbb{R}$ with respect to a point $p$ is defined by
\[ b_p(x) = \lim_{q \to \infty} \sup_{q \in M} (d(p, q) - d(q, x)), \forall x \in M, \]
where $q \to \infty$ means the distance between $q \in M$ and a fixed point in $M$ tending to infinity. It is easy to check that
\[ b_p(x) = \lim_{t \to \infty} (t - d(x, S_t(p))), \]
where $S_t(p)$ is the metric sphere of radius $t \geq 0$ centered at $p$, and the function $t \to t - d(x, S_t(p))$ is non-increasing when $t \geq d(p, x)$. The excess of $M$ at $p \in M$ is defined to be
\[ e(p) := \sup_{x \in M} (d(p, x) - b_p(x)). \]
Define
\[ e(M) := \sup_{p \in M} e(p), \]
and
\[ \varepsilon_p(x) = d(x, S_{2r_p(x)}(p)) - r_p(x). \]
Obviously, we have
\[ \varepsilon_p(x) \leq e(p) \leq e(M), \forall p, x \in M. \]

Let $M$ be a complete noncompact Riemannian manifold. Fix a point $p \in M$. For any $r > 0$, let $k_p(r) = \inf_{p \in M \setminus B(p, r)} K$, where $K$ denotes the sectional curvature of $M$, and the infimum is taken over all the sections at all points in $M \setminus B(p, r)$.

Petersen[2] conjectured that if $\text{Ric}_M \geq 0$, $\alpha_M > \frac{1}{2}$, then $M$ is diffeomorphic to $\mathbb{R}^n$. In this paper, we shall prove the following:

**Theorem 3** Let $M$ be a complete and noncompact Riemannian manifold, $\text{Ric}_M \geq 0$, $\alpha_M > \frac{1}{2}$, $k_p(r) \geq -\frac{C}{(1+r)^\alpha}$ for some constant $C > 0$ and any point $p \in M$, $0 < \alpha \leq 2$, $e(M) < \infty$. Then $M$ is diffeomorphic to $\mathbb{R}^n$. 
§2 Proof of Theorem 1 and Theorem 2

First, we introduce some lemmas which will be needed in our proof of the theorems.

**Lemma 1** ([3]) Let \( M \) be a complete, noncompact manifold, \( \text{Ric}_M \geq 0, \alpha_M > 0 \). Then there is a sequence \( \{x_i\} \subset M \), such that \((M, x_i)\) converges to \((\mathbb{R}^n, 0)\) in the pointed Gromov-Hausdorff topology.

**Lemma 2** ([4]) Let \( M \) be a complete Riemannian manifold, \( K \geq c \).

(i) Let \( \gamma_i : [0, t_i] \to M, i = 0, 1, 2 \) be minimal geodesics with \( \gamma_1(0) = \gamma_2(t_2) = p, \gamma_0(0) = \gamma_1(t_1) \) and \( \gamma_0(l_0) = \gamma_2(0) \). Then, there exist minimal geodesics \( \tilde{\gamma}_i : [0, l_i] \to M^2(c) \), where \( M^2(c) \) is the surface with constant curvature, \( i = 0, 1, 2 \) with \( \tilde{\gamma}_1(0) = \tilde{\gamma}_2(l_2), \gamma_0(0) = \tilde{\gamma}_1(t_1) \) and \( \tilde{\gamma}_0(l_0) = \tilde{\gamma}_2(0) \) such that

\[
L(\gamma_i) = L(\tilde{\gamma}_i) \quad \text{for } i = 0, 1, 2
\]

and

\[
\angle(-\gamma_1'(t_1), \gamma_0'(0)) \geq \angle(-\tilde{\gamma}_1'(t_1), \tilde{\gamma}_0'(0)),
\]

\[
\angle(-\gamma_1'(t_0), \gamma_2'(0)) \geq \angle(-\gamma_0'(l_0), \gamma_2'(0)).
\]

(ii) Let \( \gamma_i : [0, t_i] \to M, i = 1, 2 \) be two minimizing geodesics starting from \( p \). Let \( \tilde{\gamma}_i : [0, l_i] \to M^2(c), i = 1, 2 \) be minimizing geodesics starting from some point such that

\[
\angle(\gamma_1'(0), \gamma_2'(0)) = \angle(\tilde{\gamma}_1'(0), \tilde{\gamma}_2'(0)). \quad \text{Then } d(\gamma_1(t_1), \gamma_2(t_2)) \leq d_c(\gamma_1(t_1), \tilde{\gamma}_2(t_2)), \quad \text{where } d_c \text{ denotes the distance function in } M^2(c).
\]

Let \( p, q \in M \). The excess function \( e_{pq}(x) \) is defined by \( e_{pq}(x) = d(p, x) + d(q, x) - d(p, q) \).

**Lemma 3** ([5]) Let \( (M, g) \) be complete with \( \text{Ric}_M \geq -(n - 1) \lambda \) and \( \text{conj}_M \geq c_0 > 0 \). There is a constant \( C_0 = C(n, c_0) > 0 \) such that if \( \sigma_i : [0, r_i] \to M \) are minimizing geodesics from \( p \) with \( \rho = \max(r_1, r_2) \leq \frac{1}{c_0} \), then

\[
d(\sigma_1(r_1), \sigma_2(r_2)) \leq e^{C_0 \rho^{\frac{1}{2}}} |r_1 r_2|,
\]

where \( r_i = \frac{\partial}{\partial t}(0), i = 1, 2 \).

Let \( (M, g) \) be as in Lemma 3. Let \( \rho \leq \frac{1}{c_0} \), and \( \sigma_1, \sigma_2 : [0, \rho] \to M \) be minimizing geodesics from \( x \) to \( p^*, q^* \). Put \( \theta := \angle(\sigma_1'(0), \sigma_2'(0)) \). By Lemma 3, we have

\[
d(p^*, q^*) \leq 2 \rho e^{C_0 \rho^{\frac{1}{2}}} [1 - \sin^2(\frac{\theta - \theta}{2})]^{\frac{1}{2}}.
\]

thus

\[
\sin^2(\frac{\pi - \theta}{2}) \leq 2 [e^{C_0 \rho^{\frac{1}{2}}} - \frac{d(p^*, q^*)}{2 \rho}] = 2 [e^{C_0 \rho^{\frac{1}{2}}} - 1 + \frac{e_{p^* q^*}(x)}{2 \rho}].
\]

Let \( \varepsilon = \frac{1}{2} \sin \frac{\pi}{2} \). Take \( \rho = \rho(n, c_0) \leq \frac{1}{4} c_0 \) such that \( e^{C_0 \rho^{\frac{1}{2}}} \leq 1 + \varepsilon^2 \). Suppose that \( e_{p^* q^*}(x) \leq 2 \varepsilon^2 \rho \). Then we have

\[
\sin^2(\frac{\pi - \theta}{2}) \leq (2 \varepsilon)^2 = \sin^2 \frac{\pi}{8}.
\]

This implies that \( \theta \geq \frac{7}{4} \pi \).

Let \( X, Y \) be the metric spaces. A map \( f : X \to Y \) is by definition a Hausdorff \( \delta \)-approximation if

\[
|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \delta, \forall x_1, x_2 \in X,
\]

\[
B(f(X), \delta) \supset Y,
\]
where \( B(f(X), \delta) \) denotes the \( \varepsilon \)-neighbourhood of the subset \( f(X) \) in \( Y \). The Hausdorff distance \( d_H(X, Y) \) between \( X \) and \( Y \) is defined by the infimum of \( \delta \) such that there exist Hausdorff \( \delta \)-approximation maps \( f : X \to Y, g : Y \to X \).

For noncompact metric spaces we say that a pointed sequence \( (X_i, x_i) \) converges to \( (X, x) \) in the pointed Gromov-Hausdorff topology if, for all \( \gamma > 0 \), the sequence \( X_i \cap B_\gamma(x_i) \) converges to \( X \cap B_\gamma(x) \) in the Gromov-Hausdorff topology.

**Proof of Theorem 1.** By Lemma 1, there is a sequence \( \{x_i\} \subset M \), \( (M, x_i) \) converges to \( (\mathbb{R}^n, 0) \) in the pointed Gromov-Hausdorff topology. Which means that for all \( R > 0 \), the sequence \( B(x_i, R) \) converges to \( V(0, R) \) in the Gromov-Hausdorff topology, where \( V(0, R) \) denotes the ball around 0 with radius \( R \) in \( \mathbb{R}^n \). Thus by definition of Gromov-Hausdorff distance, we have \( d_{G_H}(B(x_i, R), V(0, R)) < \varepsilon_i \to 0 \). Then there exists map \( f : B(x_i, R) \to V(0, R) \) such that

1. \( f(x_i) = 0 \).
2. \( f(B(x_i, R)) \) is \( \varepsilon_i \)-dense in \( V(0, R) \).
3. \( \forall x, y \in B(x_i, R), |d_M(x, y) - d_{G_H}(f(x), f(y))| < \varepsilon_i \).

For any fixed point \( p \) in \( B(x_i, R) \), if \( d(x_i, p) < c(M) \), then \( p \) is not a critical point of \( x_i \). If \( d(x_i, p) \geq c(M) \). Let \( p' = f(p) \in V(0, R) \). \( \gamma \) be the ray connecting 0 and \( p' \), \( q' = \gamma \cap S(0, R) \), where \( S(0, R) \) denotes the sphere with radius \( R \) around 0. By (2), there exists \( q'' \in V(0, R), d_{G_H}(q', q'') < \varepsilon \), and \( q'' = f(q) \) for some \( q \in B(x_i, R) \). By (3), it is easy to see that

\[
d_M(x_i, p) + d_M(p, q) - d_M(x_i, q) < 5\varepsilon_i.
\]

Thus let \( \sigma_1, \sigma_2 \) be minimizing geodesics from \( p \) to \( x_i, q \). \( \theta = \angle(\sigma_1'(0), \sigma_2'(0)) \), \( \rho \leq \frac{\pi}{4} \), \( p^* = \sigma_1(p), q^* = \sigma_2(p) \).

By the triangle inequality, \( e_{p^*q^*}(\rho) \leq \varepsilon_i \), thus if \( 5\varepsilon_i \leq 2e^2p \), then \( \theta \geq \frac{3\pi}{4} \), and \( p \) is not a critical point of \( x_i \). Thus \( c_{x_i} \geq R \). Since we can let \( R \) arbitrarily large, we have sup \( p \in M \) \( C_p = \infty \). Q.E.D.

**Corollary 1.** Let \( M \) be a complete and noncompact manifold, \( Ric_M \geq 0, \alpha_M > 0, \) \( inj_M \geq i_0 > 0 \). Then sup \( p \in M \) \( C_p = \infty \).

**Proof.** Since \( conj_M \geq i_0 > 0 \) and \( c(M) \geq i_0 > 0 \) can both be deduced from \( inj_M \geq i_0 > 0 \), by Theorem 1, Corollary 1 is obvious. Q.E.D.

**Proof of Theorem 2.** From the proof of theorem 1, we know that for any \( R > 0 \), \( p \in B(x_i, R) \), if \( d(x_i, p) \leq c(M) \), then \( p \) is not a critical point of \( x_i \). If \( d(x_i, p) \geq c(M) \), there exists \( q \in B(x_i, R) \), such that

\[
d_M(x_i, p) + d_M(p, q) - d_M(x_i, q) < 5\varepsilon_i.
\]

Let \( \sigma_1, \sigma_2, \sigma_3 \) be minimizing geodesics from \( p \) to \( x_i, p \) to \( q \) and \( q \) to \( x_i \). \( r_i = L(\sigma_i) \) be the length of \( \sigma_i, i = 1, 2, 3 \), \( \beta = \angle(\sigma_1'(0), \sigma_2'(0)) \). By lemma 2, we have

\[
\cos \beta \leq \frac{\cosh \frac{r_2}{2} \cosh \frac{r_3}{2} - \cosh \frac{r_1}{2}}{\sinh \frac{r_2}{2} \sinh \frac{r_3}{2} + \frac{1}{2} \cosh \frac{r_2}{2} \cosh \frac{r_3}{2} - \cosh \frac{r_1}{2}}.
\]
If $\gamma_3 = \gamma_1 + \gamma_2$, by the inequality above, $\cos \beta < 0$.
Since $0 \leq r_1 + r_2 - r_3 \leq 5\varepsilon_i$, $r_1 \geq c(M) > 0$, thus if $\varepsilon_i$ is sufficiently small, $\cos \beta < 0$, $\beta > \frac{\pi}{2}$, and $p$ is not a critical point of $x_i$. So $C_{x_i} \geq R$. Since $R$ can be arbitrarily large, $\sup_{p \in M} C_p = \infty$. Q.E.D.

**Lemma 4** ([6]) Let $M$ be a complete and noncompact Riemannian manifold, $\text{Ric}_M \geq 0$, $K_M \geq -k^2$, $\alpha_M > \frac{1}{2}$. Then $e(M) \geq \iota_0 > 0$.

**Corollary 2** Let $M$ be a complete and noncompact Riemannian manifold, $\text{Ric}_M \geq 0$, $\alpha_M > \frac{1}{2}$, $K_M \geq -k^2$. Then $\sup_{p \in M} C_p = \infty$.

**Proof.** Combining theorem 2 and lemma 4, corollary is obvious. Q.E.D.

§3 Proof of the theorem 3

**Proof.** From the proof of lemma 2.1 in [7], we know that there exists a constant $\delta > 0$, if $x, p \in M$, $r_p(x) = d(p, x)$ satisfies $\frac{r_p(x)}{r_p(p)} < \delta$, then $x$ is not critical point of $p$.

Now for $e(M) = \sup_{p \in M} e(p) < \infty$, $\varepsilon_p(x) \leq e(p) \leq e(M) < \infty$, for any $p, x \in M$, thus if $r_p(x) > \left(\frac{e(M)}{\delta}\right)^{\frac{2}{k}}$, then $x$ is not a critical point of $p$. By Corollary 2, $\sup_{p \in M} C_p = \infty$. There exists $p \in M$, $C_p > \left(\frac{e(M)}{\delta}\right)^{\frac{2}{k}}$, thus $p$ has not critical point other than $p$ and $M$ is diffeomorphic to $\mathbb{R}^n$. Q.E.D.

参考文献