Justify your answers to all problems.

Notation: \( \mathbb{R} \) is the real line, \( \mathbb{C} \) is the complex plane and \( D(P, r) \subset \mathbb{C} \) is the disk of radius \( r \) centered at point \( P \).

1. Suppose \( \{f_n\}_{n=1}^{\infty} \subset L^2(\mathbb{R}) \) is a sequence that converges to 0 in the \( L^2 \) norm; in other words, 
   \[
   ||f_n||_{L^2(\mathbb{R})} = \left( \int_{-\infty}^{\infty} |f_n|^2 \, dx \right)^{1/2} \to 0.
   \]
   Prove that there exists a subsequence \( \{f_{n_k}\} \) such that \( f_{n_k} \to 0 \) almost everywhere.

2. Determine whether the following statements are true and false. If true, provide a proof. If false, prove a counter example.

   (a) If \( f(x) \) is a increasing, continuous function on the interval \([0, 1]\) such that \( f(0) = 0 \) and \( f(1) = 1 \), then there exists a set \( E \subset [0, 1] \) of positive measure such that \( f'(x) > 0 \).

   (b) If \( f(x) \) is a strictly increasing, absolutely continuous function on the interval \([0, 1]\) with \( f(0) = 0 \) and \( f(1) = 1 \), then the set \( f^{-1}(E) \cap \{x \in [0, 1] : f'(x) > 0\} \) is measurable for any measurable set \( E \subset [0, 1] \).

3. Let \( \{\varphi_k\}_{k=1}^{\infty} \) be an orthonormal basis for \( L^2(\mathbb{R}^d) \) and define \( \varphi_{k,j}(x, y) = \varphi_k(x)\varphi_j(y) \). Prove that \( \{\varphi_{k,j}\}_{k,j=1}^{\infty} \) is an orthonormal basis of \( L^2(\mathbb{R}^d \times \mathbb{R}^d) \).

4. Let \( U \subset \mathbb{C} \) be an open set containing \( D(P, r) \). Prove that if \( f : U \to \mathbb{C} \) is a holomorphic function such that \( f \) is nowhere zero on \( \partial D(P, r) \) and \( g : U \to \mathbb{C} \) is a holomorphic function sufficiently uniformly close to \( f \) on \( \partial D(P, r) \), then the number of zeros of \( f \) in \( D(P, r) \) equals the number of zeros of \( g \) in \( D(P, r) \) (counting multiplicity).

5. If \( f = u + iv \) is an entire function with the property that \( u(z) \leq 0 \) for all \( z \in \mathbb{C} \), what can you say about \( f \)?

6. If \( D(0, 1) \to \mathbb{C} \) is a function such that \( f^2 \) and \( f^3 \) are both holomorphic, prove \( f \) is holomorphic.

7. Compute the integral
   \[
   \int_0^{\infty} \frac{(\log x)^2}{1 + x^2} \, dx.
   \]