Solutions, Midterm 1, 106  
Fall 2001, Professor Sogge

1) (10 points, 2 parts)
   a) Compute \( \lim_{x \to +\infty} (\sqrt{x^2 + 3} - x) \)

   **Solution:**
   \[
   \lim_{x \to +\infty} (\sqrt{x^2 + 3} - x) = \lim_{x \to +\infty} \frac{\sqrt{x^2 + 3} + x}{\sqrt{x^2 + 3} + x} \cdot \frac{\sqrt{x^2 + 3} - x}{\sqrt{x^2 + 3} - x} = \lim_{x \to +\infty} \frac{3}{\sqrt{x^2 + 3} + x} = 0.
   \]

   b) Assign a value \( k \) so that the following function will be continuous

   \[
   f(x) = \begin{cases} 
   \frac{x+3}{x^2-x-12}, & x \neq -3 \\
   k, & x = -3.
   \end{cases}
   \]

   **Solution:** Since
   \[
   \lim_{x \to -3} \frac{x+3}{x^2-x-12} = \lim_{x \to -3} \frac{1}{x-4} = -\frac{1}{7},
   \]
   if \( k = -1/7 \) the above function will be continuous.

2) (20 points, 2 parts)
   a) Write down the precise definition of \( f' \) (i.e., the one involving limits).

   **Solution:** \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \).

   b) Using the definition of the derivative from part a), compute the equation of the tangent line to \( y = x^2 - 4x \) at the point \( x = 2 \).

   **Solution:** The slope of the tangent line is
   \[
   \lim_{h \to 0} \frac{(2+h)^2 - 4(2+h) - (2^2 - 8)}{h} = \lim_{h \to 0} \frac{4h + h^2 - 4h}{h} = 0.
   \]
   Since \( y = 2^2 - 4 \times 2 = -4 \) when \( x = 2 \), the tangent line must pass through the point \( (2, -4) \). Hence the equation of the tangent line is \( y = -4 \).

3) (20 points, 2 parts) Compute the derivatives of the following functions
   a) \( x^x \)
Solution: We first rewrite $x^x = e^{x \ln x}$. If we then use the chain rule we get that

$$\frac{d}{dx} x^x = e^{x \ln x} \frac{d}{dx} (x \ln x) = x^x (\ln x + 1).$$

b) $\cos(\sin x^2)$
Solution:

$$\frac{d}{dx} \cos(\sin x^2) = -\sin(\sin x^2) \frac{d}{dx} (\sin x^2)$$

$$= -\sin(\sin x^2) \cos(x^2) \frac{d}{dx} x^2$$

$$= -2x \sin(\sin x^2) \cos(x^2).$$

4) (15 points) Consider the curve defined near $P = (0,1)$ by the equation $y \cos x - e^{xy^2} = 0$.
Find the slope of its tangent line through $P$.
Solution: We first compute the slope of the tangent line using implicit differentiation:

$$\cos x \frac{dy}{dx} - y \sin x - e^{xy^2} (y^2 + 2xy \frac{dy}{dx}) = 0.$$

Since $x = 0$ and $y = 1$ this equation simplifies to

$$\frac{dy}{dx} - 1 = 0,$$

and so $\frac{dy}{dx} = 1$. Since the tangent line must have this slope and also pass through $(0, 1)$, its equation is $y - 1 = x$.

5) (20 points) A stone is dropped into a pond, the ripples forming concentric circles which expand. At what rate is the area of one of these circles increasing when the radius is 4 m and increasing at the rate of 0.5 m/s? (Hint: The area of a disk of radius $r$ is $\pi r^2$.)
Solution: The area is $A(r) = \pi r^2$, where $r = r(t)$ is a function of time, and we wish to determine $\frac{dA}{dt}$. We are given that $\frac{dr}{dt} = 1/2$ and $r = 4$ at the given time. If we differentiate the formula for the area we get

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$
Since this is equal to $2\pi \times 4 \times \frac{1}{2} = 4\pi$ at the given time, we conclude that the area is increasing at the rate of $4\pi$ (meters$^2$/sec).

6) (15 points) The volume $V$ of a ball of radius $r$ is given by the $V(r) = \frac{4}{3}\pi r^3$. If you determine the radius within an accuracy of 3 %, how accurate is your calculation of the volume? Show your work.

Solution: The per cent error of the measurement of $V$ is $100\Delta V/V$. By linear approximation this is approximately

$$100 \frac{V'(r) \Delta r}{V} = 100 \frac{4\pi r^2 \Delta r}{(4\pi/3) r^3} = 3\left(100 \frac{\Delta r}{r}\right).$$

Thus, the error for the calculation of the volume is three times greater than that of the radius, and so the accuracy is 9 %.