1. Find the following limits. Show your work!

(a) (10 points) \( \lim_{u \to -3} \sqrt{u^3 + 3u^2 + 1} \)

Observe that \( \lim_{u \to -3} (u^3 + 3u^2 + 1) = (\lim_{u \to -3} u^3 + 3(\lim_{u \to -3} u^2) + 1 \)

\[ = (-3)^3 + 3(-3)^2 + 1 = 1 > 0 \] so then

\[ \lim_{u \to -3} \sqrt{u^3 + 3u^2 + 1} = \sqrt{\lim_{u \to -3} (u^3 + 3u^2 + 1)} = \sqrt{1} = 1 \]

(b) (10 points) \( \lim_{x \to 1} \frac{1 - x^2}{x^2 + x - 2} \)

First, \( \lim_{x \to 1} (1-x^2) = 1 - (1)^2 = 0 \) & \( \lim_{x \to 1} (x^2 + x - 2) = 1^2 + 1 - 2 = 0 \)

so limit is of type \( \frac{0}{0} \) as \( x \to 1 \). However \( \frac{1-x^2}{x^2 + x - 2} = -\frac{(x-1)(x+1)}{(x-1)(x+2)} \)

\[ = -\frac{x+1}{x+2} \] provided \( x \neq 1 \) (which we may assume), hence

\[ \lim_{x \to 1} \frac{1-x^2}{x^2 + x - 2} = -\lim_{x \to 1} \frac{x+1}{x+2} = -\frac{1+1}{1+2} = -\frac{2}{3} \]

(c) (10 points) \( \lim_{x \to +\infty} \frac{\exp(-x) + 3 - x^2}{4x^2 - 7} \)

\[ \lim_{x \to +\infty} \frac{\exp(-x) + 3 - x^2}{4x^2 - 7} = \lim_{x \to +\infty} \frac{\exp(-x) + \frac{3}{x^2} - \frac{1}{x^2}}{4 - \frac{7}{x^2}} \]

(Dividing numerator + denominator by \( x^2 \))

Now, as \( x \to \infty \), \( \exp(-x) \to 0 \) so \( \exp(-x)/x^2 \to 0 \) as well.

Also, \( \frac{3}{x^2} \) & \( \frac{7}{x^2} \to 0 \) as \( x \to \infty \) so the limit is

\[ \frac{0 + 0 - 1}{4} = -\frac{1}{4} \]
2. Consider the function

\[ f(x) = \begin{cases} 
\exp(x), & \text{if } x \geq 0; \\
 x + k, & \text{if } x < 0,
\end{cases} \quad (1) \]

where \( k \) is a constant.

(a) (10 points) Graph the function assuming \( k = 2 \).

(b) (10 points) Find all possible values of \( k \) for which the function is continuous. \textit{Justify your answer!}

\[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \exp(x) = \exp(0) = 1, \quad \text{and} \]
\[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x + k) = 0 + k = k, \quad \text{and} \quad f(0) = \exp(0) = 1 \]

Thus, \( f \) is continuous at \( x = 0 \) when the left- and right-hand limits agree with each other and with \( f(0) \).

This happens when \( k = 1 \) only.
3. Compute and simplify all answers as much as possible.

(a) (10 points) \( \frac{d^2}{dx^2} \left( 3x^4 - \frac{\sqrt{x^2+1}}{x^2+1} + \frac{3}{2x-1} \right) \)

First derivative: \( 3(4x^3) - \frac{2x}{2\sqrt{x^2+1}} - \frac{3 \cdot 2}{(2x-1)^2} = 12x^3 - x^{(x^2+1)^{-\frac{3}{2}}} - 6(2x-1)^{-2} \)

Second derivative: \( 12(3x^2) - x \left( \frac{3}{2} \left( x^{(x^2+1)^{-\frac{3}{2}}} \right) \right) - 1 \cdot (x^2+1)^{-\frac{3}{2}} + 6 \cdot (-2) \cdot (2x-1)^{-3} \)

\[
\begin{align*}
&= 36x^2 + \frac{x^2}{\sqrt{x^2+1}} - \frac{1}{\sqrt{x^2+1}} + \frac{24}{(2x-1)^3} = 36x^2 - \frac{1}{\sqrt{(x^2+1)^3}} + \frac{24}{(2x-1)^3} \\
\end{align*}
\]

(b) (10 points) \( \frac{d}{dx} \ln(\sec x + \tan x) \).

\[
\frac{d}{dx} \ln(\sec x + \tan x) = \frac{1}{\sec x + \tan x} (\sec'x + \tan'x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x.
\]

(c) (10 points) \( \frac{d}{dx} \ln \sqrt{\frac{x-1}{x+1}} \).

\[
\frac{d}{dx} \ln \sqrt{\frac{x-1}{x+1}} = \frac{1}{2} \left( \frac{1}{x-1} - \frac{1}{x+1} \right) = \frac{1}{2} \left( \frac{1}{x-1} - \frac{1}{x+1} \right) = \frac{1}{x^2 - 1}.
\]
4. Consider the function

\[ f(x) = \frac{x^2}{x-1} \tag{2} \]

(a) (5 points) Show that the line \( x = 1 \) is an asymptote to the graph of \( y = f(x) \) by computing \( \lim_{x \to 1^+} f(x) \) and \( \lim_{x \to 1^-} f(x) \).

\[
\lim_{x \to 1^+} \frac{x^2}{x-1} = \left( \lim_{x \to 1^+} x^2 \right) \cdot \lim_{x \to 1^+} \frac{1}{x-1} = 1^2 \cdot (+\infty) = +\infty.
\]

\[
\lim_{x \to 1^-} \frac{x^2}{x-1} = \left( \lim_{x \to 1^-} x^2 \right) \cdot \lim_{x \to 1^-} \frac{1}{x-1} = 1^2 \cdot (-\infty) = -\infty.
\]

Thus, the graph of \( y = f(x) \) approaches the vertical \( x = 1 \) asymptotically when \( x \) approaches 1: from above when \( x \to 1^+ \) and from below when \( x \to 1^- \).

(b) (5 points) Show that the line \( y = x + 1 \) is another asymptote by computing \( \lim_{x \to +\infty} [f(x) - (x + 1)] \) and \( \lim_{x \to -\infty} [f(x) - (x + 1)] \).

\[
f(x) - (x+1) = \frac{x^2}{x-1} - (x+1) = \frac{1}{x-1}
\]

Now, \( \lim_{x \to +\infty} \frac{1}{x-1} = 0 \) so the graph of \( y = f(x) \) approaches the line \( y = x + 1 \) asymptotically as \( x \to +\infty \).
(c) (5 points) Sketch the graph of $y = f(x)$. (Hint: it's a hyperbola.)

(d) (5 points) Find the equation of the tangent line to the graph passing through the point $(2, 4)$ and draw this line in the graph above.

$$f'(x) = \frac{(2x)(x-1) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.$$ 

Thus, $f'(2) = \frac{2^2 - 2 \cdot 2}{(2-1)^2} = 0$ so the tangent line has slope zero, hence its equation is $y - 4 = 0$ or $y = 4$. 
