1. Compute the following limits.

(a) \( \lim_{x \to 0} \frac{e^x - 1}{\sin 2x} \).

\[
\lim_{x \to 0} \frac{e^x - 1}{\sin 2x} = \lim_{x \to 0} \frac{e^x}{2 \cos 2x} = \frac{e^0}{2 \cos 0} = \frac{1}{2}.
\]

(We used L’Hôpital in the first step since limit was of form 0/0.)

(b) \( \lim_{x \to 0^+} x^x \).

\[
\ln \left( \lim_{x \to 0^+} x^x \right) = \lim_{x \to 0^+} \ln(x^x) = \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x}.
\]

Now this is of the form \( \infty/\infty \) so apply L’Hôpital:

\[
= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = -\lim_{x \to 0} x = 0.
\]

Thus, \( \lim_{x \to 0^+} x^x = \exp(0) = 1 \).

(c) \( \lim_{x \to 1} \frac{1 - x + \ln x}{1 + \cos \pi x} \).

This is of the form 0/0 so apply L’Hôpital:

\[
\lim_{x \to 1} \frac{1 - x + \ln x}{1 + \cos \pi x} = \lim_{x \to 1} \frac{-1 + 1/x}{-\pi \sin \pi x}.
\]

The above is again of the form 0/0 so L’Hôpital once more gives:

\[
= \lim_{x \to 1} \frac{-1/x^2}{-\pi^2 \cos \pi x} = \frac{-1/1^2}{-\pi^2 \cos \pi} = \frac{1}{\pi^2}.
\]
2. Consider the function \( f(x) = \frac{x^3 - 4}{x^2} \).

(a) Compute \( f'(x) \) and analyze the regions where \( f(x) \) is increasing or decreasing.
First observe that the original function \( f(x) \) is defined whenever \( x \neq 0 \). Now, \( f'(x) = 1 + 8/x^3 \) is positive if \( x < -2 \) or \( x > 0 \) and negative only when \(-2 < x < 0\) so \( f(x) \) is increasing in the two former intervals and decreasing in the latter.

(b) Compute \( f''(x) \) and analyze the concavity of \( f(x) \).
\( f''(x) = -24/x^4 \) which is negative whenever it is defined, so the graph of \( f(x) \) is concave down.

(c) Find all asymptotes (horizontal, vertical or oblique) to the graph of \( y = f(x) \).
\( x = 0 \) is the only point where \( f(x) \) is undefined and \( f(x) \) is otherwise continuous, so looking at the one-sided limits:
\[
\lim_{x \to 0^\pm} \frac{x^3 - 4}{x^2} = -\infty
\]
(both one-sided limits are \(-\infty\)), hence \( x = 0 \) is a vertical asymptote approached by the graph of \( y = f(x) \) “in the direction of \(-\infty\)” from both sides. Also, noting that \( f(x) \) is a rational function with numerator of degree 1 more than the denominator, we separate the polynomial part:
\[
f(x) = \frac{x^3 - 4}{x^2} = x - \frac{4}{x^2},
\]
and conclude that the diagonal line \( y = x \) is an oblique asymptote to the graph. Consequently, there are no horizontal asymptotes.

(d) Sketch the graph of \( y = f(x) \).
3. A wooden beam with rectangular cross section must be cut from a log with a circular cross section of diameter 2 feet. The strength of the beam is the product of its width $w$ with the square of its height $h$. Find the optimal way to cut the beam from the log to maximize its resistance (aka, find the height and width of the most resistant beam.)

The resistance $R$ of the beam is $R = wh^2$ where, by Pythagoras’ Theorem, $h^2 + w^2 = 2^2 = 4$, thus $h^2 = 4 - w^2$ and $R = w(4 - w^2)$ for $0 \leq w \leq 2$. Note that $R$ is a continuous function defined on a closed interval so it must attain its maximum by the Extreme Value Theorem. At the endpoints $w = 0, 2$ we have $R = 0$. Checking for critical points we find $dR/dw = 4 - 3w^2$ is always well-defined, and vanishes only when $w = \sqrt{4/3}$. In this case we clearly have $R > 0$, so the value of $w = \sqrt{4/3}$ is necessarily the one attaining the maximal resistance. In this case, $h = \sqrt{4 - w^2} = \sqrt{4 - (4/3)} = \sqrt{8/3}$. 

Figure 1: Graph of $y = x - 4/x^2$ showing the oblique asymptote $y = x$.

Figure 2: Cross section of the log and the beam to be cut from it.
4. Find the area of the “boomerang” contained between the graphs of $f(x) = \frac{x}{3\pi}(3\pi - x)$ and $g(x) = -\sin x$ for $0 \leq x \leq 3\pi$.

Figure 3: The “boomerang” $0 \leq x \leq 3\pi, g(x) \leq y \leq f(x)$.

Area $= \int_0^{3\pi} \left[ \frac{x}{3\pi}(3\pi - x) - (-\sin x) \right] dx = \int_0^{3\pi} \left( x - \frac{x^2}{3\pi} + \sin x \right) dx$

$= \left[ \frac{x^2}{2} - \frac{x^3}{9\pi} - \cos x \right]_0^{3\pi} = \frac{9}{2} \pi^2 - \frac{27\pi^3}{9\pi} + 2$

$= \frac{3}{2} \pi^2 + 2$. 

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