1. p.28 #10.

Let $G$ be the group of rigid motions of a cube in $\mathbb{R}^3$. Label any two adjacent vertices $\mathcal{O}$ and $\mathcal{O}'$. Then there are 8 choices of vertices to map $\mathcal{O}$ to. Given this vertex, there are then 3 choices for $\mathcal{O}'$, since it must go to an adjacent vertex. Thus, there are $8 \times 3 = 24$ elements in $G$.

2. p.33 #5

$|G| = \text{lcm}(5, 2, 3) = 30$

3. p.33 #11

Let $\sigma$ be an $m$-cycle.

$(\Rightarrow)$ Then $\sigma$ is an $m$-cycle $\Rightarrow |\sigma| = m \Rightarrow |\sigma| = m$ for $i \in \mathbb{Z}/m\mathbb{Z}$ and $(m, i) = 1$

$(\Leftarrow)$ View $\sigma$ as a function where $\sigma$ takes $a \mapsto a + i \bmod m$

Since $|\sigma| = m$ in $\mathbb{Z}/m\mathbb{Z}$, the number of times you need to apply $\sigma$ to return to $a$ is $|\sigma| = m$

$\therefore \sigma = (1a, 2a, \ldots, ma)$. Relabel to get $\sigma = (1, 2, \ldots, m)$ so $\sigma$ is an $m$-cycle.

4. p.35 #11

Let $H(F) = \{ (a, 0) \in GL_3(F) \}$. Let $X = (a, b, c)$, $Y = (d, e, f) \in H(F)$

a) $XY = (a + d, a + e + b, c + f) \in H(F)$ is closed w.r.t. $\times$

b) $X^{-1} = (-d + b, -c + a, a + c) \in H(F)$

c) I'll let you work out it is associative. Since $H(F)$ is closed for each of $a, b, c$, $|H(F)| = 1F^3$

d) $H(2F) = \{ (1, 1), (1, i), (i, 1), (i, i), (1, 1), (1, i), (i, 1), (i, i) \}$ (\neq D_4)

Orders: 1 2 2 2 2 4

5) If $X \in H(2F)$, $X \neq I$, $|X| = n$, then $X^n = I$.

Let $X = (a, b, c)$. Then $X^n = (1, na + bnt + fac)$ when $F(a, c) \in \mathbb{R}$, $s, t, s + t = 0$

Equating entries give $na = 0$, $nc = 0 \Rightarrow a, c = 0 \Rightarrow bn = 0 \Rightarrow b = 0$

$\therefore I = X \Rightarrow X = I$.
4. Let \( G \) be a group. Define \( \psi : G \rightarrow G \) by \( x \mapsto x^{-1} \).

Then \( \psi(xy) = \psi(x) \psi(y) \mapsto (xy)^{-1} = x^{-1}y^{-1} \mapsto y^{-1}x^{-1} = \psi(x^{-1}) \psi(y^{-1}) \mapsto 1 = x^{-1}y^{-1}xy \mapsto xy = yx \). 

\( \therefore \) \( G \) is abelian.

5. Let \( |G| = 4 \).

By exhausting all possible group operations, it is easy to see \( G \) must be abelian.

\( \therefore \) by Fundamental Theorem of Finitely Generated Abelian Groups (Problem 3, p. 158), \( G \cong \mathbb{Z}_4 \) or \( \mathbb{Z}_2 \times \mathbb{Z}_2 \).

6. Define \( \phi : \mathbb{Z}/n \mathbb{Z} \rightarrow \mathbb{Z}/n \mathbb{Z} \) by \( e^\frac{2\pi k}{n} \mapsto k \mod n \).

This is clearly well-defined and surjective.

\( \phi \) is easily seen to be a homomorphism:

\( \phi \) is injective since if \( e^\frac{2\pi k}{n} = e^\frac{2\pi l}{n} \), \( k \equiv l \mod n \), i.e., \( k = n \ell \) for \( \ell \in \mathbb{Z} \).

So \( e^\frac{2\pi k}{n} = e^\frac{2\pi n\ell}{n} = (e^\frac{2\pi}{n})^\ell = 1^\ell = 1 \).

\( \therefore \) \( \phi \) is isomorphic.

7. Let \( p \) be prime. Let \( |G| = p, x \in G, xy \neq 1 \).

By Lagrange's Theorem, \( |x||y| = 1 \).

\( |x||y| = 1 \) or \( p \).

\( \therefore \) Every group of order \( p \) is cyclic, and \( G \cong \mathbb{Z}/p \mathbb{Z} \).

8. The element \((0,1)\) has order 2 in \( \mathbb{Z} \times \mathbb{Z}/2 \mathbb{Z} \), whereas no element in \( \mathbb{Z} \) has order 2.

9. Let \( p \) be prime, \( a \in \mathbb{Z} \).

\( \phi(p) = p-1 \) by (Problem 16, p. 135).

Since \( a^{p-1} = 1 \), multiply both sides by \( a \) to get \( a^p = a \) in \( \mathbb{Z}/p \mathbb{Z} \).

(PS - this is Fermat's Little Theorem: \( a^{n-1} \equiv 1 \mod n \) for \( (a,n) = 1 \).)

10. Let \( G = \mathbb{Z}/2^k \mathbb{Z} \) for \( k \geq 3 \).

1. Let \( (2^k, n) = 1 \). Then \( n \in G \) by Problem 4, pg. 10.

2. \( |G| = \phi(2^k) = 2^{k-1} \) by Problem 16, p. 135.
2. \((2^{k+1} + 1)(2^{k+1} + 1) = 2^{2k+2} + 2 \cdot 2^k + 1 \equiv 1 \mod 2^k\)

\((2^k - 1)(2^k - 1) = (2^k)^2 - 2 \cdot 2^k + 1 \equiv 1 \mod 2^k\)

4. \(\mathbb{Z}/2m\mathbb{Z}\) cyclic of order 2m. Clearly m e \(\mathbb{Z}/2m\mathbb{Z}\) is the only element of order 2 so \(\frac{m}{2}\) element.

11. Label the vertices of a rectangle \(\frac{a}{b} \cdots \frac{c}{d}\). Then the only rigid motions are \(\{1, (13)(24), (12)(23), (14)(23)\}\)

which is isomorphic to \(\mathbb{Z}_2 \times \mathbb{Z}_2\) (The Klein 4-group)

12. Let \(\mathbb{Q}/\mathbb{Z} = (\mathbb{Q}, +) / \sim\) where \(\frac{a}{b} \sim \frac{c}{d} \rightarrow \frac{a}{b} - \frac{c}{d} \in \mathbb{Z}\)

Show \(\mathbb{Q}/\mathbb{Z}\) a group: For \(\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}/\mathbb{Z}\), \(\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \in \mathbb{Q}/\mathbb{Z}\) : closed

Assoc. follows from associativity of \((\mathbb{Q}, +)\)

Identity is \(0 = \frac{0}{1}\)

Inverse of \(\frac{p}{q} = -\frac{p}{q} \in \mathbb{Q}/\mathbb{Z}\) : Group

1. For \(\tau \in \mathbb{Q}/\mathbb{Z}\), let \(\tau = \frac{p}{q}\) and WLOG, say \(\frac{p}{q} \geq 0\).

If \(\frac{p}{q} \in [0, 1]\) done, so assume not.

Then \(\frac{p}{q} - \lfloor \frac{p}{q} \rfloor \in [0, 1)\), where \(\lfloor \frac{p}{q} \rfloor\) is the floor function. Call this representation \(\tau \in [0, 1)\)

Then \(\frac{p}{q} \sim \tau'\) since \(\frac{q}{q} - \tau' = \lfloor \frac{p}{q} \rfloor \in \mathbb{Z}\).

2. Let \(n \in \mathbb{N}\). Then \(\frac{1}{n} \in \mathbb{Q}/\mathbb{Z}\), and \(\langle \frac{1}{n} \rangle = \{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, ..., \frac{n-1}{n}\}\), so \(|\langle \frac{1}{n} \rangle| = n\)

by part 1), \(\exists!\) copy of \(\mathbb{Z}/n\mathbb{Z}\) in \(\mathbb{Q}/\mathbb{Z}\)

2. Any \(\phi \in \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z})\) is completely determined where \(I\) gets mapped to \(\frac{1}{n}\) for any \(n \in \mathbb{N}\) gives all homomorphisms from \(\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}\)

\(\therefore \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}\)

\(\text{card } |\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z})| = \text{card } |\mathbb{Z}/n\mathbb{Z}| = \text{card } |\mathbb{N}| = \text{countable}\)

4. Any \(\phi \in \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})\) must take an element \(\frac{1}{n}\) to 0, since every element in \(\mathbb{Q}/\mathbb{Z}\) has finite order.

and 0 is the only element of \(\mathbb{Z}\) with finite order.

\(\therefore \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) = 0\)

\(\text{card } |\{0\}| = 1\)

Note: see remark in ex 2 p 387 on Pontrjagin Dual Group