Abstract Algebra - HW #5 Solutions

1. p. 85 #3

Let $A$ be abelian, $B \triangleleft A$. Then $ab, bB \in A/B$, $ab \cdot bB = abB = b(abB) = bB \cdot aB$. $B/B$ is abelian.

For example, of non-abelian group $G$ containing proper normal subgroup $N$ s.t. $G/N$ is abelian, let $G = Q_8$, $N = \langle i \rangle$.

Then $Q_8/\langle i \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which is abelian.

2. p. 86 #6

Let $\phi: \mathbb{R}^+ \to \{\pm 1\}$ def by $x \mapsto \frac{x}{|x|}$.

$\phi$ is a homomorphism since $\phi(xy) = \frac{xy}{|xy|} = \frac{x}{|x|} \cdot \frac{y}{|y|} = \phi(x) \cdot \phi(y)$.

Fibers of $\phi$ are $\phi^{-1}(1) = \{x \in \mathbb{R}^+ | x > 0\}$ and $\phi^{-1}(-1) = \{x \in \mathbb{R}^+ | x < 0\}$.

3. p. 87 #20

Let $G = \mathbb{Z}/24\mathbb{Z}$, $G = \langle 6 \rangle$.

By 3rd Iso Thm, $G \cong (\mathbb{Z}/24\mathbb{Z})/(\mathbb{Z}/12\mathbb{Z}) \cong \mathbb{Z}/12\mathbb{Z}$, which gives part c).

a) trivially follows and use the formula that $|G| = 161 = \frac{|G|}{(6, a)} = \frac{12}{(6, a)}$.

4. p. 101 #1

I'm done this in section! Let $\psi: \text{GL}_n(F_q) \to \mathbb{F}_q^\times$ be homomorphism defined by $A \mapsto \det A$.

Then $\ker \psi = \text{SL}_n(F_q)$, so by 1st Iso Thm, $\text{GL}_n(F_q)/\text{SL}_n(F_q) \cong \mathbb{F}_q^\times$ since $\psi$ is surjective.

Thus $|\text{GL}_n(F_q)/\text{SL}_n(F_q)| = |\mathbb{F}_q^\times| = q - 1$.

5. See Cor 9 p. 125

4. Let $G$ group, $H, K \subseteq G$ s.t. $|G/H| = |G:/K = 2$, and $H \triangledown K = \{e\}$.

The lattice of $G$ is then $G/H \quad \text{and by the 2nd Iso Thm, } |H| = |K| = 2$.

By example 2 pg 71, both $H \subseteq G$ and $K \subseteq G$, so by Thm 9 p. 121, $G = H \times K$ so $1H = 1H_1 / 1H_1 / 2 = K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
5. Define a map \( G \to \langle [i : j], \ [i : j^2] \rangle \) defined by \( i \mapsto [i : j], j \mapsto [i : j^2] \).
Recalling that a presentation for \( G = \langle i, j \mid i^2, j^2, i^3 = j^3 = 1, i^2j = ji \rangle \) it is an easy
verification that \( \psi \) is an isomorphism. Then \( G \) is the only element of order 2,
which is also the center, and since every subgroup of \( G \) has order 2, they are normal (with
the exception of the center, which is clearly normal).

6. Draw lattice:

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\[
\begin{array}{c}
\text{gen} \\
\downarrow \\
G
\end{array}
\begin{array}{c}
\downarrow \\
H
\end{array}
\begin{array}{c}
\downarrow \\
\text{HK}
\end{array}
\begin{array}{c}
\downarrow \\
H \cap K
\end{array}
\begin{array}{c}
\downarrow \\
H \cap K
\end{array}
\begin{array}{c}
\downarrow \\
1
\end{array}
\]
\]
\]
\]
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Assume \( K \neq H \). Then \( H \leq HK \) (proper containment)

\[\exists 2^{nd} \text{ Iso Thm., } K : \text{H} = \frac{p}{q} \]

\[|G : H| = |G : \text{H} \cap H| = p, \text{ so since } N = H K,
\]

\[|H : \text{H} \cap H| = 1 \rightarrow |H : H| = p \text{ and } |G : H K| = 1\]

\[G = H K\]

7. Look at map \( \pi : H \to \mathbb{Z}/n \) defined by \( h \mapsto hN \)

Then \( |\pi(H)| = \frac{|H|}{|H \cap N|} \) since \( \pi(H) \) is a quotient of \( H \)

and \( |\pi(H)| = \frac{|H|}{|H \cap N|} \) by Lagrange.

\[\therefore \text{since } (|H|, |G : H|) = 1, \ |\pi(H)| = 1 \therefore H \text{ gets mapped to the identity coset, so } H \leq N\]

8. \((aH)^n = a^n \cdot H = H \therefore aH \leq H\]

9. Let \( \varphi : A \times B \to A / c \times B / d \) be natural quotient map. Then \( \varphi \) is surjective and \( ker \varphi = (c \times d) \)

\[\therefore \text{by 1st Iso Thm., } A \times B / (c \times d) \cong A / c \times B / d\]

10. 1) \[G \text{ is finite, } [G : K] = \frac{|G|}{|K|}, \text{ so since } \frac{|G|}{|K|} = \frac{|H|}{|K|} = \frac{|G : H|}{[G : H] [H : K]}\]

2) If \( [G : H] [H : K] \) is finite, say \( [G : H] = p, [H : K] = q, \) then have cosets \( g H, g_2 H, \ldots, g_q H \) which partition \( G \)
and \( H \cap K, \ldots, K \) (cosets of \( K \) in \( H \)) partition \( H \). Then let \( g \in g H = \bigcup_{j=1}^{p} g_j H \), so \( g \in g_j H \).

\[|G : K| \text{ is finite}\]

If \(|G : K| = n < \infty, g_1 K, g_2 K, \ldots, g_n K \) partition \( G \). Pick cosets in \( H \), say \( g_1, g_2, \ldots, g_n K \leq H \)

Then there are finitely many \((n - q)\) cosets \( g H, H \). \[|G : H| = 1 \therefore |G : H| / |K| \text{ will be finite as well.}\]
Abstract Algebra HW #5

1. equality proved if \([G:K]=\infty\) or \([G:H][H:K]=\infty\)

Now, if all indices are finite, say \(n=[G:K]\) and \([G:H][H:K]=p^aq^b\).

considering everything mod \(K\), \(G/K=\{gK\mid g\in G\}\) is finite, so by part a)

\[|G/K| = |G:H| \cdot |H/K| = p^a \cdot q^b\]

so by the 3rd Iso Thm, \([G:K] = [G:H][H:K]\) as desired.

11. Let \(G = \{z \in \mathbb{C} \mid |z|=1\}\) be \(\mathbb{C}\) prime.

1. \(\zeta : G \to G\) def by \(z \mapsto z^p\) is surjective since given any \(z = e^{i\theta} \in G\), consider \(e^{i\theta} \in G\).

Then \(\zeta(e^{i\theta}) = e^{i\theta^p} = z\) as desired.

2. Then by first iso thm, \(G/\ker \zeta \simeq G\)