Practice Midterm Exam 1

1. Find all solutions \( y = y(x) \) to the following initial value problems (remember to include domain):

   (a) (10 points)
   \[
   \begin{cases}
   y' = (1 + y^2) x \\
   y(0) = 1
   \end{cases}
   \]

   (b) (10 points)
   \[
   \begin{cases}
   y' = y^{2/3} \\
   y(0) = 0
   \end{cases}
   \]

2. Put the following matrices in canonical form (i.e., Jordan normal form).

   (a) (10 points)
   \[
   A_1 = \begin{pmatrix} -1 & 1 \\ -9 & 5 \end{pmatrix}
   \]

   (b) (10 points)
   \[
   A_2 = \begin{pmatrix} 1 & 2 \\ -4 & -3 \end{pmatrix}
   \]

3. Determine a \( 2 \times 2 \) linear system of ODEs which has the following properties:

   (a) (10 points) The phase portrait contains a stable line \( y = 3x \) and an unstable line \( x = 0 \).

   (b) (10 points) The phase portrait contains the ellipse \( 5x^2 - 4xy + y^2 = 20 \).

4. (20 points) Find the general solution to the following \( 3 \times 3 \) linear system:

   \[
   \mathbf{Y}' = \begin{pmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{pmatrix} \mathbf{Y}
   \]

5. Consider the following one-parameter family of autonomous ODEs

   \[
   y' = F_a(y) = \frac{y}{1 + y^2} - ay
   \]

   (a) (10 points) Draw the bifurcation diagram for this family of ODEs.

   (b) (10 points) Show that there is a value \( a_0 \) so that if \( a_- < a_0 \) and \( a_+ > a_0 \), then the systems \( y' = F_{a_-}(y) \) and \( y' = F_{a_+}(y) \) are not topologically conjugate (Hint: Do not try and solve the ODEs explicitly).