Problem #1. Find a particular solution of the following second order ODEs:

a) \(x'' + x = e^t\)
b) \(x'' - x = e^t\)

Solution. Several people used a complicated variation of parameters method to solve these, so I thought I’d illustrate how much easier it is to use undetermined coefficients instead.

a) Note that in this case our characteristic polynomial (obtained from setting \(x = e^{rt}\) and plugging in to the homogeneous equation) is \(r^2 + 1 = 0\). Thus the system has eigenvalues \(\pm i\). Since 1 is not an eigenvalue, we simply try the solution \(Ae^t\) (since the RHS is just \(e^t\)). Plugging this in, we obtain \(Ae^t + Ae^t = e^t\). Thus \(A = 1/2\), and we have the particular solution \(\frac{1}{2}e^t\) (See how easy that was? No integrals required!).

b) Now our characteristic polynomial has become \(r^2 - 1 = 0\), so we have eigenvalues \(\pm 1\). Since 1 is an eigenvalue with multiplicity 1, we modify our guess. That is, try \(Ate^t\). Plugging in yields \(2Ae^t + Ate^t - Ate^t = e^t\). So once again \(A = 1/2\) works, and we have the particular solution \(\frac{1}{2}te^t\).

Problem #3. Consider a mass on a spring whose motion is determined by

\[x'' + 5x' + 4x = 0.\]

a) Determine the initial conditions \(x_0, v_0\) which allow one to stop the mass completely after time \(t = \pi\) by a single blow with a hammer at time \(t = \pi\) (i.e., with forcing \(a\delta_\pi\)).

b) What about if you are allowed a second hammer blow at time \(t = 2\pi\) and want to completely stop the mass after time \(t = 2\pi\) (i.e., the forcing is \(a\delta_\pi + b\delta_{2\pi}\))?

Solution.

a) A lot of people made mistakes on this one, by either making the wrong change of variables or getting an incorrect differential equation from their change of variables. Here’s what I did:

The condition that the mass stops completely after time \(t = \pi\) is equivalent to setting the initial condition \(x(\pi + 1) = x'(\pi + 1) = 0\) (or at any time after \(t = \pi\)). I don’t want to think too hard about what will actually happen at \(t = \pi\), so I’ll use this instead.

Now, let’s make a change of variables to make our lives easier. I set \(y(t) = x(\pi + 1 - t)\). That way \(y(0) = y'(0) = 0\). Also, \(y'(t) = -x'(\pi + 1 - t)\), but \(y''(t) = x''(\pi + 1 - t)\). Thus, \(y\) satisfies the differential equation

\[y'' - 5y' + y = a\delta_\pi(\pi + 1 - t) = a\delta_0(1 - t) = a\delta_0(t - 1) = a\delta_1(t)\]
Laplacing everything, we end up with the expression
\[ Y(s) = \mathcal{L}\{Y\} = \frac{ae^{-s}}{s^2 - 5s + 4} = \frac{ae^{-s}}{3} \left( \frac{1}{s - 4} - \frac{1}{s - 1} \right) \]
using partial fractions. Inverse Laplacing yields \( y(t) = \frac{a}{3}H_1(t) \left( e^{4(t-1)} - e^{t-1} \right) \). Thus
\[ x(t) = \frac{a}{3} \left( 1 - H_\pi(t) \right) (e^{4(\pi-t)} - e^{\pi-t}) \]Differentiating at every \( t \neq \pi \), we have
\[ x'(t) = \frac{a}{3} \left( 1 - H_\pi(t) \right) (e^{\pi-t} - 4e^{4(\pi-t)}) \]Thus \( x(0) = x_0 = \frac{a}{3} (e^{4\pi} - e^{\pi}) \) and
\[ x'(0) = v_0 = \frac{a}{3} (e^{\pi} - 4e^{4\pi}) \].

b) We take the same strategy, but instead use the initial condition \( x(2\pi+1) = x'(2\pi+1) = 0 \) and the change of variables \( y(t) = x(2\pi + 1 - t) \). Then our IVP is \( y(0) = y'(0) = 0 \) and
\[ y' - 5y' + 4y = a\delta_{\pi+1} + b\delta_1 \]
The same process yields
\[ x(t) = \frac{1}{3} \left( a \left( 1 - H_\pi(t) \right) \left( e^{4(\pi-t)} - e^{\pi-t} \right) + b \left( 1 - H_{2\pi}(t) \right) \left( e^{4(2\pi-t)} - e^{2\pi-t} \right) \right) \]
Thus we have \( x(0) = x_0 = \frac{a}{3} (e^{4\pi} - e^{\pi}) + \frac{b}{3} (e^{8\pi} - e^{2\pi}) \) and
\[ x'(0) = v_0 = \frac{a}{3} (e^{\pi} - 4e^{4\pi}) + \frac{b}{3} (e^{2\pi} - 4e^{8\pi}) \].

**Problem #8.** Let \( f \) be a non-negative piecewise continuous function and let \( F(s) \) be its Laplace transform.

a) Show that if \( f \) is bounded by a constant \( C \), then \( F(s) \) is defined on at least \((0, \infty)\) and \( F(s) \leq \frac{C}{s} \).

b) Show that if there is a \( C \geq 0 \) so that \( C t^n \leq f(t) \) and \( F(s) \) is defined on \((0, \infty)\), then, for such \( s \),
\[ \frac{Cn!}{s^{n+1}} \leq F(s) \]

**Solution.**

a) I was kind of surprised to see so many people not even use the fact that \( f \) is non-negative in their solutions to this problem. It is absolutely essential to use this; otherwise the statement is not true! I’ll show you what I mean.

By definition, \( F(s) = \int_0^\infty e^{-st} f(t) \, dt \). On the one hand, \( f \leq C \) implies that
\[ \int_0^\infty e^{-st} f(t) \, dt \leq \int_0^\infty e^{-st} C \, dt = \mathcal{L}\{C\} = \frac{C}{s} \]for any \( s > 0 \). On the other hand, \( f \geq 0 \) implies that \( 0 = \int_0^\infty 0 \, dt \leq \int_0^\infty e^{-st} f(t) \, dt \). Thus, we have that \( 0 \leq F(s) \leq \frac{C}{s} \) for every \( s > 0 \). It is only because we have both inequalities that we can conclude that \( F(s) \) converges for all \( s > 0 \).

b) This just comes down to doing integration by parts a whole bunch. We have that
\[ F(s) = \int_0^\infty e^{-st} f(t) \, dt \geq \int_0^\infty C e^{-st} t^n \, dt = - \frac{C t^n e^{-st}}{s} \bigg|_0^\infty + \frac{n}{s} \int_0^\infty C t^{n-1} e^{-st} \, dt \]
The left term is 0 since we are assuming $s > 0$ and we proceed with the same method with the right term. Eventually we get

$$F(s) \geq \frac{n!}{s^n} \int_0^\infty Ce^{-st} dt = \frac{Cn!}{s^{n+1}}$$

which is exactly what we wanted. \qed