110.108: Calculus 1
Exam 1 Solutions

1. (21 points) Find the following limits.
   a) \( \lim_{x \to -2} 3x^3 - 2x^2 + 5 \)
      \[ \lim_{x \to -2} 3x^3 - 2x^2 + 5 = 3(-2)^3 + 2(-2)^2 + 5 = -11 \]
   b) \( \lim_{x \to -2} \frac{4 - x}{\sqrt{x} - 2} \)
      \[ \lim_{x \to -2} \frac{4 - x}{\sqrt{x} - 2} = \lim_{x \to -2} \left( \frac{4 - x}{\sqrt{x} - 2} \right) = \lim_{x \to -2} \frac{(4 - x)(\sqrt{x} + 2)}{x - 4} = \lim_{x \to -2} - (\sqrt{x} + 2) = -4 \]
   c) \( \lim_{x \to 0} \frac{\sin 5x}{x} \)
      \[ \lim_{x \to 0} \frac{\sin 5x}{x} = \lim_{x \to 0} \frac{5 \sin 5x}{5x} = 5 \cdot 1 = 5 \]

2. (21 points) Find the first derivative of each of the following functions. You do not have
   to simplify your answers.
   a) \( f(x) = 4x^5 - 3x^4 + 2 \)
      \[ f'(x) = 20x^4 - 12x^3 \]
   b) \( g(x) = \sec 3x \tan 4x \)
      \[ g'(x) = 3 \sec 3x \tan 3x \tan 4x + 4 \sec 3x \sec^2 4x \]
   c) \( h(t) = \frac{t^5 - 3t^3}{t^2 + 1} \)
      \[ h'(t) = \frac{(5t^4 - 9t^2)(t^2 + 1) - 2t(t^5 - 3t^3)}{(t^2 + 1)^2} \]

3. (8 points) Give an \( \epsilon, \delta \) proof for \( \lim_{x \to 2} 3x + 4 = 10 \).

   Let \( \epsilon > 0 \). Let \( \delta = \epsilon/3 \). Suppose \( 0 < |x - 2| < \epsilon/3 \). Then \( 3|x - 2| < \epsilon \). But \( 3|x - 2| = |3x - 6| = |3x + 4 - 10| \). Thus, \( |3x + 4 - 10| < \epsilon \). Since \( |3x + 4 - 10| < \epsilon \) whenever \( 0 < |x - 2| < \delta \), we may conclude that \( \lim_{x \to 2} 3x + 4 = 10 \).
4. (7 points) Is the function

\[ g(x) = \begin{cases} 
2x^2 + 2 & \text{if } x < 1 \\
2x + 4 & \text{if } x \geq 1 
\end{cases} \]

continuous at \( x = 1 \)? Justify your answer using the definition of continuity.

Note that \( \lim_{x \to 1^-} g(x) = 2 \cdot 1^2 + 2 = 4 \) and \( \lim_{x \to 1^+} g(x) = 2 \cdot 1 + 4 = 6 \). Thus, \( \lim_{x \to 1} g(x) \) does not exist because \( \lim_{x \to 1^-} g(x) \neq \lim_{x \to 1^+} g(x) \). Hence, \( g(x) \) is not continuous at \( x = 1 \).

5. (24 points) A rubber ball is thrown upward from a window 32 feet above the ground at a speed of 16 ft/sec. Show your work in answering the questions below. (\( g = 32 \text{ ft/sec}^2 \))

a) When will the ball hit the ground?

The position at time \( t \) is \( y(t) = -16t^2 + 16t + 32 \). The ball will hit the ground when \( y(t) = 0 \):

\[-16t^2 + 16t + 32 = 0\]

\[-16(t^2 - t - 2) = 0\]

\[-16(t - 2)(t + 1) = 0\]

\[t = 2 \text{ or } t = -1.\]

The ball hits the ground at \( t = 2 \) seconds.

b) What is the maximum height reached by the ball?

The ball reaches its maximum height when \( v(t) = 0 \). Differentiating \( y(t) \), we obtain \( v(t) = -32t + 16 \). Setting \( v(t) = 0 \) yields

\[-32t + 16 = 0 \Rightarrow t = 1/2.\]

The position at \( t = 1/2 \) is

\[y(1/2) = -16(1/2)^2 + 16(1/2) + 32 = 36.\]

The maximum height is 36 feet.

c) Each time the ball hits the ground, it rebounds at \( 1/2 \) its impact speed. What will the speed of the ball be immediately after the second time it hits the ground?

The velocity of the ball the first time it hits the ground is

\[v(2) = -32(2) + 16 = -48.\]

The velocity immediately after the first bounce is 24 feet per second. The position of the ball \( t \) seconds after the first bounce is given by

\[y_2(t) = -16t^2 + 24t.\]
The ball hits the ground the second time when $y_2(t) = 0$:

$$-16t^2 + 24t = 0 \Rightarrow t = 0 \text{ or } t = 3/2.$$  

The velocity $t$ seconds after the first bounce is $v_2(t) = -32t + 24$. The velocity of the ball as it hits the ground the second time is $v_2(3/2) = -24$ ft/sec. The speed of the ball immediately after the second bounce is 12 ft/sec.

6. (19 points) a) Find $\frac{dy}{dx}$ for $3\sqrt{x^3} + \sqrt{y} = 12$. Show your work.

The equation is the same as $3x^{3/2} + y^{1/2} = 12$. Differentiate the equation implicitly:

$$\frac{9}{2}x^{1/2} + 1/2y^{-1/2} \frac{dy}{dx} = 0.$$  

Solve the equation for $\frac{dy}{dx}$:

$$\frac{1}{2}y^{-1/2} \frac{dy}{dx} = -\frac{9}{2}x^{1/2}$$

$$\frac{dy}{dx} = -\frac{9}{\sqrt{x}} \sqrt{y}.$$  

b) Find the equation of the tangent line to $3\sqrt{x^3} + \sqrt{y} = 12$ at the point $(1,81)$. Show your work.

The slope of the tangent line is $\frac{dy}{dx}(x,y) = (1,81) = -9\sqrt{1}\sqrt{81} = -81$. The equation of the tangent line is

$$y - 81 = -81(x - 1)$$

or

$$y = -81x + 162.$$  

c) Find all points on the graph of $3\sqrt{x^3} + \sqrt{y} = 12$ where the tangent line is horizontal. Show your work.

The tangent line is horizontal when its slope is 0, i.e., when $\frac{dy}{dx} = 0$:

$$-9\sqrt{x}\sqrt{y} = 0 \Rightarrow x = 0 \text{ or } y = 0.$$  

The point where $x = 0$ has $y$-coordinate determined by:

$$3\sqrt{0^3} + \sqrt{y} = 12 \Rightarrow y = 144.$$  

The point where $y = 0$ has $x$-coordinate determined by:

$$3\sqrt{x^3} + \sqrt{0} = 12 \Rightarrow x = \sqrt[3]{16}.$$  

The points where the tangent line is horizontal are $(0,144)$ and $(\sqrt[3]{16},0)$. 