Start with \( y = f(x) \), some unknown functional relation between 2 variables, where
\( x \)- independent variable
\( y \)- dependent variable

Such an equation (or sets of them are) called a mathematical model when the variables represent measurable quantities in some application (usually set up to study some unknown entity \( y \) based on its relationship to something controllable \( x \)).

If \( y = f(x) \) is known, then we can simply study its properties (usus calculus).

Often, though, we do not know \( y = f(x) \), but we do have information about some properties, like derivatives, for example.
Examples

\[ \frac{dy}{dx} = ky, \quad k \in \mathbb{R}. \]

\[ F = ma \quad \text{(Newton's 2nd Law of Motion)}. \]

\[ f'(x) = x - e^{x^2} \quad \text{(restatement of)} \]
\[ \text{Find } \int (k - e^{x^2}) \, dx. \]

\[ \frac{d^2 \theta}{dt^2} + \frac{g}{L} \sin \theta = 0 \]

\[ \text{Pendulum}. \]

De: An ordinary differential equation (ODE) \[ \text{is an equation involving an unknown function between two variables and some of its derivatives.} \]

Note: "Ordinary" means that the unknown function is a function of one independent variable.
Ex. The heat equation (a 3-space)

\[ \frac{du}{dt} = x \left( \frac{d^2u}{dx^2} + \frac{du}{dy^2} + \frac{du}{dz^2} \right) \]

is a partial diff. eqn since u is a func. of more than 1 independent variable.

Def. The order of an oDE is the same as the order of the highest derivative that appears in the equation:

Ex. (I) and (II) are 1st order oDEs.
(III) and (IV) are 2nd order oDEs. (Do you see why?)

Def. The general form of an n-th order oDE is

\[ F(x, y, y', \ldots, y^{(n)}) = 0 \]  \hspace{1cm} (x)

where
- \( x \)- indep. var. (time?)
- \( y \)- dep. var. (unknown func. \( y = f(x) \))
- \( y^{(i)} \) is the \( i \)th derivative of \( y = f(x) \)
- \( F \) is some expression in \( x, y, y', \ldots, y^{(n)} \)
Note: Sometimes we can solve for the highest derivative:

\[(x)\quad y^{(n)} = Q(x, y, y', \ldots, y^{(n-1)})\]

But not always:

\[\text{ex. } y^{(5)} + \sin y^{(3)} = y^{(2)} \text{ cannot be written like } (x)\). But for (x), \(F = y^{(5)} + \sin y^{(3)} - y^{(2)}\).

Def. A function \(F(x, \ldots, x_n, y_1, \ldots, y_m)\) is linear in the variables \(y_1, \ldots, y_m\) if

\[F(x, \ldots, x_n, y_1, \ldots, y_m) = \alpha_0(x) + \sum\limits_{i=1}^{m} \alpha_i(x) y_i\]

where the \(\alpha_i(x), i = 0, \ldots, m\) are arbitrary.

\(\text{Note: For an ODE to be linear, it must be linear in } y, y', \ldots, y^{(n)}\), and can be written as:

\[\alpha_n(x) y^{(n)} + \cdots + \alpha_1(x) y' + \alpha_0(x) y = g(x)\]
Examples

1. \((\sin x) y' + (\ln x) y = \tan e^x\) is linear of 1st order ODE.

2. \(y'' + xy' + \sin y = 0\) is not linear.

3. \(y'' y + y' = 0\) is not linear.

Suppose \(y' = f(t,y)\) is a 1st order linear ODE (written like \((\times\times\)));

Then there exist functions \(p(t)\), \(q(t)\) so that

\(f(t,y) = p(t)y + q(t)\), and the ODE can be written as

\(y' + p(t)y = q(t)\).

This form will be very important to understanding how to study this type of ODE.

ex. Given \((\sin x) y' + (\ln x) y = \tan e^x\), identify \(p(t)\).

Solution: Divide by \(\sin x\) to get,

\(y' + \frac{(\ln x)}{(\sin x)} y = \frac{\tan e^x}{\sin x}\), \(p(t) = \frac{\ln x}{\sin x}\).
A solution to $(\mathbf{A})$ on $I=(a,b) \subset \mathbb{R}$ is any function $y = f(x)$ that satisfies the equation $(\mathbf{A})$.

Example: \[
\frac{dy}{dt} = \frac{p}{2} - 450 \text{ is solved by } p(t) = 900 + ce^{5t} \quad \forall c \in \mathbb{R}.
\]
* How do we know? Try it.
* How did we find it? Keep listening.
* What to make of the parameter $c$?

Example: Sometimes a solution is only known implicitly:

Show $x^2 + y^2 - 5 = 0$ solves $\frac{dy}{dx} = -\frac{x}{y}$. Here only locally can we solve for $y(x)$.

Example: Solve $x''(t) + \frac{k}{m} x(t) = 0$.

Example: Solve $y'(x) = x - e^{x^2}$.