Very generally, a first-order ODE of the form
\[ \frac{dy}{dt} = f(t, y) \] (x)
will have \( f \) as a function of both \( t \) and \( y \) and will not be solvable.

However, with some additional structure to \( f \), there are methods to solve. In Chapter 2, we explore some of these.

First type of structure (Section 2.1): Linear

\[ f(t, y) = -p(t)y + q(t) \]

for some continuous functions \( p(t) \) and \( q(t) \).

Then (x) can be rewritten
\[ y' = -p(t)y + q(t) \]
or
\[ (\forall x) \quad y' + p(t)y = q(t) \]
This new form exposes a structure that facilitates calculation. No LHS is almost the total derivative of a function. To make it so, we multiply the ODE by an expression called an integrating factor.

**Def:** An integrating factor is a term that when multiplied to an expression renders the expression integrable.

To understand what we are looking for, look at the patterns here:

Let \( y \) be a function of \( t \). Then, for any other differ function of \( t \), \( f(t) \), we have

\[
\frac{d}{dt} \left[ f(t) y \right] = f(t) y' + f'(t) y \quad \text{by Prod. Rule}
\]
And also \( \frac{d}{dt} [e^{\phi(t)} y] = e^{\phi(t)} y' + e^{\phi(t)} f(t) y \)

\[= e^{\phi(t)} [y' + f(t) y]. \]

We do this just to look for patterns. In this case, we see an important one: Inside the bracket, \([y' + f(t) y]\) looks very close to the LHS of \((**): y' + p(t) y = q(t)\).

In fact, they are precisely the same when \( f(t) = p(t) \), or \( f(t) = \int p(t) dt \).

So we do one more calculation for a pattern:

\[ \frac{d}{dt} [e^{\int p(t) dt} y] = e^{\int p(t) dt} y' + \frac{d}{dt} [e^{\int p(t) dt}] y \]

\[= e^{\int p(t) dt} y' + e^{\int p(t) dt} p(t) y \]

\[= e^{\int p(t) dt} [y' + p(t) y], \]

precisely the LHS of \((**): y' + p(t) y = q(t)\).
This is useful because, if we take $y' + p(t)y = q(t)$ and multiply the entire equation by $e^{\int p(t) dt}$, then the LHS becomes easily integrable.

Call $e^{\int p(t) dt}$ the integrating factor of

\[ y' + p(t)y = q(t). \]

Challenge Q: It turns out, any antiderivative of $p(t)$ will give the same effect. Why?

Let's play this out and see just how the integrating factor is helpful.

Solve \[ y' + p(t)y = q(t). \]
Step 1: Multiply each term by $e^{s_P t}$.

\[ e^{s_P t} \left[ y' + p(t) y = q(t) \right] \]

\[ e^{s_P t} y' + e^{s_P t} p(t) y = e^{s_P t} q(t) \]

\[ \frac{d}{dt} \left[ e^{s_P t} y \right] = e^{s_P t} q(t). \]

Step 2: Integrate with respect to $t$.

\[ \int \frac{d}{dt} \left[ e^{s_P t} y \right] dt = \int e^{s_P t} q(t) dt \]

\[ e^{s_P t} y = \int e^{s_P t} q(t) dt + C \]

Step 3: Solve for $y$.

\[ y(t) = e^{-s_P t} \left[ \int e^{s_P t} q(t) dt + C \right] \].
Notes

(1) Theoretically, we can always do this. Practically, the integrating factor $e^{\int P(x)dx}$ is pretty easy to calculate, especially usually.

(2) You do not need to memorize any thing of the form of steps. Just remember the steps.

(3) Any first derivative of $p(x)$ will do, since (2) they all only differ by a constant. You are multiplying the entire equation by the factor.

Ex. Suppose $p(x) = 2t$. Then $e^{\int 2tdt} = e^{t^2} = e^t$.

If indeed you chose $e^{\int 2tdt} = e^{t^2} = e^t$, then

$e^{t^2 + c} = e^{t^2} e^c = e^{t^2} K$, for $K e^c$ constant.

Then $K e^{t^2}[g(x) + p(x)g - q(x)]$ is same as $e^{t^2}[g' + p(x)g - q(x)]$ as far as solutions are concerned.
Sure examples

I. Solve $ty' - 2y = t^2 e^{-2t}$

**Strategy:** This is linear so we use the int. factor $e^{\int p(t) dt}$ to solve using the 3 steps above.

**Solution:** Place the ODE in standard form

$$y' - \frac{2}{t} y = t^2 e^{-2t}.$$ 

This gives us $p(t) = -\frac{2}{t}$, so the int. factor is

$$e^{\int \frac{-2}{t} dt} = e^{\ln t^2} = t^{-2}.$$ 

**Step 1:** Multiply ODE by int. factor.

$$t^{-2} \left[ y' - \frac{2}{t} y = t^2 e^{-2t} \right]$$

$$t^{-2} y' - \frac{2}{t} t^{-2} y = e^{-2t}$$

$$\frac{d}{dt} [t^{-2} y] = e^{-2t}.$$ 

**Step 2:** Integrate w.r.t $t$.

$$\int \frac{d}{dt} [t^{-2} y] dt = t^{-2} y + c_1 = \int e^{-2t} dt = -\frac{1}{2} e^{-2t} + c_2$$

$$t^{-2} y = -\frac{1}{2} e^{-2t} + K$$

**Step 3:** Solve for $y(t)$:

$$y(t) = -\frac{1}{2} t^2 e^{-2t} + K t^2$$

This function solves the ODE.
Check to see if this is correct:

\[
\begin{align*}
(t e^{-2t} + t^2 e^{-2t} + 2kt) - \frac{2}{4} \left( -\frac{1}{2} t^2 e^{-2t} + kt^2 \right) &= t^2 e^{-2t} \\
\Rightarrow t e^{-2t} + t^2 e^{-2t} + 2kt + t e^{-2t} - 2kt &= t^2 e^{-2t} \\
t e^{-2t} &= t^2 e^{-2t}
\end{align*}
\]

\( \checkmark \)

(4) \( x + 2tx = t^3 \). Solve this.

**Strategy:** Use the integrating factor on this linear ODE to integrate through to an expression for \( x(t) \).

**Solution:** This ODE is linear, with \( p(t) = 2t \).

Thus the int. factor is

\[
e^{\int p(t) dt} = e^{\int 2tdt} = e^{t^2}
\]

**Step 1:** Multiply ODE by int. factor:

\[
e^{t^2} [x + 2tx = t^3]
\]

\[
e^{t^2} x + 2te^{t^2} x = t^3 e^{t^2}
\]

\[
\frac{d}{dt} [e^{t^2} x] = t^3 e^{t^2}
\]
Step 2: Integrate with respect to t.

\[ \int \frac{d}{dt} \left[ e^{t^2} \right] dt = e^{t^2} x + C_1 = \int e^{t^2} dt \]

- Integrate by substitution. Let \( s = t^2 \), then \( ds = 2t \, dt \).

\[ \int t^3 e^{t^2} dt \]

\[ \frac{1}{2} \int s e^s ds \]

- Apply integration by parts. Let \( u = s \), \( dv = e^s \, ds \), \( du = ds \), \( v = e^s \).

\[ \frac{1}{2} (se^s - \int e^s ds) = \frac{1}{2} (se^s - e^s) + C_2 \]

Combine constants to get:

\[ e^{\frac{t^2}{2}} x = \frac{1}{2} e^{t^2} (t^2 - 1) + C \]

Step 3: Solve for \( x(t) \).

This is the general solution to \( x + 2tx = t^3 \).

\[ x(t) = \frac{1}{2} t^2 - \frac{1}{2} + Ke^{-t^2} \]

Check this:

\[ (t - 2Kte^{-t^2}) + 2t \left( \frac{1}{2} t^2 - \frac{1}{2} + Ke^{-t^2} \right) = t^3 \]

\[ t - 2Kte^{-t^2} + t^3 - t + 2Kte^{-t^2} = t^3 \]

\[ t^3 = t^3 \]  

It works.
Solve \( \frac{dx}{ds} = \frac{x}{s} - s^2 \), for \( s > 0 \)

Here the ODE is again linear (note \( s \) is the independent variable), and \( p(s) = -\frac{1}{s} \).

The integrating factor is then

\[
\exp \left( \int p(s) \, ds \right) = e^{\int -\frac{1}{s} \, ds} = e^{-\ln s} = s^{-1}
\]

Multiply through standard form of ODE to get

\[
\frac{1}{s} \left[ \frac{dx}{ds} - \frac{x}{s} = -s^2 \right] \Rightarrow \frac{1}{s} \frac{dx}{ds} - \frac{x}{s^2} = -s
\]

\[
\frac{d}{ds} \left[ \frac{1}{s} \cdot x \right] = -s
\]

Integrate with \( s \) to get

\[
\frac{1}{s} \cdot x = \int (-s) \, ds + C = -\frac{s^2}{2} + C
\]

Solve for \( x(s) \):

\[
x(s) = \frac{-s^3}{2} + Cs
\]

This is the general solution to ODE

Check it:

\[
\left( -\frac{3}{2} s^2 + C \right) = \frac{1}{s} \left( \frac{-s^3}{2} + Cs \right) - s^2
\]

\[
\frac{dx}{ds}
\]

\[
-\frac{3}{2} s^2 + C = -\frac{s^2}{2} + C - s^2
\]

\[
-\frac{3}{2} s^2 = -\frac{3}{2} s^2 \quad \text{✓ It is correct}
\]
Ⅳ Find the general solution to
\[ t(y'-y) = (1+t^2)e^t \quad \text{on} \quad t > 0. \]
Here, try to see why this is linear, with
\[ p(t) = -t. \]
The solution is
\[ y(t) = e^t(\ln t + \frac{t^2}{2} + c) \]
This solution is drawn up in a separate
document on the example problems on
the website.

Ⅴ Solve \( \frac{dp}{dc} = \frac{p}{2} - 450 \) using an integrating
factor.

Solution: This is an exercise. You already
know the answer.