II. Structure type: Separable

Suppose for \( y' = f(t, y) \) that
\[
f(t, y) = g(t) \cdot h(y)
\]
for 2 functions \( g(t) \) and \( h(y) \).

Then we say the ODE is separable
(we can separate RHS into the product of 2 functions; one of \( t \) alone and the other of \( y \) alone).

Then the ODE is \( y' = g(t) \cdot h(y) \), and we can write
\[
\frac{1}{h(y)} \frac{dy}{dt} = g(t)
\]

Since \( y \) is a function of \( t \), both sides are functions of \( t \) and we can integrate \( w.r.t. \) \( t \).
\[ \int \left( \frac{1}{h(y)} \frac{dy}{dt} \right) dt = \int g(t) dt \]

The general solution to this kind of ODE is then found by integrating alone.

\[ \text{ex. Find the general solution to } \frac{dy}{dx} = xy^2 \]

(Note: This ODE is separable, but not linear!)

Solution: Separate the variables: \( \frac{1}{y^2} \frac{dy}{dx} = x \)

Then integrate both sides with \( x \):

\[ \int \frac{1}{y^2} \frac{dy}{dx} \, dx = \int x \, dx \]

\[ -\frac{1}{y} = \frac{x^2}{2} + C = \frac{x^2 + K}{2} \]

\[ \Rightarrow y = \frac{-2}{x^2 + K} \]

Note: While this is the general solution, particular solutions will require more than just a choice of \( K \).
This function works since for \( y = \frac{-2}{x^2 + k} \)

\[ y' = \frac{2}{(x^2 + k)^2} \cdot 2x = x\left(\frac{2}{x^2 + k}\right)^2 = x\left(\frac{-2}{x^2 + k}\right)^2 = xy^2 \quad \checkmark \]

Notes

1. The LHS of (4) is interesting:

\[ \int \left(\frac{1}{h(y)} \cdot \frac{dy}{dx}\right) \, dx \]

is the antiderivative of \( \frac{1}{h(y)} \) as a function of \( t \) in the example,

\[ \int \frac{1}{y^2} \frac{dy}{dx} \, dx = \frac{-1}{y} + C. \]

To see this, rewrite LHS(4) using \( u \) as the dependent variable:

\[ \int \frac{1}{h(u(x))} \cdot \frac{du}{dx} \, dx \]

looks just like the integral one would find in a substitution problem:

Let \( y = u(x) \), \( dy = u'(x) \, dx = \frac{du}{dx} \, dx \)

then \( \int \frac{1}{h(u(x))} \frac{du}{dx} \, dx = \int \frac{1}{h(y)} \, dy \) and you can integrate wrt \( y \) directly,

In our example above:

\[ \int \frac{1}{y^2} \frac{dy}{dx} \, dx = \int \frac{1}{y^2} \, dy = \frac{-1}{y} + C. \]
Strictly speaking one does not simply cross out the \( dx \)'s. But it does look that way.

2. The book uses a slightly different formula:

\[
y' = \frac{dy}{dx} = f(x, y)
\]

Any \( y' = \frac{dy}{dx} = f(x, y) \) can be written

\[M(x, y) + N(x, y) \frac{dy}{dx} = 0\]

(think \( M = -f \) and \( N = 1 \), but this may sometimes not be the only way).

Now if \( M(x, y) = M(x) \) and \( N(x, y) = N(y) \)

we get \( M(x) + N(y) \frac{dy}{dx} = 0 \) or

\[N(y) \frac{dy}{dx} = M(x) \]

and \( \frac{dy}{dx} \) is separable.

3. In differential form

\[M(x) + N(y) \frac{dy}{dx} = 0\]

may be presented

\[M(x) dx + N(y) dy = 0\]

or

\[N(y) dy = -M(x) dx.\]
Integrating the differentials yields

\[-\int M(x) \, dx = \int N(y) \, dy\]

Ex. We may sometimes see \( y' = xy^2 \) as

\[ \frac{dy}{dx} = xy^2, \text{ or } \frac{dy}{dx} - xy^2 = 0, \text{ or } \frac{dy}{y^2} = x \, dx \]

The notation is different, but the ODE is the same.

4) Sometimes, a solution is known only implicitly:

Ex. in book: \( \frac{dy}{dx} = \frac{x^2}{1 - y^2} \)

Solution is \( \frac{-x^3}{3} + y - \frac{y^3}{3} = K \)

New curves are solutions to a general eqn with \( x \) and \( y \) (not necessarily a function).

\( K = 0 \) level set
Q: How does one use this information to find an explicit solution?

Q: What is the domain of the solution?

Q: How do we know which piece to pick?

For \( y' = \frac{x^2}{1-y^2} \), \( y(1) = 0 \), the solution is \( y(x) \)

where \(-\frac{x^2}{3} + y - \frac{y^3}{3} = 0\), but the function \( y(x) \)
is only defined up to the vertical tangent lines. Hence we have \( y^2 = 1 \), or \( y = \pm 1 \).

Here when \( y = 1 \), \(-\frac{x^2}{3} + 1 - \frac{1}{3} \Rightarrow x = \pm \sqrt{2} \).

\[ y = -1 \Rightarrow x = -\sqrt{2} \]

Caution: A solution to an ODE is a function (even when defined implicitly) that includes its domain!

\[ \frac{d}{dx} \left( \frac{-x^2}{3} + y - \frac{y^3}{3} = 0 \right) \text{ on interval} \]

\[ -\sqrt{2} < x < \sqrt{2} \text{. Not valid.} \]
Back to $y' = xy^2$ with its general solution $y(x) = \frac{-2}{x^2 + k}$. This is an example of a general solution.

But for an IVP, we will also need a domain for which the solution is continuous.

$\text{IVP: } y' = xy^2, \ y(0) = 1.$

Here, the particular solution has $k$-value $k = 2$.

But $y(x) = \frac{-2}{x^2 - 2}$ does not solve $y' = xy^2, \ y(0) = 1$.

Only the continuous piece that contains the initial value is the solution.

The solution to $y' = xy^2, \ y(0) = 1$ is
\[ y(x) = \frac{-2}{x^2 - 2} \text{ on } (-\sqrt{2}, \sqrt{2}) \text{ only}.\]

The solution to $y' = xy^2, \ y(2) = -1$ is $y(x) = \frac{-2}{x^2 - 2} \text{ on } (\sqrt{2}, \infty) \text{ only}.$

The proper domain is absolutely necessary to specifying a solution.
One more example.

example (ex 4 pg 37)

Solve the IVP \( t y' + 2ty = 4t^2 \) for

1. \( y(-1) = 1 \)
2. \( y(-1) = 2 \)
3. \( y(0) = 0 \)
4. \( y(0) = 1 \)

Here method of int. factors yields

\[ y(t) = t^2 + \frac{c}{t^2} \]

as the general soln.

1. \( y(-1) = 1 = (-1)^2 + \frac{c}{(-1)^2} \Rightarrow c = 0 \)
   \[ y(t) = t^2 \quad \text{for} \quad t \in (-\infty, \infty) \]

2. \( y(-1) = 2 = (-1)^2 + \frac{c}{(-1)^2} \Rightarrow c = 1 \)
   \[ y(t) = t^2 + \frac{1}{t^2} \quad \text{for} \quad t \in (-\infty, 0) \]

3. Cannot plus in 0. But the pt \((0,0)\) is on an integral curve of IVP. It is on the curve \( y = t^2 \) or \((-\infty, \infty)\).

4. No pt \( t=0, y=1 \) is not on any integral curve. No IVP \( ty' + 2ty = 4t^2, y(0)=1 \)
   has no solution. What gives??
The domain of the IVP solution in $\Box$ is $(-\infty, \infty)$.

The domain of IVP in $\Box$ is $(-\infty, 0)$.

(Though only 1 piece of $y(t) = t^2 + \frac{1}{t}$ is the one that includes the pt. $x$.)

$\Rightarrow$ Solution to IVP $ty' + 2y = 4t^2$, $y(0) = 1$ is

\[
y(t) = t^2 + \frac{1}{t^2} \text{ on } (-\infty, 0)
\]

Careful here.