New Structure Type: Autonomous

Suppose in \( y' = f(x, y) \), it is only a function of \( y \): \( y' = f(y) \)

Such an ode is separable, but \( \frac{1}{f(y)} \frac{dy}{dx} = 1 \) may still be hard to integrate.

An ode of the form \( y' = f(y) \) is called autonomous; \( t \) is not explicitly present in the equation.

\[ \begin{align*}
  \exists &\quad x = x^{\frac{1}{k}} \quad \text{if} \quad k \neq 0 \\
  y' &= ky \quad \text{if} \quad k \leq 0 \\
  \frac{dx}{dt} &= x(1-x) \quad (\text{Logistic eqn}).
\end{align*} \]

Here, even without solving, properties of autonomous odes allow for effective study.
Properties of autonomous $y' = f(y)$

1. Structure of slope field.
   - Slope field doesn't change in time direction.
   - Different $t$'s have same look.
   - Every vertical slice looks the same.
   - Every horizontal slice is an isocline - a curve along which all slopes of solution curves are the same.

2. Existence and uniqueness.
   - Since $f(y)$ is only a function of $y$,
     - Existence of solutions is assured when $f(y)$ is continuous.
     - Uniqueness of solutions is assured when $f'(y) = \frac{df}{dy} \neq 0$ is continuous.

(Here, $\frac{df}{dy}(t,y) = \frac{df}{dy} = f'(y)$ since $t$ does not occur as a variable in $f$).

Conclusion: No crossing of solutions where $f(y)$ and $f'(y)$ are defined.
3. Equilibrium Solutions

At any place \( y_0 \) where \( f(y_0) = 0 \), then \( y'(y) = 0 \) here, and thus \( y(t) = y_0 \) is a constant solution (or equilibrium, or steady-state solution).

Its graph is a horizontal line and is an isoline.

\[
\begin{align*}
\text{ex} & \quad z' = z(1-z) \\
\text{here} \quad z(t) & = 0 \quad \begin{cases} 
z' > 0 & \text{if } z < 0 \\
\text{or } & z > 1 \\
\text{or both} & \text{if } z = 1 \\
\text{equilibrium solutions} & z' = 0 \\
\end{cases}
\end{align*}
\]

And in between the equilibria, the sign of \( z' \) does not change. Hence solutions

\( \text{are trapped between equilibria, and} \)
\( \text{always travel in the same direction.} \)
In the example, we can say the following without solving:

1. Solutions exist and are unique everywhere (the ODE is a first order linear).

2. Equilibrium only at $x = 0$ and $x = 1$.

3. Any solution that passes through $0 < x_0 < 1$ will tend toward the equilibrium $x(t) = 1$.

   Any solution that starts at $x_0 < 0$ will tend to $-\infty$.

   Any solution that starts at $x_0 > 1$ will tend to $\infty$.

Here we can say \( \lim_{x \to x_0} x(t) = \begin{cases} 
1 & x_0 > 0 \\
0 & x_0 = 0 \\
-\infty & x_0 < 0
\end{cases} \)

4. Phase line

Any vertical slice through the slope field gives you all information about long term behavior of solutions.
Deb. For \( y' = f(y) \), the set \( \{ y \in \mathbb{R} | f(y) = 0 \} \) is the set of \underline{equilibrium solutions} for the ODE.

Phase line of \( z' = z(1-z) \).

Here, the phase line is a schematic that determines all long-term behavior of the autonomous \( z' = f(z) \).

\[ f(z) = z(1-z) \]
Critical pts (equilibrium solutions) can be classified by how solutions behave around them:

Let $y^*$ be a critical pt for $y' = f(y)$, and let $N_\epsilon(y^*) = \{ y \in \mathbb{R} \mid |y-y^*| < \epsilon \}$ be an $\epsilon$-neighborhood of $y^*$.

(a) If there is a $\epsilon > 0$ where for all $y \in N_\epsilon(y^*)$
\[ \lim_{t \to \infty} y(t) = y^* \implies y^* \text{ is asymptotically stable} \]

(b) If there is a $\epsilon > 0$ where for all $y \in N_\epsilon(y^*)$
\[ \lim_{t \to -\infty} y(t) = y^* \implies y^* \text{ is unstable} \]

(c) If asympt. stable on one side and unstable on the other, then $y^*$ is semi-stable.
ex. \( z' = z(1-z) \), here critical pts at \( z = 0, 1 \). And

the \( z(t) = 1 \) is asymptotically stable and \( z(t) = 0 \) is unstable.

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ex. \( y' = (1-y)^2(y-4) \)

Here, critical pts at \( y = 1, 4 \).

Phase line is
(check at in each interval formed by critical pts.

Graph of \( f(y) = (1-y)^2(y-4) \)

\[ \lim_{t \to \infty} y(t) = \begin{cases} 4 & y_0 > 4 \\ 4 & y_0 = 4 \\ 1 & 1 \leq y_0 < 4 \\ -\infty & y_0 < 1 \end{cases} \]