Consider the autonomous \( \dot{y} = f(a, y) \) where 

\[ \text{"a" is a parameter (an unknown constant).} \]

The number and classification of equilibrium may depend on the value of "a".

\[ \dot{y} = ay - y^2 = y/(a-y^2) \]

Here, for \( a < 0 \) and \( a > 0 \) the number of equilibrium are different (also the type).

\[ a = -1 < 0 \]

\[ a = 1 > 0 \]

Here \( f(1, y) = y/(1-y^2) = y/(1-y)(1+y) \)

and equilibrium exist at \( y = -1, 0, 1 \). \( y(0) \equiv 0 \) is unstable here.

It is asymptotically stable.
We can study how the parameter affects equilibrium via a bifurcation diagram: a graph of equilibrium in relation to parameter value in the ay-plane for $y' = f(x, y)$.

Properties:

- Each vertical slice here is the phase line of $y' = f(x, y)$ for that value of $\varepsilon$.

- As $\varepsilon$ varies, equilibrium track out curves of fixed pts. found by solving $f(x, y) = 0$.

- Special values of $\varepsilon$ where the number of equilibria and/or the stability change are called bifurcation values of $\varepsilon$.

- Here, $\varepsilon = \infty$ correspond to the solutions to $f(x, y) = y'(a-y^2) = 0$, or to $y = 0$, $x = y^2$, or $y = \pm \sqrt{a}$. 
• Here, the only bifurcation value of $a$ is $a = 0$.

• Here we use solid lines for all equilibria, and vertical arrows to denote stability. Red use solid for asymptotically stable curves, and dotted for unstable curves.

• This kind of bifurcation is called a pitchfork bifurcation... why?

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ex. $\dot{y} = a - y^2$

\begin{align*}
\text{Lines of equilibria:} \\
\text{solve } a = y^2 \\ 
y = \pm \sqrt{a}, y = -\sqrt{a} \\
\text{only for } a > 0
\end{align*}

Called a saddle-node or crotch bifurcation
example Basic model of a laser (simple)

\[ \dot{n} = (\alpha N_0 - k) n - \alpha n^2 \]

This models a basic simple laser, where

\( n(t) \) = # of photons at time \( t \)

\( N_0, k, \alpha \) are positive constants.

We study how the eqn is affected by \( N_0 \geq 0 \)

There are equilibrium at

\[ n (\alpha N_0 - k - \alpha n) = 0 \]

necessarily \( n = 0 \)

\( \Lambda = N_0 - \frac{k}{\alpha} \)

\[ \text{(i) when } N_0 < \frac{k}{\alpha}, \]

\( (\alpha N_0 - k) < 0, \)

so \( \dot{n} < 0 \).

\( n(t) = 0 \) is a sink.

\[ \text{(ii) when } N_0 > \frac{k}{\alpha}, \alpha N_0 - k > 0 \Rightarrow \text{ for small } n, \]

\( \alpha N_0 - k - \alpha n > 0, \) or \( N_0 - \frac{k}{\alpha} - n > 0, \) so

for small \( n, \dot{n} > 0 \), etc.
Change tack

Recall for any equation involving \( x, y \),

- we can bring all terms to one side of the equation and create an equivalent equation \( y(x, y) = 0 \)

for \( y(x, y) \) a function of 2 variables. Then the curve in the \( x,y \)-plane satisfying this equation is called the 0-level set of \( y \).

ex. \( y^2 = 1 - x^2 \). We view this eqn as the 0-level set of \( y \) and \( y(x, y) = x^2 + y^2 - 1 = 0 \).

- We can view \( y \) as an implicit function of \( x \).

In either case, the graph of the original equation (or the \( y(x, y) = 0 \)) is a curve in \( x,y \)-plane that in general will not look like a function.
We can calculate the tangent lines to this graph via differentiation in either interpretation.

\[ x^2 + xy^2 = 4, \text{ or } \Psi(x, y) = 0, \quad \Psi(x, y) = x^2 + xy^2 - 4 \]

**Implicit Diff**
\[
\frac{d}{dx} (x^2 + xy^2 = 4) \Rightarrow 2x + y^2 + 2xy \frac{dy}{dx} = 0
\]

**Calc III**
\[
\frac{dy}{dx} (x, y) = \frac{\frac{dy}{dx}}{\frac{dx}{dy}} = \frac{\frac{dy}{dx}}{\frac{dx}{dy}} y'
\]

when we think of \( y \) as an implicit func of \( x \).

Suppose a first-order ODE is of the form
\[
M(x, y) + N(x, y) y' = 0 \quad (*)
\]

Then (*) and (*) are the same under the condition that there exists a function \( \Psi(x, y) \), where

1. \( \frac{d\Psi}{dx}(x, y) = M(x, y) \)
2. \( \frac{d\Psi}{dy}(x, y) = N(x, y) \)

So that
\[
M(x, y) + N(x, y) y' = 0 = \frac{d\Psi}{dx} = \frac{d\Psi}{dx} + \frac{d\Psi}{dy} y'
\]
If this is the case, then the ODE \( y' = f(x) \) can be rewritten as \( \frac{dy}{dx} = 0 \), or
\[ y(x, y) = C, \text{ a constant}. \]
Thus, solving the ODE at least implicitly.

Existence.

Notice that \( 2x + y^2 + 2xy y' = 0 \) is of the form \( M(x, y) + N(x, y) y' = 0 \) with
\[ M(x, y) = 2x + y^2 \]
\[ N(x, y) = 2xy. \]

But we also can see that the function \( y(x, y) = x^2 + y^2 \) has the partials
\[ \frac{dy}{dx}(x, y) = 2x + y^2 \quad \frac{dy}{dy}(x, y) = 2xy. \]

Hence, \( 2x + y^2 + 2xy y' = 0 \) can be written
\[ \frac{dy}{dx} = 0 = \frac{1}{x^2 + 2xy^2} \]

16. We assume that \( y \) is an implicit function of \( x \).
Here we can (assuming \( y \) is an implicit function of \( x \)) integrate \( \frac{dy}{dx} (x, y) = 0 \) w.r.t \( x \) to get

\[ \int \frac{dy}{dx} (x, y(x)) \, dx = \int_0^x \, dx \]

\( y(x_0) = x^2 + xy^2 = C \)

This is the general implicit solution to

\[ 2x + y^2 + 2xy' = 0 \]

Q: How do we know such a \( y(x_0) \) may exist and if so how to find it?

Calc III: Let \( y(x, y) \) have continuous partial derivatives in some open region. Then

\[ \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial y}{\partial x} \right) \]

i.e. mixed partials are equal.