Answer all five questions. The first two questions are short answer. Fully justify your answer for the last three questions.

**Question 1.** (10 points; True/False). Every bounded sequence has a convergent subsequence; no proof is required.

**Question 2.** (10 points; Short answer). Give an example of a set that is both open and closed; no proof is required.

**Question 3.** (25 points). Suppose that $E$ is a non-empty compact set. Show that $\text{sup}(E)$ is contained in $E$.

**Question 4.** (25 points). Suppose that for each $\lambda$ in a set $\Lambda$, we have a positive real number $a_\lambda > 0$. Suppose also that for any natural number $n$ and any $\lambda_1, \ldots, \lambda_n \in \Lambda$ we have

$$\sum_{i=1}^{n} a_{\lambda_i} < 1.$$ 

Prove that the set $\Lambda$ is at most countable (i.e, is either countable or finite).

**Question 5.** (30 points). Suppose that a sequence $y_n$ is defined iteratively by $y_0 = 1$ and then

$$y_{n+1} = \frac{1}{2 + y_n}.$$ 

(a) Compute the next three terms $y_1$, $y_2$, and $y_3$.

(b) Prove that the sequence $y_n$ converges.