Harmonic maps from a simplicial complex and geometric rigidity

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Abstract

Abstract. We study harmonic maps from an admissible flat simplicial complex to a non-positively curved Riemannian manifold. These maps are shown to be $C^\infty$ at the interfaces of the top-dimensional simplices in addition to satisfying a balancing condition. If we assume that the domain is a 2-complex satisfying certain geometric and combinatoric conditions, then the regularity, the balancing condition, and a Bochner formula lead to rigidity and vanishing theorems for harmonic maps.

1 Introduction

Harmonic maps are critical points of the energy functional. The energy of a map \( \varphi : X \to Y \) between two spaces \( X \) and \( Y \) is defined to be the integral over the domain space of the energy density function which measures the total stretch of the map at each point of \( X \). In the case when \( X \) and \( Y \) are smooth Riemannian manifolds, the energy density function is the squared norm of the differential of the map.

One of the highlights of the harmonic map theory has been in its successful applications to study representation of discrete groups. Suppose \( \Gamma \) is a fundamental group of a manifold \( X \) acting on a space \( Y \) by \( \rho : \Gamma \to \text{Isom}(Y) \). The idea is to associate the action with an equivariant harmonic map \( \tilde{f} : \tilde{X} \to Y \) where \( \tilde{X} \) is the universal cover of \( X \). Once the existence is established, one can use the curvature assumptions on the domain and the target spaces to make strong statements about \( \tilde{f} \) and hence about the representation \( \rho \).

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To illustrate this, let $X$ be a compact Riemannian manifold of non-negative Ricci curvature and with fundamental group $\Gamma$ and $N$ be a complete Riemannian manifold of non-positive sectional curvature. Consider a representation $\rho : \Gamma \to \text{Isom}(N)$ and let $\tilde{f} : \tilde{X} \to N$ be a $\Gamma$–equivariant harmonic map. Such a map $\tilde{f}$ exists as long as the action $\rho$ does not fix an equivalence class of rays in $N$ and $N$ has negative sectional curvature or $N$ has nonpositive sectional curvature and is locally compact. The Eells-Sampson Bochner formula implies

$$\frac{1}{2} \Delta |\nabla \tilde{f}|^2 = |
abla d\tilde{f}|^2 + <d\tilde{f}(\text{Ric}_X(e_k)), d\tilde{f}(e_k)>_{\tilde{T}N}$$

$$- <R^N(d\tilde{f}(e_k), d\tilde{f}(e_l))d\tilde{f}(e_l), d\tilde{f}(e_k)>_{\tilde{T}N}$$

where $\text{Ric}_X$ and $R^N$ are the Ricci and sectional curvatures of $X$ and $N$ respectively. The right hand side of the equation above is non-negative. Furthermore, Stoke’s theorem says

$$\int_X \Delta |\nabla \tilde{f}|^2 = 0$$

from which we conclude that each term on the right hand side of equation (1) is also zero. In particular, $\nabla d\tilde{f} = 0$, i.e. the map $\tilde{f}$ is totally geodesic. The representation $\rho : \Gamma \to \text{Isom}(N)$ is then said to be rigid.

Further rigidity formulas were discovered by Siu, Corlette, and others; see for further examples [Si], [C], and [MSiY]. In the seminal work of Gromov-Schoen [GS] and subsequently Korevaar-Schoen [KS1] [KS2], the situation in which $Y$ is only a complete metric space rather than a smooth manifold was considered. This enabled them to prove super-rigidity of $p$-adic representations along the lines of [C].

In a different direction, one can ask if it is possible to allow the domain $X$ to be singular, for example a simplicial complex. This idea goes back to the work of Garland [G] and was subsequently elaborated by several groups of authors (cf. [BS], [Gr], [IN], [W1], [W2] and [Z]) in connection with Kazhdan property (T). For the nonlinear versions, the key idea is to define a combinatorial version of harmonic maps and relate them via a Bochner formula to a combinatorial analogue of curvature on $X$. This is in essence a refinement of Garland’s notion of $p$-adic curvature. (For a definition of combinatorial harmonic maps, see [J] and [W1],[W2].)
The actual notion of a harmonic map on a polyhedral domain, rather than its combinatorial counterpart, was first introduced in [Ch] and was further developed in [EF], [DM] and [M]. In the special case when $X$ is a flat $n$-dimensional simplicial complex, it was shown in [DM] and [M] that the harmonic map is Lipschitz across the edges. In the case of a 2-dimensional flat simplicial complex, the harmonic map has particular growth rate at the vertices depending on the order of the map. (See Section 2, Theorems 1 and 2 for precise statements of the results.)

The first goal of this paper is to further improve the regularity of harmonic maps defined on polyhedral domains. In particular, we show that the harmonic map must be smooth across the strict $(n-1)$-skeleton and satisfy a natural balancing condition. (See Theorem 5 and Corollary 6).

The second goal of the paper is to generalize the Bochner technique described earlier in the introduction to the case when $X$ is a simplicial complex. Here, we restrict to the case $\dim X = 2$ and introduce weights (see the beginning of Section 2 for a precise definition.) This imposes no restriction as far as applications to group theory is concerned. We show that under the assumption that $|\nabla \tilde{f}|$ is bounded, Stoke’s formula (2) still holds. By combining this with a simplex-wise Bochner formula (1), we obtain that $\tilde{f}$ is totally geodesic (see Theorem 8).

It therefore remains to establish the assumption on $X$ for which harmonic maps from $X$ are forced to satisfy the condition $|\nabla \tilde{f}| \leq C$. It is not hard to see that the latter condition is equivalent to the condition that $\text{ord}_p \tilde{f} \geq 1$ for all points $p$ in $X$, which in turn is equivalent to the condition that the first nonzero eigenvalue of the Laplacian of the link of $p$, $\text{Lk}(p)$, is $\geq 1$. Of course the $\text{Lk}(p)$ is a graph and we can easily relate the spectrum of the Laplacian on $\text{Lk}(p)$ to the spectrum of the discrete Laplacian (cf. Proposition 13 and Corollary 14). In particular, we show that the condition that the first nonzero eigenvalue being $> 1$ is equivalent to the first nonzero eigenvalue of the discrete Laplacian being $> \frac{1}{2}$ which is precisely the condition appearing in the combinatorial approach (cf. [BS], [IN], [W1], [W2], [Z]). This allows us to deduce the main theorem in [W1], that if $\Sigma^n$ is compact simplicial $n$-complex with admissible weight whose first nonzero eigenvalue of each link of a vertex is $> \frac{1}{2}$ then any isometric action of $\Gamma$ on a complete simply connected manifold of nonpositive sectional curvature has a fixed point, as a direct consequence of the Eells-Sampson Bochner formula for polyhedral domains. Furthermore, our approach enables us to prove rigidity in the borderline case when the eigenvalue of the discrete laplacian is $= \frac{1}{2}$. In
particular, this implies the existence of a fixed point in the case when \( \pi_1 \Sigma \) acts on a complete simply connected negatively curved manifold and the first nonzero eigenvalue of each link of a vertex of \( \Sigma \) is \( \geq \frac{1}{2} \).

We now turn to the organization of the paper. Throughout the rest of the paper, \( X \) will be a compact, admissible and flat simplicial complex and \( (N, g) \) a complete Riemannian manifold. Section 2 is a review of the results in [DM] about the existence and regularity of harmonic maps. The only difference in this paper is that our maps depend on weights \( w(F) \) associated to the \( n \)-simplices \( F \) of \( X \). This slight modification, which from the point of view of the analysis amounts to taking certain weighted Sobolev spaces associated to the complex, is necessary in order to cover all \( p \)-adic buildings (cf. for example [SW]). We therefore talk about the \( w \)-energy of a map or \( w \)-harmonic maps, but this imposes no real analytical difficulty. Section 4 contains our main regularity result and the balancing condition. More precisely, we show:

**Theorem** (cf. Theorem 5) Let \( f : X \to (N^m, g) \) be a \( w \)-harmonic map. For any point \( p \in X^{(n-1)} - X^{(n-2)} \), there is a neighborhood \( \Omega \) of \( p \) so that for any (closed) \( n \)-dimensional simplex \( F \), the restriction of \( f \) to \( F \cap \Omega \) is a \( C^\infty \) map.

**Theorem** (cf. Corollary 6) Let \( f : X \to (N^m, g) \) be a \( w \)-harmonic map. For any point \( p \in X^{(n-1)} - X^{(n-2)} \), let \( F_1, \ldots, F_J \) be the \( n \)-simplices containing \( p \) and let \( E = \cap_{j=1}^J F_j \). Choose coordinates \((x^1, \ldots, x^n)\) on \( X \) near \( p \) so that \( E \) corresponds to the equation \( x^n = 0 \) and coordinates \((y^1, \ldots, y^m)\) on \( N \) near \( f(p) \). Set \( f^n_j = y^a \circ f|_{F_j} \). Then

\[
\sum_{j=1}^J w(F_j) \frac{\partial f^n_j}{\partial x^n}(x^1, \ldots, x^{n-1}, 0) = 0
\]

where \( w(F_j) \) are the weights associated to \( F_j \).

We also prove:

**Theorem** (cf. Theorem 8) Let \( f : X \to (N, g) \) be a \( w \)-harmonic map where \( \dim X = 2 \), \( N \) has nonpositive sectional curvature and \( |\nabla f| \) is a bounded function. Then \( f \) is totally geodesic on each simplex of \( X \). Furthermore, if \( N \) has negative sectional curvature, then \( f \) is a constant map.
Finally in section 4, we relate the question of regularity of the harmonic map with spectral theory of graphs. In particular, we show:

**Theorem** (cf. Theorem 12) Suppose that $X$ is a 2-complex such that every nonzero eigenvalue of the link of every vertex in $X$ satisfies $\lambda \geq 1$. If $f : X \rightarrow (N, g)$ is a $w$-harmonic map into a complete Riemannian manifold of nonpositive sectional curvature, then $f$ is totally geodesic on each 2-simplex of $X$. In particular, this implies that if the sectional curvature of $N$ is negative, then $f$ is a constant map. If the eigenvalues satisfy the stronger condition $\lambda > 1$ then $f$ is a constant map.

We also establish the equivalence of the eigenvalue condition in Theorem 12 with the one appearing in the combinatorial approach. More precisely, we show:

**Theorem** (cf. Corollary 14) The condition $\lambda \geq (>)1$ in the previous theorem is equivalent to the condition that the first nonzero eigenvalue of the discrete Laplacian being $\geq (>)\frac{1}{2}$.

By taking $(\Sigma^n, c)$ an arbitrary compact $n$-dimensional simplicial complex, by reducing the weights $c$ to its 2-skeleton $X = \Sigma^{(2)}$ and by applying the previous two theorems on $X$, we immediately obtain the main theorem in [W1].

**Corollary** (cf. Corollary 15). Let $(\Sigma^n, c)$ be a compact simplicial complex with admissible weights. Assume that the first nonzero eigenvalue of the link of every vertex is $> \frac{1}{2}$. Then $\pi_1(\Sigma) = \Gamma$ has property $F$; i.e. any isometric action of $\Gamma$ on a complete, simply connected manifold of nonpositive sectional curvature has a fixed point.

We also have the following extension:

**Corollary** Let $(\Sigma^n, c)$ as above and assume that the first nonzero eigenvalue of the link of every vertex is $\geq \frac{1}{2}$. Then any isometric action of $\pi_1(\Sigma) = \Gamma$ on a complete, simply connected manifold of negative sectional curvature has a fixed point.

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2 Definitions and known results

A simplicial complex of dimension \( n \) is referred to as a \( n \)-complex. A connected locally finite \( n \)-complex is called admissible (cf. [Ch] and [EF]) if the following two conditions hold:

(i) \( X \) is dimensionally homogeneous, i.e., every simplex is contained in a \( n \)-simplex, and

(ii) \( X \) is \((n - 1)\)-chainable, i.e., every two \( n \)-simplices \( A \) and \( B \) can be joined by a sequence \( A = F_0, e_0, F_1, e_1, \ldots, F_{k-1}, e_{k-1}, F_k = B \) where \( F_i \) is a \( n \)-simplex and \( e_i \) is a \((n - 1)\)-simplex contained in \( F_i \) and \( F_{i+1} \).

The boundary \( \partial X \) of \( X \) is the union of all simplices of dimension \( n - 1 \) which is contained in only one \( n \) dimensional simplex. We call a \( n \)-complex flat if for each \( k \)-simplex \( F \) is isometric to the convex hull of \( k + 1 \) equidistant points of distance 1 in \( \mathbb{R}^k \) and every \( l \)-simplex \( L \) incident to a \( k \)-simplex \( K \) \((l < k)\) can be seen as a totally geodesic subset of \( \bar{K} \). In the sequel, all complexes are admissible, flat, compact and without boundary. An isometric action of a group \( \Gamma \) is a homomorphism \( \rho : \Gamma \to \text{Isom}(\mathbb{R}^k) \). Let \( \Gamma = \pi_1(X) \). A map \( \tilde{\varphi} : \tilde{X} \to N \) is said to be equivariant if

\[
\rho(\gamma) \tilde{\varphi}(p) = \tilde{\varphi}(\gamma p)
\]

for \( \gamma \in \Gamma \) and \( p \in X \). If \( \Gamma \) acts freely and properly discontinuously on \( N \), then the map \( \tilde{\varphi} \) is a lift of the map \( \varphi : X \to N/\Gamma \). By identifying \( X \) with a fundamental domain of \( \tilde{X} \), we can think of \( \tilde{\varphi} \) also being defined on \( X \).

In order to include certain important examples appearing in \( p \)-adic geometry (e.g. \( p \)-adic buildings), we will assume that for each \( n \)-dimensional simplex \( F \) in \( X \), we have an associated weight \( w(F) > 0 \) and we define the \( w \)-measure \( d\mu_w \) by setting

\[
d\mu_w = w(F)dx
\]

where \( dx \) is the standard Lebesgue measure on \( F \). We define the \( w \)-energy \( E_w(\tilde{\varphi}) \) of a map \( \tilde{\varphi} : \tilde{X} \to (N, g) \) as

\[
E_w(\tilde{\varphi}) = \int_X |\nabla \tilde{\varphi}|^2 d\mu_w = \sum_F w(F) \int_F |\nabla \tilde{\varphi}|^2 dx
\]
where $\sum_F$ indicates the sum over all $n$-dimensional simplices $F$ of $X$ and $|\nabla \tilde{\varphi}|^2$ is defined as usual; i.e.

$$|\nabla \tilde{\varphi}|^2 = \sum_{k=1}^n g\left(\frac{\partial \tilde{\varphi}}{\partial x^k}, \frac{\partial \tilde{\varphi}}{\partial x^k}\right).$$

Of course, if $w(F) = 1$ for all $F$, then we recover the usual notion of harmonicity defined in [DM]. For the sake of notational simplicity, we will fix weights $w(F)$ on $F$ and we will denote $d\mu = d\mu_w$, $E = E_w$, etc.

A map $\tilde{f} : \tilde{X} \to N$ is said to be $w$-harmonic if $E(\tilde{f}) \leq E(\tilde{\varphi})$ for all equivariant maps $\tilde{\varphi} : \tilde{X} \to N$.

The following existence and regularity results for $w$-harmonic maps from a 2-complex into a non-positively curved metric space follows by minor modification of the arguments presented in [DM] and [M]. (In [DM] and [M], we only considered weight function $w$ so that $w(F) = 1$ for all 2-simplices $F$ of $X$.)

**Theorem 1** Let $X$ be a 2-complex with $\Gamma = \pi_1(X)$, $Y$ be a complete metric space of non-positive curvature and $\rho : \Gamma \to \text{Isom}(Y)$ be an isometric action of $\Gamma$. Assume that $\rho$ does not fix an equivalent class of rays. If sectional curvature of $N$ is negative or if the sectional curvature of $N$ is nonpositive and $N$ is locally compact, then there exists an equivariant $w$-harmonic map $\tilde{f} : X \to Y$.

**Theorem 2** Let $X$ be a 2-complex, $Y$ a complete metric space of non-positive curvature and $f : X \to Y$ a $w$-harmonic map. Then $f$ is Lipschitz continuous away from the 0-simplices of $X$ with the Lipschitz bound dependent only on the total $w$-energy of $f$ and the distance to the 0-simplices. Let $p$ be a 0-simplex and $q$ be the order of $f$ at $p$. (The definition of order is given in Section 4.) Then there exists $\sigma > 0$ so that

$$|\nabla f|^2(q) \leq Cr^{2q-2}$$

for all $q \in B_{\sigma}(p)$ where $C$ depends on $E(f)$ and $r = d_X(p,q)$. More generally, if $X$ is a $n$-complex and $Y$ and $f$ are as above, then $f$ is Lipschitz continuous away from the $(n-2)$-simplices.

### 3 Regularity results

Let $X$ be a $n$-complex and $N$ a complete Riemannian manifold as above. For $0 \leq k \leq n$, let $X^{(k)}$ denote the $k$-skeleton of $X$. Given $p \in X^{(n-1)}$ –
\(X^{(n-2)}\), choose an \((n-1)\)-simplex \(E\) containing \(p\) and let \(P \in U \mapsto (y^1, \ldots, y^m) \in \mathbb{R}^m\) be a local coordinate system of a neighborhood \(U\) of \(f(p)\). Assume \(g\) is given by \((g_{\alpha\beta})\) in terms of this coordinate system. Choose a neighborhood \(V \subset X\) of \(p\) sufficiently small so that \(V\) does not intersect any \((n-2)\)-simplex and \(f(V)\) is compactly contained in this coordinate patch. If \(F_1, \ldots, F_J\) are the \(n\)-simplices of \(X\) intersecting \(V\) and \(E\) is a \(\epsilon\)-neighborhood of \(E\), we will use the coordinate system \(q \in V \cap (F_j \cup E) \mapsto (x^1, \ldots, x^n) \in \mathbb{R}^n\) for \(j = 1, \ldots, J\) so that a point in \(E\) is given by \((x^1, \ldots, x^{n-1}, 0)\) and a point in \(V \cap E\) is given by \((x^1, \ldots, x^n)\) with \(0 \leq x^n < \epsilon\). Let \(E_\epsilon\) be the \(\epsilon\)-neighborhood of \(E\). Furthermore, for a map \(f : V \to N\),

\[
\begin{align*}
  f(x^1, \ldots, x^n) &= (f^1(x^1, \ldots, x^n), \ldots, f^m(x^1, \ldots, x^n))
\end{align*}
\]

in \(V\) and let \(f^\alpha_j = f^{\alpha|F_j}\) for \(\alpha = 1, \ldots, m\) and \(j = 1, \ldots, J\).

**Theorem 3** Let \(F_1, \ldots, F_J, E, E_\epsilon, (x^1, \ldots, x^n), \) and \((f^1, \ldots, f^m)\) as above. If \(f : V \to N\) is a harmonic map, then for any Lipschitz function \(\eta : V \to \mathbb{R}\) with compact support and any \(\alpha = 1, \ldots, m\),

\[
\begin{align*}
  \lim_{\epsilon \to 0} \sum_{j=1}^J w(F_j) \int_{F_j \cap \partial E_\epsilon} \eta(x_1^1, \ldots, x_1^{n-1}, \epsilon) \frac{\partial f^\alpha_j}{\partial x^k}(x_1^1, \ldots, x_1^{n-1}, \epsilon) dx^1 \cdots dx^{n-1} = 0.
\end{align*}
\]

**Proof.** Let \(\varphi = (\varphi^1, \ldots, \varphi^m) : V \subset X \to \mathbb{R}^m\) be a Lipschitz continuous map with compact support. For sufficiently small \(t\), define \(f_t : V \subset X \to U\) by setting

\[
\begin{align*}
  f_t = f + t\varphi = (f^1 + t\varphi^1, \ldots, f^m + t\varphi^m).
\end{align*}
\]

The \(w\)-energy of \(f_t\) in \(V\) is

\[
\begin{align*}
  E(f_t; V) &\quad= \sum_{k=1}^n \sum_{\alpha, \beta=1}^m \int_V g_{\alpha\beta}(f(x) + t\varphi(x)) \left( \frac{\partial f^\alpha}{\partial x^k} + t \frac{\partial \varphi^\alpha}{\partial x^k} \right) \left( \frac{\partial f^\beta}{\partial x^k} + t \frac{\partial \varphi^\beta}{\partial x^k} \right) d\mu
\end{align*}
\]

and since \(f = f_0\) is \(w\)-energy minimizing,

\[
\begin{align*}
  0 &\quad= \frac{d}{dt} E(f_t; V)|_{t=0} \\
 &\quad= 2 \sum_{k=1}^n \sum_{\alpha, \beta=1}^m \int_V g_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \frac{\partial \varphi^\beta}{\partial x^k}
\end{align*}
\]
\begin{align*}
\int_{\mathcal{V}} \sum_{k=1}^{n} \sum_{\alpha,\beta,\gamma=1}^{m} g_{\alpha\beta,\gamma}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \frac{\partial f^\beta}{\partial x^k} \varphi^\gamma \, d\mu \\
= 2 \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_{\mathcal{V}} \frac{d}{dx^k} \left( g_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \varphi^\beta \right) \\
- 2 \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_{\mathcal{V}} g_{\alpha\beta}(f(x)) \frac{\partial^2 f^\alpha}{\partial(x^k)^2} \varphi^\beta \, d\mu \\
- 2 \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_{\mathcal{V}} g_{\alpha\beta,\gamma}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \frac{\partial f^\gamma}{\partial x^k} \varphi^\beta \, d\mu \\
+ \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_{\mathcal{V}} g_{\alpha\beta,\gamma}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \frac{\partial f^\beta}{\partial x^k} \varphi^\gamma \, d\mu \\
= 2 \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_{\mathcal{V}} \frac{d}{dx^k} \left( g_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \varphi^\beta \right) \, d\mu \\
- 2 \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_{\mathcal{V}} g_{\alpha\beta}(f(x)) \frac{\partial^2 f^\alpha}{\partial(x^k)^2} \varphi^\beta \, d\mu \\
- \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_{\mathcal{V}} g_{\gamma\beta,\alpha}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \frac{\partial f^\gamma}{\partial x^k} \varphi^\beta \, d\mu
\end{align*}

Let \( \eta_\alpha = \sum_{\beta=1}^{m} g_{\alpha\beta} \varphi^\beta \). Then \( \varphi^\beta = \sum_{\alpha=1}^{m} g^{\alpha\beta} \eta_\alpha \) and we see that the last two terms above equal

\begin{align*}
- \int_{\mathcal{V}} \sum_{\alpha=1}^{m} 2 \Delta f^\alpha \eta_\alpha + \sum_{k=1}^{n} \sum_{\alpha,\beta,\gamma,\delta=1}^{m} g^{\delta\beta} (g_{\alpha\beta,\gamma} + g_{\gamma\beta,\alpha} - g_{\alpha\gamma,\beta}) \frac{\partial f^\alpha}{\partial x^k} \frac{\partial f^\gamma}{\partial x^k} \varphi^\delta \, d\mu
\end{align*}
\[-2\int_V \left( \sum_{\alpha=1}^n \triangle f^\alpha \eta_\alpha + 2 \sum_{\alpha,\gamma,\delta=1}^n \Gamma^\alpha_{\alpha\gamma}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \frac{\partial f^\gamma}{\partial x^k} \eta_\delta \right) d\mu = 0.\]

The last equality to zero is because \( f \) is a smooth harmonic map in the interior of each \( n \)-simplex \( F \) and we have the pointwise equality,

\[
\triangle f^\alpha + \sum_{\beta,\gamma=1}^n \Gamma^\alpha_{\beta\gamma}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \frac{\partial f^\gamma}{\partial x^k} = 0
\]

in \( F_j \cap V \). Therefore, by the monotone convergence theorem and the fact that \( f \) is Lipschitz away from the \((n-2)\)-skeleton by Theorem 2, we conclude

\[
0 = \sum_{k=1}^n \sum_{\alpha,\beta=1}^m \int_{V} \frac{d}{dx^k} \left( g_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \varphi^\beta(x) \right) d\mu
\]

\[
= \lim_{\epsilon \to 0} \sum_{k=1}^n \sum_{\alpha,\beta=1}^m \int_{V-E_\epsilon} \frac{d}{dx^k} \left( g_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \varphi^\beta(x) \right) d\mu
\]

\[
= \lim_{\epsilon \to 0} \sum_{j=1}^J w(F_j) \sum_{k=1}^n \sum_{\alpha,\beta=1}^m \int_{F_j-E_\epsilon} \frac{d}{dx^k} \left( g_{\alpha\beta}(f_j(x)) \frac{\partial f_j^\alpha}{\partial x^k} \varphi^\beta(x) \right) dx.
\]

Since \( \varphi^\beta \) has compact support in \( V \),

\[
\sum_{\alpha,\beta=1}^m \int_{F_j-E_\epsilon} \frac{d}{dx^k} \left( g_{\alpha\beta}(f_j(x)) \frac{\partial f_j^\alpha}{\partial x^k} \varphi^\beta(x) \right) dx = 0
\]

for \( k = 1, \ldots, n-1 \) and \( j = 1, \ldots, J \). Hence,

\[
0 = \lim_{\epsilon \to 0} \sum_{j=1}^J w(F_j) \sum_{\alpha,\beta=1}^m \int_{F_j-E_\epsilon} \frac{d}{dx^k} \left( g_{\alpha\beta}(f_j(x)) \frac{\partial f_j^\alpha}{\partial x^k} \varphi^\beta(x) \right) dx
\]

\[
= \lim_{\epsilon \to 0} \sum_{j=1}^J w(F_j) \sum_{\alpha,\beta=1}^m \int_{F_j \cap \partial E_\epsilon} g_{\alpha\beta}(f_j(x^1, \ldots, x^{n-1}, \epsilon)) \frac{\partial f_j^\alpha}{\partial x^n} \varphi^\beta(x^1, \ldots, x^{n-1}, \epsilon) dx.
\]

Let \( \eta : V \to \mathbb{R} \) be a Lipschitz continuous compactly supported function and \( \varphi^\beta = g^{\beta\alpha_0} \eta \) for \( \alpha_0, \beta \in \{1, \ldots, m\} \). Then

\[
\sum_{\alpha,\beta=1}^m g_{\alpha\beta} \frac{\partial f_j^\alpha}{\partial x^n} \varphi^\beta = \sum_{\alpha,\beta=1}^m g_{\alpha\beta} g^{\beta\alpha_0} \eta \frac{\partial f_j^\alpha}{\partial x^n}
\]

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\[
\sum_{\alpha=1}^{m} \delta_{\alpha \alpha} \eta \frac{\partial f^\alpha_j}{\partial x^n} = \eta \frac{\partial f^\alpha_0}{\partial x^n}
\]
and therefore

\[
\lim_{\varepsilon \to 0} \sum_{j=1}^{J} w(F_j) \int_{\partial \mathcal{E}_\varepsilon} \eta(x^1, ..., x^{n-1}, \varepsilon) \frac{\partial f^\alpha_0}{\partial x^n}(x^1, ..., x^{n-1}, \varepsilon) dx = 0.
\]

Q.E.D.

We now study the weak solution \( f \) of \( \triangle f = F \) for \( L^2 \) function \( F \) in \( n \)-complex near a \((n-1)\)-simplex and away from the \((n-2)\)-simplices. Below, we show that most second order derivatives of the weak solution are locally \( L^2 \). The proof is adapted to our situation from the standard argument for solutions of second order elliptic equations (cf. [GT]).

**Proposition 4** Let \( p \in X^{(n-1)} - X^{(n-2)} \), \( V \) and \((x^1, ..., x^n)\) be as in the paragraph preceding Theorem 3. If \( f, F : V \to \mathbb{R} \) are functions so that \( f \) is \( W^{1,2}(V) \), \( F \in L^2(V) \), and

\[
- \int_V \nabla f \cdot \nabla \varphi d\mu = \int_V F \varphi d\mu
\]

for any \( \varphi \in W^{1,2}(V) \), then \( \frac{\partial f}{\partial x_k} \in W^{1,2}(\Omega) \) for \( k = 1, ..., n-1 \) and any \( \Omega \subset V \) and

\[
\sum_{k=1}^{n-1} \int_{\Omega} \left| \frac{\partial^2 f}{\partial x_k \partial x^l} \right|^2 d\mu \leq C \left( \int_V F^2 d\mu + \int_V |\nabla f|^2 d\mu \right)
\]

for \( k = 1, ..., n-1 \) and \( l = 1, ..., n \) where \( C \) depends only on \( \Omega \).

**Proof.** For any \( \varphi \in W^{1,2}(V) \), let

\[
\circ_h \varphi(x^1, ..., x^n) = \frac{\varphi(x^1, ..., x^{k-1}, x^k + \xi, x^{k+1}, ..., x^n) - \varphi(x^1, ..., x^n)}{h}
\]

for \( k = 1, ..., n-1 \). Note that \( \circ_h \varphi \) cannot be defined for \( k = n \), \((x^1, ..., x^n)\) with \( x^n < |h| \) and \( h > 0 \). By the fundamental theorem of calculus,

\[
\circ_h \varphi = \frac{1}{h} \int_0^h \frac{\partial}{\partial x_k} \varphi(x^1, ..., x^{k-1}, \xi, x^{k+1}, ..., x^n) d\xi
\]
and, by Jensen’s inequality,
\[ |\phi_h \varphi|^2 \leq \frac{1}{h} \int_0^h \left| \frac{\partial \varphi}{\partial x^k}(x^1, \ldots, \xi^{k-1}, \xi, x^{k+1}, \ldots, x^{n-1}, \xi) \right|^2 d\xi. \]

For \( \Omega \subset \subset V \), let \( \Omega_h \) be the \( h \)-neighborhood of \( \Omega \). Integrating the above inequality over \( \Omega \), we obtain,
\[
\int_\Omega |\phi_h \varphi|^2 d\mu \leq \frac{1}{h} \int_0^h \int_{\Omega_h} \left| \frac{\partial \varphi}{\partial x^k} \right|^2 d\mu d\xi \leq \int_{\Omega_h} \left| \frac{\partial \varphi}{\partial x^k} \right|^2 d\mu.
\]

Thus, if \( \varphi \) has compact support, then
\[
\int_V |\phi_h \varphi|^2 d\mu \leq \int_V \left| \frac{\partial \varphi}{\partial x^k} \right|^2 d\mu.
\]
(3)

For \( g \in W^{1,2}(V) \) with compact support, we have
\[
\int_V \nabla (\phi_h f) \cdot \nabla g d\mu = - \int_V \nabla f \cdot \nabla (\phi_h g) d\mu = \int_V F(\phi_h g) d\mu \leq \frac{1}{2\epsilon} \int_V F^2 d\mu + \epsilon \int_V |\phi_h g|^2 d\mu \leq \frac{1}{2\epsilon} \int_V F^2 d\mu + \epsilon \int_V \left| \frac{\partial g}{\partial x^k} \right|^2 d\mu
\]
by inequality (3).

Let \( \eta : V \to \mathbb{R} \) be a smooth function so that \( 0 \leq \eta \leq 1 \), \( \eta \) is identically equal to 1 in \( \Omega \), equal to 0 in a neighborhood of \( \partial V \), and \( |\nabla \eta| \leq C \). Set \( g = \eta^2 (\phi_h f) \). Then
\[
\frac{\epsilon}{2} \int_V \left| \frac{\partial g}{\partial x^k} \right|^2 d\mu = \frac{\epsilon}{2} \int_V \left| 2\eta \frac{\partial \eta}{\partial x^k} (\phi_h f) + \eta^2 (\phi_h \frac{\partial f}{\partial x^k}) \right|^2 d\mu \leq 4\epsilon \int_V \eta^2 \left| \frac{\partial \eta}{\partial x^k} \right|^2 d\mu + \epsilon \int_V \eta^2 |\phi_h \frac{\partial f}{\partial x^k}|^2 d\mu \leq 4\epsilon \int_V \eta^2 |\nabla \eta|^2 (\phi_h f)^2 d\mu + \epsilon \int_V \eta^2 |\nabla (\phi_h f)|^2 d\mu
\]
and hence
\[
\int_V \nabla (\phi_h f) \cdot \nabla g d\mu \leq \frac{1}{2\epsilon} \int_V F^2 d\mu + 4\epsilon \int_V \eta^2 |\nabla \eta|^2 (\phi_h f)^2 d\mu + \epsilon \int_V \eta^2 |\nabla (\phi_h f)|^2 d\mu. \quad (4)
\]
Furthermore,
\[- \int_V \nabla (\phi_h f) \cdot \nabla g \, d\mu = - \int_V \nabla (\phi_h f) \cdot \nabla (\eta^2 \phi_h f) \, d\mu\]
\[= - \int_V \eta^2 |\nabla (\phi_h f)|^2 + 2 \eta (\phi_h f) \nabla (\phi_h f) \cdot \nabla \eta \, d\mu.\]

Thus,
\[\int_V \eta^2 |\nabla (\phi_h f)|^2 \, d\mu = \int_V \nabla (\phi_h f) \cdot \nabla g \, d\mu + 2 \int_V \eta (\phi_h f) \nabla (\phi_h f) \cdot \nabla \eta \, d\mu\]
\[\leq \int_V \nabla (\phi_h f) \cdot \nabla g \, d\mu + \epsilon \int_V \eta^2 |\nabla (\phi_h f)|^2 \, d\mu + \frac{1}{\epsilon} \int_V (\phi_h f)^2 |\nabla \eta|^2 \, d\mu\]
\[\leq \frac{1}{2\epsilon} \int_V F^2 \, d\mu + 4 \epsilon \int_V \eta^2 |\nabla \eta|^2 (\phi_h f)^2 \, d\mu\]
\[+ 2 \epsilon \int_V \eta^2 |\nabla (\phi_h f)|^2 \, d\mu + \frac{1}{\epsilon} \int_V (\phi_h f)^2 |\nabla \eta|^2 \, d\mu\]
where the second inequality follows from inequality (4). This implies
\[(1 - 2\epsilon) \int_V \eta^2 |\nabla (\phi_h f)|^2 \, d\mu \leq \frac{1}{2\epsilon} \int_V F^2 \, d\mu + (4 \epsilon + \frac{1}{\epsilon}) \int_V (\phi_h f)^2 |\nabla \eta|^2 \, d\mu.\]

Fix \( \Omega = \{\eta = 1\} \) and \( \epsilon \) sufficiently small. Then
\[\int_\Omega |\nabla (\phi_h f)|^2 \, d\mu \leq \int_V \eta^2 |\nabla (\phi_h f)|^2 \, d\mu\]
\[\leq C_1 \int_V F^2 \, d\mu + C_2 \int_V |\nabla \eta|^2 (\phi_h f)^2 \, d\mu\]
\[\leq C_1 \int_V F^2 \, d\mu + C'_2 \int_V |\nabla f|^2 \, d\mu \quad (5)\]
where \( C_1, C'_2 \) depends on \( \eta \) and the second inequality follows from inequality (3). By the weak compactness of bounded sets in \( L^2(V \cap F_j) \), there exists \( G = (G^1, ..., G^n) \) with \( G^l_j = G^l|_{V \cap F_j} \in L^2(V \cap F_j) \) for \( l = 1, ..., n \) and \( j = 1, ..., J \) so that
\[\lim_{h \to 0} \int_V \varphi \cdot \nabla (\phi_h f) \, d\mu = \lim_{h \to 0} \sum_{j=1}^J w(F_j) \int_{V \cap F_j} \varphi \cdot \nabla (\phi_h f) \, dx\]
\[= \sum_{j=1}^J w(F_j) \int_{F_j \cap V} \varphi \cdot G_j \, dx\]
\[= \int_V \varphi \cdot G \, d\mu\]
for any $\varphi = (\varphi^1, ..., \varphi^n)$ with $\varphi^l \in C^\infty_c(V \cap F_j)$. Additionally, by the dominated convergence theorem,

$$\lim_{h \to 0} \int_V \varphi \cdot \nabla(\phi_h f) d\mu = - \lim_{h \to 0} \int_V (\phi_{-h} \varphi) \cdot \nabla f d\mu = - \int_V \frac{\partial \varphi}{\partial x^k} \cdot \nabla f d\mu,$$

hence

$$\int_V \varphi \cdot G d\mu = - \int_V \frac{\partial \varphi}{\partial x^k} \cdot \nabla f d\mu.$$ 

Thus,

$$\frac{\partial^2 f_j}{\partial x^k \partial x^l} = G^l_j \in L^2(V \cap F_j)$$

for $k = 1, ..., n - 1$ and $l = 1, ..., n$ which implies $\frac{\partial f}{\partial x^k} \in W^{1,2}(V)$. Furthermore, equation (5) and the lower semicontinuity of the $L^2$ norm under a weak convergence implies

$$\sum_{k=2}^{n-1} \int_{\Omega} \left| \frac{\partial^2 f}{\partial x^k \partial x^l} \right|^2 d\mu \leq C_1 \int_V F^2 d\mu + C_2 \int_V |\nabla f|^2 d\mu$$

where $C_1, C_2$ depend only on $\Omega$. Q.E.D.

**Theorem 5** Let $f : \mathbb{C} \times X \to (\mathbb{N}^n, g)$ be a $w$-harmonic map. For any point $p \in V \cap X^{(n-1)} - X^{(n-2)}$, there is a neighborhood $\hat{\Omega} \subset \subset V$ of $p$ so that for any $n$-simplex $F$, the restriction $f$ to the closure of $F \cap \hat{\Omega}$ is a $C^\infty$ map.

**Proof.** We use the notation $E, E_\epsilon, (x^1, ..., x^n)$, and $(f^1, ..., f^m)$ given in the paragraph before the statement of Theorem 3 and $\phi_h$ given in the proof of Proposition 4. Since $f$ is harmonic on each $n$-simplex $F$, $f$ is smooth and satisfies

$$\Delta f^\alpha + \sum_{l=1}^{n} \sum_{\beta,\gamma=1}^{m} \Gamma^\alpha_{\beta \gamma}(f(x)) \frac{\partial f^\beta}{\partial x^l} \frac{\partial f^\gamma}{\partial x^l} = 0 \tag{6}$$

in the interior of $F$. Let $\varphi$ be a Lipschitz function with compact support in $V$. Since $f$ is Lipschitz, equation (6) implies $\varphi \Delta f^\alpha$ is $L^1$. Thus, by the dominated convergence theorem,

$$\int_V \varphi \Delta f^\alpha d\mu = \lim_{\epsilon \to 0} \int_{V - E_\epsilon} \varphi \Delta f^\alpha d\mu$$

$$= \lim_{\epsilon \to 0} \sum_{k=1}^{n} \int_{V - E_\epsilon} \frac{\partial}{\partial x^k} (\varphi \frac{\partial f^\alpha}{\partial x^k}) d\mu - \lim_{\epsilon \to 0} \int_{V - E_\epsilon} \nabla \varphi \cdot \nabla f^\alpha d\mu.$$
Again, since $\varphi$ is compactly supported in $V$,

$$
\int_{V_{\varepsilon}} \frac{\partial}{\partial x^k}(\varphi \frac{\partial f^\alpha}{\partial x^k}) d\mu = 0
$$

for $k = 1, \ldots, n - 1$. Hence,

$$
\int_{V} \varphi \triangle f^\alpha d\mu = \lim_{\varepsilon \to 0} \sum_{j=1}^{J} w(F_j) \int_{F_j \cap \partial \varepsilon} \varphi \frac{\partial f^\alpha}{\partial x^k}(x^1, \ldots, x^{n-1}, \varepsilon) dx^1 \ldots dx^n
$$

$$
- \int_{V} \nabla \varphi \cdot \nabla f^\alpha d\mu
$$

$$
= 0 - \int_{V} \nabla \varphi \cdot \nabla f^\alpha d\mu
$$

where the last equality follows from Theorem 3. Therefore,

$$
- \int_{V} \nabla \varphi \cdot \nabla f^\alpha d\mu = - \sum_{l=1}^{n} \sum_{\beta, \gamma = 1}^{m} \int_{V} \Gamma_{\beta \gamma}^\alpha(f(x)) \frac{\partial f^\beta}{\partial x^l} \frac{\partial f^\gamma}{\partial x^l} \varphi d\mu.
$$

(7)

and applying Proposition 4, we deduce that $\frac{\partial f^\alpha}{\partial x^k} \in W^{1,2}(\Omega)$ for $k = 1, \ldots, n - 1$ and $\Omega \subset \subset X$. From equality (7), we obtain

$$
- \int_{V} \nabla(\partial f^\alpha/\partial x^k) \cdot \nabla \varphi d\mu = - \int_{V} \nabla f^\alpha \cdot \nabla (\varphi_k \varphi) d\mu
$$

$$
= \int_{V} \sum_{l=1}^{n} \sum_{\beta, \gamma = 1}^{m} \Gamma_{\beta \gamma}^\alpha(f(x)) \frac{\partial f^\beta}{\partial x^l} \frac{\partial f^\gamma}{\partial x^l} (\partial f^\alpha/\partial x^k) \varphi d\mu
$$

$$
= - \int_{V} \left( \varphi \left( \sum_{l=1}^{n} \sum_{\beta, \gamma = 1}^{m} \Gamma_{\beta \gamma}^\alpha(f(x)) \frac{\partial f^\beta}{\partial x^l} \frac{\partial f^\gamma}{\partial x^l} \right) \right) \varphi d\mu
$$

for $\varphi \in W^{1,2}(X)$ compactly supported in $\Omega$. Letting $h \to 0$, we obtain

$$
- \int_{V} \nabla(\partial f^\alpha/\partial x^k) \cdot \nabla \varphi d\mu = - \int_{V} \frac{\partial}{\partial x^k} \left( \sum_{l=1}^{n} \sum_{\beta, \gamma = 1}^{m} \Gamma_{\beta \gamma}^\alpha(f(x)) \frac{\partial f^\beta}{\partial x^l} \frac{\partial f^\gamma}{\partial x^l} \right) \varphi d\mu
$$

for $k = 1, \ldots, n - 1$. Since $\frac{\partial f^\alpha}{\partial x^k} \in W^{1,2}(\Omega),$

$$
\frac{\partial}{\partial x^k} \left( \sum_{l=1}^{n} \sum_{\beta, \gamma = 1}^{m} \Gamma_{\beta \gamma}^\alpha(f(x)) \frac{\partial f^\beta}{\partial x^l} \frac{\partial f^\gamma}{\partial x^l} \right)
$$

$$
= \sum_{l=1}^{n} \sum_{\beta, \gamma = 1}^{m} \Gamma_{\beta \gamma, \delta}^\alpha(f(x)) \frac{\partial f^\beta}{\partial x^l} \frac{\partial f^\gamma}{\partial x^l} \frac{\partial f^\delta}{\partial x^k} + 2 \sum_{l=1}^{n} \sum_{\beta, \gamma = 1}^{m} \Gamma_{\beta \gamma}^\alpha(f(x)) \frac{\partial^2 f^\beta}{\partial x^l \partial x^k} \frac{\partial f^\gamma}{\partial x^l}
$$
is in $L^2(\Omega)$ for $k = 1, ..., n - 1$ since $f$ is Lipschitz by Theorem 2. Again, apply Proposition 5, to obtain $\frac{\partial^2 f}{\partial x^k \partial x^l} \in W^{1,2}(\Omega')$ for any $\Omega' \subset \subset \Omega$ and for $k, l = 1, ..., n - 1$. We can inductively continue to prove

$$\frac{\partial^s f}{\partial x^{l_1} \cdots \partial x^{l_s}}$$

is in $W^{1,2}(\hat{\Omega})$ for $l_1, ..., l_s = 1, ..., n - 1$, any integer $s$, and $\hat{\Omega} \subset \subset V$. By $W^{1,2}$-trace theory,

$$\frac{\partial^s f}{\partial x^{l_1} \cdots \partial x^{l_s}}|_{E \cap \hat{\Omega}}$$

is in $L^2(E \cap \hat{\Omega})$; in other words, $f^\alpha$ restricted to $E \cap \hat{\Omega}$ is in $W^{s,2}$. By the Sobolev embedding theorem, this implies $f^\alpha$ restricted to $E \cap \hat{\Omega}$ is in $C^r$ for any $r < s - \frac{n-1}{2}$. By the elliptic boundary regularity theorem, we finally conclude that $f^\alpha \in C^{r-1}(\hat{\Omega})$ for any $r = 1, 2, ...$ (cf. [GT], Theorem 6.19). q.e.d.

**Corollary 6** Let $V, F_1, ..., F_j, E, (x^1, ..., x^n), (f^1, ..., f^\alpha), f^\alpha_j$ be as in the paragraph proceeding Theorem 3. If $f : V \rightarrow N$ is a harmonic map, then

$$\sum_{j=1}^{J} w(F_j) \frac{\partial f^\alpha_j}{\partial x^n}(x^1, ..., x^{n-1}, 0) = 0$$

for all $\alpha = 1, ..., m$ and all $(x^1, ..., x^{n-1}, 0) \in V \cap E$.

**Proof.** This pointwise equality follows immediately from Theorem 3 and the regularity result of Theorem 5. Q.E.D.

For a 2-complex $X$, the balancing condition along the 1-simplex stated in Corollary 6 implies that Stoke’s theorem can be applied the integral of $\Delta |\nabla f|^2$ as long as $|\nabla f|$ is a bounded function in $X$.

**Theorem 7** Let $X$ be a 2-complex and $f : X \rightarrow (N, g)$ be a w-harmonic map. If there exists $C$ so that $|\nabla f| \leq C$, then

$$\int_X \Delta |\nabla f|^2 d\mu = 0.$$
Proof. The Bochner formula gives
\[
\frac{1}{2} \Delta |\nabla f|^2 (x, y) = |\nabla df|^2 - R^N (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} >
\]
where \( R^N (\cdot, \cdot) \leq 0 \) by hypothesis. In particular, \( \Delta |\nabla f|^2 \geq 0 \) and the monotone convergence theorem implies
\[
0 \leq \int_X \Delta |\nabla f|^2 d\mu = \lim_{\sigma \to 0} \int_{X - \bigcup_{v \in V_F} B_\sigma (v)} \Delta |\nabla f|^2 d\mu.
\]
Now applying Stoke’s theorem to each 2-simplex \( F \) of \( X \), we have
\[
0 \leq \int_X \Delta |\nabla f|^2 d\mu = \lim_{\sigma \to 0} \sum_F w(F) \int_{\partial F - \bigcup_{v \in V_F} B_\sigma (v)} \frac{\partial}{\partial \eta} |\nabla f|^2 ds + \lim_{\sigma \to 0} \sum_v \sum_{j} \int_{\partial B_\sigma (v) \cap F} \frac{\partial}{\partial \eta} |\nabla f|^2 ds
\]
where \( \sum_F \) and \( \sum_v \) indicates the sum over all 2-simplices \( F \) and 0-simplices \( v \) of \( X \) respectively and \( \eta \) is the outward pointing normal to \( F - \bigcup_{v \in V_F} B_\sigma (v) \). For an edge point \( p \in X \), let \( V, (x^1, x^2) = (x, y), (f^1, ..., f^m), F_1, ..., F_J \) and \( f^\alpha_j \) be as in the paragraph proceeding Theorem 3. Then for every \((x, 0) \in V\)
\[
\sum_{j=1}^J w(F_j) \frac{\partial}{\partial \eta} |\nabla f_j|^2 (x, 0)
\]
\[
= - \sum_{j=1}^J w(F_j) \frac{\partial}{\partial y} |\nabla f_j|^2 (x, 0)
\]
\[
= \sum_{j=1}^J w(F_j) \frac{\partial}{\partial y} \left( \sum_{\alpha, \beta=1}^m g_{\alpha \beta} (f_j (x, y)) \left( \frac{\partial f^\alpha_j}{\partial x} \frac{\partial f^\beta_j}{\partial x} + \frac{\partial f^\alpha_j}{\partial y} \frac{\partial f^\beta_j}{\partial y} \right) \right) \bigg|_{(x, y) = (x, 0)}
\]
Now recall that the function \( \phi_j : F_j \to C \) defined by
\[
\phi_j (x, y) = \sum_{\alpha, \beta=1}^m g_{\alpha \beta} (f_j (x, y)) \left( \frac{\partial f^\alpha_j}{\partial x} \frac{\partial f^\beta_j}{\partial x} - \frac{\partial f^\alpha_j}{\partial y} \frac{\partial f^\beta_j}{\partial y} - 2 i \frac{\partial f^\alpha_j}{\partial x} \frac{\partial f^\beta_j}{\partial y} \right) (x, y)
\]
is a holomorphic function \([S]\). By Corollary 6,
\[
\text{Im} \sum_{j=1}^{J} w(F_j) \phi_j(x, 0) = 2 \sum_{\alpha, \beta=1}^{n} g_{\alpha\beta}(f_1(x, 0)) \frac{\partial f_1^\alpha}{\partial x}(x, 0) \sum_{j=1}^{J} w(F_j) \frac{\partial f_j^\beta}{\partial y}(x, 0) = 0.
\]
Here, we have used the fact that \(f\) is smooth and hence \(f_j^\alpha(x, 0) = f_1^\alpha(x, 0)\) and \(\frac{\partial f_j^\alpha}{\partial x}(x, 0) = \frac{\partial f_1^\alpha}{\partial x}(x, 0)\) for each \(j = 1, \ldots, J\). Hence
\[
\phi(x, y) = \sum_{j=1}^{J} w(F_j) \phi_j(x, y)
\]
extends across \(y = 0\) as a holomorphic function. By the Cauchy-Riemann equation,
\[
0 = - \left( \frac{\partial \text{Im}\phi}{\partial x} \right) (x, 0) = \left( \frac{\partial \text{Re}\phi}{\partial y} \right) (x, 0) = \sum_{j=1}^{J} w(F_j) \frac{\partial}{\partial y} \left( \sum_{\alpha, \beta=1}^{m} g_{\alpha\beta}(f_j(x, y)) \left( \frac{\partial f_j^\alpha}{\partial x} \frac{\partial f_j^\beta}{\partial x} - \frac{\partial f_j^\alpha}{\partial y} \frac{\partial f_j^\beta}{\partial y} \right) \right) |_{(x,y)=(x,0)} \]
Hence,
\[
\sum_{j=1}^{J} w(F_j) \frac{\partial}{\partial y} |\nabla f_j|^2(x, 0) = 2 \sum_{j=1}^{J} w(F_j) \frac{\partial}{\partial y} \left( \sum_{\alpha, \beta=1}^{m} g_{\alpha\beta}(f_j(x, y)) \frac{\partial f_j^\alpha}{\partial x} \frac{\partial f_j^\beta}{\partial x} \right) |_{(x,y)=(x,0)} \]
\[
= 2 \sum_{j=1}^{J} w(F_j) \sum_{\alpha, \beta=1}^{m} g_{\alpha\beta, \gamma}(f_j(x, 0)) \frac{\partial f_j^\alpha}{\partial x} \frac{\partial f_j^\beta}{\partial x} \frac{\partial f_j^\gamma}{\partial y}(x, 0) \]
\[
+ 4 \sum_{j=1}^{J} w(F_j) \sum_{\alpha, \beta=1}^{m} g_{\alpha\beta}(f_j(x, 0)) \frac{\partial f_j^\alpha}{\partial x} \frac{\partial^2 f_j^\beta}{\partial x \partial y}(x, 0) \]
Again, by the smoothness of \(f\) which implies \(f_j(x, 0) = f_1(x, 0)\) and \(\frac{\partial f_j^\alpha}{\partial x}(x, 0) = \frac{\partial f_1^\alpha}{\partial x}(x, 0)\), we have
\[
\sum_{j=1}^{J} w(F_j) \frac{\partial}{\partial y} |\nabla f_j|^2(x, 0)
\]
\[
= 2 \sum_{\alpha, \beta = 1}^{m} g_{\alpha \beta \gamma}(f_j(x,0)) \frac{\partial f_j}{\partial x} \frac{\partial f_j}{\partial y} \left( \sum_{j=1}^{J} w(F_j) \frac{\partial f_j}{\partial y}(x,0) \right) \\
+ 4 \sum_{\alpha, \beta = 1}^{m} g_{\alpha \beta}(f_j(x,0)) \frac{\partial f_j}{\partial x}(x,0) \frac{\partial}{\partial x} \left( \sum_{j=1}^{J} w(F_j) \frac{\partial f_j}{\partial y}(x,0) \right) (x,0).
\]

By Corollary 6,
\[
\sum_{j=1}^{J} w(F_j) \frac{\partial f_j}{\partial y}(x,0) = 0.
\]

Thus, we conclude
\[
\sum_{F} w(F) \int_{\partial F - \bigcup_{v \in \Gamma} B(\sigma)} \frac{\partial}{\partial \eta} |\nabla f|^2 ds = 0 \tag{11}
\]
for any \( \sigma > 0 \). This implies
\[
0 \leq \int_{X - \bigcup_{v \in \Gamma} B(\sigma)} \Delta |\nabla f|^2 d\mu = \sum_{F} w(F) \sum_{v} \int_{\partial B(\sigma) \cap F} \frac{\partial}{\partial \eta} |\nabla f|^2 ds \tag{12}
\]
by equation (8). With \( r \) denoting the distance from the vertex \( v \),
\[
\int_{\partial B(\sigma) \cap F} \frac{\partial}{\partial r} |\nabla f|^2 ds = \sigma \frac{d}{d\sigma} \left( \int_{\partial B(\sigma) \cap F} |\nabla f|^2 ds \right).
\]
Since \( \frac{\partial}{\partial \eta} = -\frac{\partial}{\partial r} \), equation (12) implies
\[
\frac{d}{d\sigma} \left( \sum_{F} w(F) \sum_{v} \frac{1}{\sigma} \int_{\partial B(\sigma) \cap F} |\nabla f|^2 ds \right) \leq 0
\]
and
\[
\sigma \mapsto \sum_{F} w(F) \sum_{v} \frac{1}{\sigma} \int_{\partial B(\sigma) \cap F} |\nabla f|^2 ds
\]
is monotone non-increasing. On the other hand, the hypothesis implies that
\[
\frac{1}{\sigma} \int_{\partial B(\sigma) \cap F} |\nabla f|^2 ds \leq C'
\]
for some constant \( C' \). Thus,
\[
L = \lim_{\sigma \to 0} \sum_{F} w(F) \sum_{v} \frac{1}{\sigma} \int_{\partial B(\sigma) \cap F} |\nabla f|^2 ds
\]
exists. Let

\[
G(\epsilon) = \begin{cases} 
0 & \text{for } \epsilon = 0 \\
\sum_{F} w(F) \sum_{v} \int_{\partial B_{\epsilon}(v) \cap F} |\nabla f|^2 ds & \text{for } \epsilon \in (0, \frac{1}{2}] 
\end{cases}
\]

Since $|\nabla f|^2$ is smooth away from the vertices, $G$ is a differentiable function on $(0, \frac{1}{2})$. Furthermore, since $|\nabla f|$ is bounded, $G$ is continuous on $[0, \frac{1}{2}]$. Therefore, for $\sigma_i \to 0$, we can choose $\epsilon_i \in (0, \sigma_i)$ by the mean value theorem so that

\[
\frac{d}{d\epsilon} \left( \sum_{F} w(F) \sum_{v} \int_{\partial B_{\epsilon_i}(v) \cap F} |\nabla f|^2 ds \right) |_{\epsilon = \epsilon_i} = \frac{1}{\sigma_i} \sum_{F} w(F) \sum_{v} \int_{\partial B_{\epsilon_i}(v) \cap F} |\nabla f|^2 ds.
\]

Then

\[
\begin{align*}
\sum_{F} w(F) \sum_{v} \int_{\partial B_{\epsilon_i}(v) \cap F} \frac{\partial}{\partial r} |\nabla f|^2 ds &= \sum_{F} w(F) \sum_{v} \frac{\epsilon_i}{\epsilon} \frac{d}{d\epsilon} \left( \frac{1}{\epsilon} \int_{\partial B_{\epsilon_i}(v) \cap F} |\nabla f|^2 ds \right) |_{\epsilon = \epsilon_i} \\
&= \frac{d}{d\epsilon} \left( \sum_{F} w(F) \sum_{v} \int_{\partial B_{\epsilon_i}(v) \cap F} |\nabla f|^2 ds \right) |_{\epsilon = \epsilon_i} \\
&\quad - \sum_{F} w(F) \sum_{v} \frac{1}{\epsilon_i} \int_{\partial B_{\epsilon_i}(v)} |\nabla f|^2 ds \\
&= \sum_{F} w(F) \sum_{v} \frac{1}{\sigma_i} \int_{\partial B_{\sigma_i}(v) \cap F} |\nabla f|^2 \\
&\quad - \sum_{F} w(F) \sum_{v} \frac{1}{\epsilon_i} \int_{\partial B_{\epsilon_i}(v)} |\nabla f|^2 ds
\end{align*}
\]

Letting $i \to 0$, we obtain

\[
0 \leq \int_{X} \Delta |\nabla f|^2 d\mu = \lim_{i \to 0} \sum_{F} w(F) \sum_{v} \int_{\partial B_{\epsilon_i}(v)} \frac{\partial}{\partial r} |\nabla f|^2 ds = L - L = 0.
\]

Q.E.D.

We are now ready to prove the following:
Theorem 8  Suppose $X$ is a 2-complex and $(N, g)$ a complete Riemannian manifold of nonpositive sectional curvature. If $f : X \to (N, g)$ is a $w$-harmonic map and $|\nabla f|$ is bounded, then $f$ is totally geodesic on each simplex of $X$. Furthermore, if the sectional curvature of $N$ is strictly negative, then $f$ is a constant map.

Proof. By Theorem 7,

$$\int_X \triangle |\nabla f|^2 d\mu = 0.$$ 

On the other hand, for any point on the interior of a 2-simplex, the Bochner formula implies

$$\frac{1}{2} \triangle |\nabla f|^2 \geq |\nabla df|^2 \geq 0 \quad (13)$$

by the hypothesis on the flatness of $X$ and the nonpositive curvature of $N$. Therefore $|\nabla df|^2 = 0$ and $f$ is totally geodesic on each simplex. The second statement follows because the second inequality of (13) is a strict inequality if $f$ is nonconstant and the sectional curvature of $N$ is negative. Q.E.D.

4  Tangent maps and eigenvalues of the link

By Theorem 8, a harmonic map is totally geodesic provided that its the energy density function is bounded. This behavior can be guaranteed by certain assumption on the link of a vertex which is defined in terms of a spectral theory on graphs.

Let $G$ be a graph. We denote the edges and vertices of $G$ by $e_1, ..., e_L$ and $v_1, ..., v_K$ respectively. For each $k = 1, ..., K$, let $\mathcal{E}_k$ be the set of edges incident to vertex $v_k$. We identify each edge $e_l$ with the interval $[0, \pi/3]$ (the $\pi/3$ corresponding to the fact that all our 2-simplices are equilateral triangles) and we assume that each edge $e_l$ has an associated weight $\hat{w}_l = \hat{w}(e_l)$. In the case $G = Lk(v)$ where $v$ is a vertex of a 2-dimensional simplex $X$ with weights $w(F_j)$, we define

$$\hat{w}(e_l) = w(F_l)$$

where $F_l$ is the join (i.e. convex hull) of $v$ and $e_l$. Returning to the case of a general graph $G$ with weights $\hat{w}_l$, $l = 1, ..., L$, we let $\mathcal{G}$ to be the set
of functions $\varphi : G \to \mathbb{R}$ so that $\varphi_l = \varphi|_{e_l}$ is smooth up to the endpoints and

$$\sum_{e_l \in \mathcal{E}_k} \hat{w}_l \frac{\partial \varphi_l}{\partial \eta}(v_k) = 0 \quad (14)$$

where $\frac{\partial}{\partial \eta}$ is the outward pointing unit normal at the vertex $v_k$. On each edge $e_l$, define the measure $\hat{w}_l d\tau$, where $d\tau$ is the Lebesgue measure on $e_l = [0, \pi/3]$ and let $d\nu$ be the measure on $X$ so that $d\nu|_{e_l} = \hat{w}_l d\tau$.

**Lemma 9** For any $\varphi \in G$,

$$\int_G \psi \varphi'' d\nu = - \int_G \psi' \varphi' d\nu$$

where $\psi : G \to \mathbb{R}$ is a continuous function so that $\psi_l = \psi|_{e_l}$ is $C^1$.

**Proof.** By integration by parts, we get

$$\int_G \psi \varphi'' d\nu = \sum_{l=1}^{L} \hat{w}_l \int_0^{\pi/3} ((\psi_l \varphi'_l)' - \psi'_l \varphi'_l) d\tau$$

$$= \sum_{l=1}^{L} \hat{w}_l \left( \psi_l(0) \frac{d\varphi_l}{d\eta}(0) + \psi_l(\pi/3) \frac{d\varphi_l}{d\eta}(\pi/3) - \int_0^{\pi/3} \psi'_l \varphi'_l d\tau \right)$$

$$= \sum_{l=1}^{L} \hat{w}_l \left( \psi_l(0) \frac{d\varphi_l}{d\eta}(0) + \psi_l(\pi/3) \frac{d\varphi_l}{d\eta}(\pi/3) \right) - \int_G \psi'_l \varphi'_l d\tau.$$  

On the other hand,

$$\sum_{l=1}^{L} \hat{w}_l \left( \psi_l(0) \frac{d\varphi_l}{d\eta}(0) + \psi_l(\pi/3) \frac{d\varphi_l}{d\eta}(\pi/3) \right)$$

$$= \sum_{k=1}^{K} \psi_l(v_k) \sum_{e_l \in \mathcal{E}_k} \hat{w}_l \frac{d\varphi_l}{d\eta}(v_k) = 0$$

by equation (14). Q.E.D.

**Definition** The *eigenfunction* and *eigenvalue* (of the Laplacian $\triangle$ on $G$) is a function $\varphi \in \mathcal{G}$, not identically 0, and $\lambda \in \mathbb{R}$ so that for $\varphi_l = \varphi|_{e_l}$, 

$$\varphi'''_l + \lambda \varphi_l = 0.$$
Lemma 10 The eigenvalue $\lambda$ of $G$ is positive.

Proof. By Lemma 9,

$$\lambda \int_G \varphi^2 d\nu = - \int_G \varphi \varphi'' d\nu = \int (\varphi')^2 d\nu$$

which immediately implies $\lambda > 0$. Q.E.D.

Next, we show that a tangent map of a $w$-harmonic map at 0-simplex $v$ defines an eigenfunction on the link of $v$. First, we review the notion of tangent map $f^*$ of $f$. For a more detailed discussion for the special case of $w$ so that $w(F) = 1$, see [DM]. The general case considered here follows by a trivial modification of the argument in [DM]. Let $p \in X$ and $d_g$ be the distance function on $N$ induced by its metric $g$. Define

$$E(\sigma) = \int_{B_\sigma(p)} |\nabla f|^2 d\mu$$

$$I(\sigma) = \int_{\partial B_\sigma(p)} d_g^2(f, f(p)) ds,$$

$$\mu(\sigma) = (I(\sigma) \sigma^{-1})^{-\frac{1}{2}}$$

and

$$\varrho = \lim_{\sigma \to 0} \frac{\sigma E(\sigma)}{I(\sigma)}$$

where $ds$ is the measure on $\partial B_\sigma(p)$ so that $ds|_{F \cap \partial B_\sigma(p)} = w(F) d\theta$ where $d\theta$ is the Lebesgue measure on $\partial B_\sigma(p)$. The above limit always exists as the quotient appearing on the left is a monotone non-decreasing function of $\sigma$. This limit is called the order of $f$ at $p$.

Let $p \in X$. The star $\text{St}(p)$ is the union of all simplices whose closure contains $p$. Let $\sigma$ sufficiently small so that $B_\sigma(p) \subset \text{St}(p)$, let $B_1$ be $B_\sigma(p)$ rescaled by a factor of $\frac{1}{\sigma}$ so that it has radius 1, and $S : B_1 \to B_\sigma(p)$ be the natural identification map defined by the scaling. Define the $\sigma$-blow up map of $f$ at $p$ as the map $\sigma f : B_1 \to (N, g_\sigma)$ given by

$$\sigma f(p) = f \circ S(p)$$

and where $(N, g_\sigma)$ is the manifold $(N, g)$ rescaled by a factor of $\mu(\sigma)^2$, i.e. $g_\sigma = \mu(\sigma)^2 g$. The map $\sigma f$ converges locally uniformly in the pull-back sense (see [KS2] and [DM]) to a $w$-harmonic homogeneous map $f_* = (f_*^1, ..., f_*^m) : B_1 \to \mathbb{R}^m$ of order $\varrho$. By using polar coordinates
(r, θ) on each face F incident to p so that r measures the distance from the vertex p, we can write

\[ f_\alpha^*(r, \theta) = r^\varphi f_\alpha^*(1, \theta) \]

for \( \alpha = 1, \ldots, m \).

**Lemma 11** Let \( p \) be a vertex of \( X \) and \( f_* : B_1 \to \mathbb{R}^n \) be a tangent map of \( f \) at \( p \). The function \( \varphi_* : G \to \mathbb{R} \) defined by letting \( G = \partial B_1 \) and \( \varphi_*(\theta) = f_*(1, \theta) \) is an eigenfunction of \( G \) with eigenvalue \( \varphi^2 \).

**Proof.** Let \( v \) be a vertex in \( \text{Lk}(p) \). Let \( F_l, l = 1, \ldots, L \), be the faces of \( B_1 \) which contain \( v \). Without the loss of generality, we arrange the polar coordinates \((r, \theta)\) on \( F_l \) so that \((1, 0)\) corresponds to \( v \). By Corollary 6,

\[ \sum_{l=1}^{L} w(F_l) \frac{\partial f_\alpha^*}{\partial \theta}(1, 0) = 0 \]

where \( f_\alpha^* = f_*|_{F_l} \), which is equivalent to

\[ \sum_{e \in E} u_l \frac{\partial f_\alpha^*}{\partial \eta}(v) = 0 \]

where \( E \) is the set of edges of \( \text{Lk}(p) \) containing \( v \). Furthermore, \( f_\alpha^* \) is a \( w \)-harmonic function on each face and hence

\[ 0 = \Delta f_\alpha^* \]

\[ = \frac{\partial^2 f_\alpha^*}{\partial r^2} + \frac{1}{r} \frac{\partial f_\alpha^*}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f_\alpha^*}{\partial \theta^2} \]

\[ = \rho(q - 1) r^{q-2} \varphi_\alpha^* + \rho r^{q-2} \varphi_\alpha^* + r^{q-2} (\varphi_\alpha^*)'' \]

\[ = (\rho^2 \varphi_\alpha^* + (\varphi_\alpha^*)') r^{q-2} \]

which shows \((\varphi_\alpha^*)'' + \rho^2 \varphi_\alpha^* = 0\). Q.E.D.

By combining Lemma 11 with Theorem 8, we obtain:

**Theorem 12** Suppose that \( X \) is a 2-complex such that every nonzero eigenvalue of the link of every vertex in \( X \) satisfies \( \lambda \geq 1 \). (i) If \( f : X \to (N, g) \) is a \( w \)-harmonic map into a complete Riemannian manifold.
of nonpositive sectional curvature, then \( f \) is totally geodesic on each 2-simplex of \( X \). In particular, this implies that if the sectional curvature of \( N \) is negative, \( f \) is a constant map. (ii) If the eigenvalues satisfy the stronger condition \( \lambda > 1 \) then \( f \) is a constant map.

**Proof.** If \( \varrho \) is the order of \( f \) at a vertex \( p \), then by Lemma 11, \( \varrho^2 \) is an eigenvalue. Hence \( \varrho \geq 1 \) by assumption and the first assertion of the theorem follows from Theorems 2 and 8. Let \( (r, \theta) \) be the polar coordinate where \( r \) measures the distance from a vertex \( p \). Since \( f \) is totally geodesic, we have \( f(r, \theta) = rf(1, \theta) \). If \( f \) is not identically constant then \( f \) is of order 1 at \( p \). Again, by Lemma 11, the second assertion of the theorem follows immediately. Q.E.D.

Finally, we need to characterize the assumption on the complex \( X \) for which the eigenvalue assumption of Theorem 12 is satisfied. For this, we need to divert our attention to the notion of the discrete Laplacian on a graph. Again, let \( G \) be a weighted graph as in the beginning of this section. Let \( V(G) = \{ v_1, \ldots, v_K \} \) denote the vertex set of \( G \) and let \( A(G) \cong \mathbb{R}^K \) denote the space of functions \( \varphi : V(G) \to \mathbb{R} \). Given \( v_i, v_j \in G \), we set

\[
\hat{w}_{ij} = \begin{cases} 
\hat{w}_s & \text{if } v_i \text{ is adjacent to } v_j \text{ and } v_iv_j = e_s \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
d_i = \sum_{j=1}^K \hat{w}_{ij}.
\]

Finally, the discrete Laplacian is defined to be the linear operator

\[
\triangle^\text{disc} : A(G) \to A(G)
\]

defined as

\[
(\triangle^\text{disc} \varphi)(v_i) = \sum_{i,j=1}^K \frac{\hat{w}_{ij}}{d_i} (\varphi(v_i) - \varphi(v_j)).
\]

Notice that \( \triangle^\text{disc} \) is self-adjoint with respect to the inner product

\[
< \varphi, \psi > = \sum_{i=1}^K d_i \varphi(v_i) \psi(v_i)
\]
and

\[ < \triangle^{\text{disc}} \varphi, \psi > = \sum_{i=1}^{K} \sum_{i,j=1}^{K} \hat{w}_{ij}(\varphi(v_i) - \varphi(v_j))(\psi(v_i) - \psi(v_j)). \]

The next proposition relates the spectra of \( \triangle \) and \( \triangle^{\text{disc}} \).

**Proposition 13** A real number \( \lambda \neq 9k^2 \), \( k = 1, 2, \ldots \), is an eigenvalue of \( \triangle \) if and only if \( 1 - \cos \left( \frac{\sqrt{\lambda} \pi}{3} \right) \) is an eigenvalue of \( \triangle^{\text{disc}} \).

**Proof.** Assume \( \lambda > 0 \) is an eigenvalue of \( \triangle \) with eigenfunction \( \varphi \), \( \lambda \neq 9k^2 \). Then for each \( l = 1, \ldots, L \), \( 0 \leq x \leq \pi/3 \),

\[ \varphi_l(x) = \frac{\varphi_l(0) \sin(\sqrt{\lambda}(\pi/3 - x)) + \varphi_l(\pi/3) \sin \sqrt{\lambda}x}{\sin(\sqrt{\lambda} \pi/3)}. \]

We claim that the balancing condition implies that \( \varphi|_{V(G)} : V(G) \to \mathbb{R} \) is an eigenfunction of \( \triangle^{\text{disc}} \) with eigenvalue \( 1 - \cos \sqrt{\lambda} \). Indeed, fix \( v_i \in V(G) \). Then

\[
0 = \sum_{e_i \in E_i} \frac{\partial \varphi_l}{\partial \eta}(0) \\
= \sum_{e_i \in E_i} w_l \left( -\varphi_l(0) \sqrt{\lambda} \cos(\sqrt{\lambda}(\pi/3 - x)) + \varphi_l(\pi/3) \sin \sqrt{\lambda}x \right)_{|x=0} \\
= \frac{\sqrt{\lambda}}{\sin \left( \frac{\sqrt{\lambda} \pi}{3} \right)} \sum_{e_i \in E_i} w_l \left( -\varphi_l(0) \cos \left( \frac{\sqrt{\lambda} \pi}{3} \right) + \varphi_l \left( \frac{\pi}{3} \right) \right) \\
= \frac{\sqrt{\lambda}}{\sin \left( \frac{\sqrt{\lambda} \pi}{3} \right)} \left\{ -\cos \left( \frac{\sqrt{\lambda} \pi}{3} \right) \left( \sum_{e_i \in E_i} w_l \right) \cdot \varphi(v_i) + \sum_{e_i \in E_i} w_l \varphi_l \left( \frac{\pi}{3} \right) \right\} \\
= \frac{\sqrt{\lambda}}{\sin \left( \frac{\sqrt{\lambda} \pi}{3} \right)} \left\{ -\cos \left( \frac{\sqrt{\lambda} \pi}{3} \right) d_i \varphi(v_i) + \sum_{j \neq i} \hat{w}_{ij} \varphi(v_j) \right\} \\
= \frac{\sqrt{\lambda}}{\sin \left( \frac{\sqrt{\lambda} \pi}{3} \right)} \left\{ \left( 1 - \cos \left( \frac{\sqrt{\lambda} \pi}{3} \right) \right) d_i \varphi(v_i) + \sum_{i,j} \hat{w}_{ij} \varphi(v_j) - \varphi(v_i) \right\},
\]

hence

\[
\sum_{i,j} \hat{w}_{ij} \frac{\varphi(v_j) - \varphi(v_i)}{d_i} = \left( 1 - \cos \left( \frac{\sqrt{\lambda} \pi}{3} \right) \right) \varphi(v_i)
\]
and the proof of the claim is complete. By reversing our argument the converse is also true. Q.E.D.

**Corollary 14** Any nonzero eigenvalue of $\triangle$ is $\geq (>)1$ if and only if the first nonzero eigenvalue of $\triangle^{\text{disc}}$ is $\geq (>)\frac{1}{2}$.

**Proof.** It follows immediately from Proposition 13 and the equivalence $1 - \cos \left( \frac{2}{3} \sqrt{\lambda} \right) \geq (>)\frac{1}{2}$ if and only if $\lambda \geq (>)1$ for $\lambda \geq 0$. Q.E.D.

Now let $\Sigma^n$ be a compact $n$-dimensional simplicial complex with an admissible weight $c$ (see Definition 2.1 of [W1] or [SW] for a precise definition). Let $w$ be the induced weights on the 2-skeleton $\Sigma^{(2)} = X$. By applying Theorem 12 and Corollary 14 to $X$ with weights $w$, we immediately obtain as a corollary the main theorem of [W1] (cf. [W1] Theorem 1.1)

**Corollary 15** Let $(\Sigma^n, c)$ be a compact simplicial complex with admissible weight. Assume that the first nonzero eigenvalue of the link of vertex is $> \frac{1}{2}$. Then $\pi_1 \Sigma = \Gamma$ has property $F$; i.e. any isometric action of $\Gamma$ on a complete simply connected manifold of nonpositive sectional curvature has a fixed point.

**References**


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