WEEK 4 - HOMEWORK

I. Let $G$ be a group, $N$ a normal subgroup of $G$, and $\pi$ the canonical surjection

$$\pi : G \longrightarrow G/N.$$ 

Assuming that $N$ is abelian, let us define

$$\phi : G/N \longrightarrow \text{Aut}(N),$$

by $\phi(\pi(x))(n) := xnx^{-1}$, for all $x \in G$ and $n \in N$.

1. Show that $\phi$ is a well-defined function and that $\phi$ is in fact a group morphism.
2. Show that, if there exists a subgroup $H$ of $G$ such that the restriction $\pi \mid_H$ of $\pi$ to $H$ gives a group isomorphism

$$\pi \mid_H : H \longrightarrow G/N,$$

then the group $G$ is isomorphic to the internal semidirect product of $N$ and $H$ as well as the (external) semidirect product of $N$ and $G/N$ with respect to $\phi$.

II. (1) Show that, if a group $G$ admits a cyclic tower ending with the trivial subgroup, then $G$ is finitely generated. (i.e. there exists a finite subset $S$ of $G$, such that $G = \langle s \mid s \in S \rangle$.)

2. Give an example of an abelian group which does not admit a cyclic tower ending with the trivial subgroup.

3. Give an example of a group which admits an abelian tower that cannot be refined to a cyclic tower of subgroups.

III. (1) Compute the commutator of $S_4$.
(2) Compute the commutator of $S_n$, for $n \geq 5$.
(3) Compute the commutator of $A_n$, for $n \geq 5$.

IV. Show that a group $G$ is solvable if and only if there exists a natural number $n$ such that the $n$-th iterated commutator $G^{(n)}$ of $G$ is trivial.