LINEAR ALGEBRA (MATH 110.201)

HOMEWORK 7

Due date: Friday, 11 March at the beginning of the lecture. In order to make sure that graders can get assignments back to everyone on time and not derail the homework schedule, no late homework will be accepted.

Instructions: Be sure to write your name, section number, and TA’s name on your solution set. Please staple your solutions together before turning them in. When writing up your solutions, explain the logic behind your solutions, and write clearly (as if you are explaining your solution to a classmate). Points will be allocated for conceptual clarity, as well as accuracy in your calculations.

Exercise 1. Find a basis of $W = \text{Span}(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \end{bmatrix})$.

Exercise 2. Describe all values $\lambda$ for which the following vectors a basis of $\mathbb{R}^3$:
\[
\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} \lambda \\ 1 \\ \lambda^2 \end{bmatrix}
\]

Exercise 3. Consider the following matrix:
\[
A = \begin{bmatrix}
-1 & 2 & 0 & 1 & 1 \\
2 & -4 & 1 & 1 & 1 \\
3 & -6 & -1 & 2 & 3
\end{bmatrix}
\]

(1) Find a basis for $\text{Ker}(A)$. Before solving (2) below, what do you expect to find for $\text{dim}(\text{Im}(A))$?

(2) Find a basis for $\text{Im}(A)$.

Exercise 4. Find a basis for the hyperplane in $\mathbb{R}^5$ defined by the equation
\[2X_1 + 3X_2 - 4X_3 + X_4 - X_5 = 0\]

Exercise 5. Suppose that $A$ is a $4 \times 4$ matrix, and that there is a vector $\vec{v}$ in $\mathbb{R}^4$ with the property that $A^4 \vec{v} = \vec{0}$ but $A^3 \vec{v} \neq \vec{0}$. Justify why the vectors $\vec{v}, A\vec{v}, A^2\vec{v}, A^3\vec{v}$ are a basis of $\mathbb{R}^4$.

Exercise 6. Do exercises 2, 4, 6, 19, 21, and 23 in the section 3.4 of the textbook.
Exercise 7. Consider the vectors
\[ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

(1) Explain why the vectors above are a basis of \( \mathbb{R}^3 \).

(2) For each of the following vectors \( \vec{x} \), write down the coordinate vector of \( \vec{x} \) with respect to the basis above:
\[ \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \]

(3) Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the linear transformation projecting a vector onto the \( XY \) plane. What is the matrix of \( T \) with respect to the basis above?

Exercise 8. Suppose that you are given a \( 6 \times 6 \) matrix \( A \), which are told can be written as a product
\[ A = BC \]
where \( B \) is \( 6 \times 2 \) and \( C \) is \( 2 \times 6 \). Explain, with complete justification, why \( A \) cannot be invertible. Hint: If \( A \) is invertible, then we must have \( \text{Ker}(A) = \{0\} \).