1. (1)
The set of all functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(1) = 0 \) is a subspace.

Let \( f_1, f_2 \) be 2 arbitrary such functions, \( r_1, r_2 \) be arbitrary real numbers.

Then the function \( r_1 f_1 + r_2 f_2 \) as a function defined by
\[
r_1 f_1 + r_2 f_2(x) = r_1 f_1(x) + r_2 f_2(x)
\]
is also in that set because \( r_1 f_1 + r_2 f_2(1) = r_1 f_1(1) + r_2 f_2(1) = 0 + 0 = 0 \)

The function \( f_0(x) = 0 \) is the zero element in \( F(\mathbb{R}, \mathbb{R}) \), also satisfies \( f_0(1) = 0 \),
thus in the set.

This subset is closed under addition and scaler multiplication, and contains the zero element. thus is a subspace.

(2)

The set of all functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(1) = 1 \) is not a subspace obviously because the zero function is not in this set.

3. (1)
The set of all matrices \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]
with \( a + b + c + d = 0 \) is a subspace because

for any 2 arbitrary such matrices \[
\begin{bmatrix}
a_1 & b_1 \\
c_1 & d_1
\end{bmatrix}
\text{ and } \begin{bmatrix}
a_2 & b_2 \\
c_2 & d_2
\end{bmatrix}
\]
satisfying
\[
a_1 + b_1 + c_1 + d_1 = 0, a_2 + b_2 + c_2 + d_2 = 0
\]
and arbitrary real numbers \( r_1, r_2 \).

The matrix \[
\begin{bmatrix}
a_1 & b_1 \\
c_1 & d_1
\end{bmatrix}
\text{ + } \begin{bmatrix}
a_2 & b_2 \\
c_2 & d_2
\end{bmatrix} = \begin{bmatrix}
r_1 a_1 + r_2 a_2 & r_1 b_1 + r_2 b_2 \\
r_1 c_1 + r_2 c_2 & r_1 d_1 + r_2 d_2
\end{bmatrix}
\]
will satisfy
\[
(r_1 a_1 + r_2 a_2) + (r_1 b_1 + r_2 b_2) + (r_1 c_1 + r_2 c_2) + (r_1 d_1 + r_2 d_2) = r_1(a_1 + b_1 + c_1 + d_1) + r_2(a_2 + b_2 + c_2 + d_2) = 0
\]
so it’s also in this set.

Also the zero matrix \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]
also satisfies this condition, thus in this set.

It’s a linear subspace.

(2)

This set of all matrices \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]
with \( a + d = 0 \) is a subspace because

for any 2 arbitrary such matrices \[
\begin{bmatrix}
a_1 & b_1 \\
c_1 & d_1
\end{bmatrix}
\text{ and } \begin{bmatrix}
a_2 & b_2 \\
c_2 & d_2
\end{bmatrix}
\]
satisfying
\[
a_1 + d_1 = 0, a_2 + d_2 = 0
\]
and arbitrary real numbers \( r_1, r_2 \).

The matrix \[
\begin{bmatrix}
a_1 & b_1 \\
c_1 & d_1
\end{bmatrix}
\text{ + } \begin{bmatrix}
a_2 & b_2 \\
c_2 & d_2
\end{bmatrix} = \begin{bmatrix}
r_1 a_1 + r_2 a_2 & r_1 b_1 + r_2 b_2 \\
r_1 c_1 + r_2 c_2 & r_1 d_1 + r_2 d_2
\end{bmatrix}
\]
will satisfy
\[
(r_1 a_1 + r_2 a_2) + (r_1 d_1 + r_2 d_2) = r_1(a_1 + d_1) + r_2(a_2 + d_2) = 0
\]
so it’s also in this set.
Also the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ also satisfies this condition, thus in this set. It’s a linear subspace.

This is not a linear subspace obviously because the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not in it.

5. We can prove by contradiction.

say it’s not linearly independent, then there’re non-trivial relations. There are numbers $r_1, r_2, r_3, r_4$ such that not all of them is zero

and $r_1 v_1 + r_2 (v_1 + 2v_2) + r_3 (v_1 + 2v_2 + 3v_3) + r_4 (v_1 + 2v_2 + 3v_3 + 4v_4) = 0$

namely

$(r_1 + r_2 + r_3 + r_4) v_1 + (2r_2 + 2r_3 + 2r_4) v_2 + (3r_3 + 3r_4) v_3 + 4r_4 v_4 = 0$

firstly $r_4$ must be zero, otherwise we would have $4r_4 \neq 0$ which provides a non-trivial relation for the vectors $v_1, v_2, v_3, v_4$, contradicting the fact that they are linearly independent.

now know $r_4 = 0$

we have

$(r_1 + r_2 + r_3) v_1 + (2r_2 + 2r_3) v_2 + 3r_3 v_3 = 0$

similarly, we have $r_3 = 0$, otherwise we would have a nontrivial relation for the vectors $v_1, v_2, v_3$, contradicting the fact that they are linearly independent.

So now $r_3 = r_4 = 0$,

we have

$(r_1 + r_2) v_1 + 2r_2 v_2 = 0$

again, we argue as above, we must have $r_2 = 0$, otherwise contradicting that $v_1, v_2$ linearly independent.

thus $r_2 = r_3 = r_4 = 0$

so we have $r_1 v_1 = 0$

This implies $r_1 = 0$ as well.

namely $r_1 = r_2 = r_3 = r_4$ which contradicting our choice of these numbers at the beginning that not all of them are zero.

So by this contradiction, these 4 vectors must be linearly independent.

(This might not be the best way of doing this problem, you’re welcome to come up with more elegant way of doing it)

6. Use the standard basis for the space $M_{2 \times 2}(\mathbb{R})$,

then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is represented as $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ under this basis.

For 3(1)

The subspace is actually the kernel of the linear map

$L : M_{2 \times 2}(\mathbb{R}) \to \mathbb{R}

\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a + b + c + d$

the matrix is $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, which is already in RREF, with $b, c, d$ being free variables.
A basis under the standard basis is \[
\begin{pmatrix}
-1 \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
0 \\
1
\end{pmatrix}
\]
written as actual matrices, this basis is \[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 1
\end{bmatrix}
\]
For 3(2)
The subspace is actually the kernel of the linear map
\[L : M_{2 \times 2}(\mathbb{R}) \to \mathbb{R}\]
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \mapsto a + d
\]
the matrix is \[
\begin{bmatrix}
1 & 0 & 0 & 1
\end{bmatrix}
\]
with \(b, c, d\) being free variables.
A basis under the standard basis is \[
\begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
-1 \\
0 \\
0 \\
0
\end{pmatrix}
\]
written as actual matrices, this basis is \[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
7. \(W \neq P_3(\mathbb{R})\) because the polynomial \(f_0(x) = x\) is in \(P_3(\mathbb{R})\) but not in \(W\) since \(f_0(-1) = -1 \neq 0\).

(2)
choose arbitrary \(f_1(x) = a_1 + b_1x + c_1x^2 + d_1x^3\), \(f_2(x) = a_2 + b_2x + c_2x^2 + d_2x^3\)
in \(W\), and arbitrary real numbers \(r_1, r_2\).
\(f_1, f_2\) in \(W\) means \(f_1(-1) = a_1 - b_1 + c_1 - d_1 = 0, f_2(-1) = a_2 - b_2 + c_2 - d_2 = 0\)
So \(r_1f_1 + r_2f_2(x) = (r_1a_1 + r_2a_2) + (r_1b_1 + r_2b_2)x + (r_1c_1 + r_2c_2)x^2 + (r_1d_1 + r_2d_2)x^3\)
also satisfies
\(r_1f_1 + r_2f_2(-1) = (r_1a_1 + r_2a_2) - (r_1b_1 + r_2b_2) + (r_1c_1 + r_2c_2) - (r_1d_1 + r_2d_2)\)
\(= r_1(a_1 - b_1 + c_1 - d_1) + r_2(a_2 - b_2 + c_2 - d_2) = 0\)
\(r_1f_1 + r_2f_2(x)\) is also in \(W\).
Also the zero polynomial \(f_0(x) = 0\) is also in \(W\) because \(f_0(-1) = 0\).

So \(W\) is a subspace.

(3)
use \(\{1, x, x^2, x^3\}\) as a basis for \(P_3(\mathbb{R})\).
Then \(W\) is the kernel of the matrix \(\begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}\), which is already in RREF.
we can get a basis under the that basis.
\[
\begin{pmatrix}
1 \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
-1 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
0 \\
1
\end{pmatrix}
\]
as polynomials the basis is \(\{1 + x, 1 - x^2, 1 + x^3\}\).
The dimension of \(W\) is 3.

10. From properties of linear transformation, that it commutes with addition and scaler multiplication.
\(T(2 - 3X + 5X^2) = 2T(1) - 3T(X) + 5T(X^2)\)
\[ = 2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + 5 \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ 2 & -3 \end{bmatrix} \]