LINEAR ALGEBRA (MATH 110.201)

HOMEWORK 11

Due date: Friday, 22 April at the beginning of the lecture. In order to make sure that graders can get assignments back to everyone on time and not derail the homework schedule, no late homework will be accepted.

Instructions: Be sure to write your name, section number, and TA’s name on your solution set. Please staple your solutions together before turning them in. When writing up your solutions, explain the logic behind your solutions, and write clearly (as if you are explaining your solution to a classmate). Points will be allocated for conceptual clarity, as well as accuracy in your calculations.

Exercise 1. Calculate the determinant of the following matrix:
\[
\begin{bmatrix}
0 & 1 & 2 \\
-2 & 3 & -1 \\
4 & 0 & 1
\end{bmatrix}
\]

Is it invertible? If so, write down its inverse using determinants.

Exercise 2. Calculate the determinant of the following matrix:
\[
\begin{bmatrix}
3 & 3 & 1 & -1 \\
0 & 0 & 2 & 0 \\
4 & 1 & -9 & 3 \\
-5 & 2 & 1 & 0
\end{bmatrix}
\]

Is it invertible? If so, write down its inverse using determinants.

Exercise 3. Calculate the determinant of the following matrix:
\[
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
2 & 1 & -1 & 0 & 0 \\
0 & 2 & 1 & -1 & 0 \\
0 & 0 & 2 & 1 & -1 \\
0 & 0 & 0 & 2 & 1
\end{bmatrix}
\]
Exercise 4. Calculate the determinant of the following matrix:
\[
\begin{bmatrix}
10 & 9 & 8 & 7 & 6 \\
0 & 8 & 0 & 0 & 0 \\
0 & 7 & 6 & 5 & 4 \\
0 & 6 & 0 & 4 & 0 \\
0 & 5 & 0 & 3 & 2 \\
\end{bmatrix}
\]

Exercise 5. Show the following statements for $2 \times 2$ matrices $A$ and $B$.
1. $\det(AB) = \det(A) \det(B)$.
2. Show how to deduce $\det(AB) = \det(BA)$ from (1).
3. Show that if $A$ is invertible, then $\det(A^{-1}) = 1/\det(A)$.
4. Show that $\det(A) = \det(A^T)$.
5. If $\lambda$ is a real number, then $\det(\lambda A) = \lambda^2 \det(A)$.

Exercise 6. Suppose that $A$ is an anti-symmetric $17 \times 17$ matrix. Show that $\det(A) = 0$.

Exercise 7. Compute the determinant of the following matrix:
\[
\begin{bmatrix}
a & b & c & d & e \\
f & g & h & i & j \\
k & l & 0 & 0 & 0 \\
m & n & 0 & 0 & 0 \\
p & q & 0 & 0 & 0 \\
\end{bmatrix}
\]

Exercise 8. (1) Find an invertible $2 \times 2$ matrix with integer coefficients (i.e., coefficients in the set \{\ldots, -2, -1, 0, 1, 2 \ldots\}) whose inverse does not have integer coefficients.
2. Suppose that $A$ is an $n \times n$ invertible matrix with integer coefficients, and suppose that $A^{-1}$ also has integer coefficients. Show that $\det(A) = \pm 1$.
3. If $A$ is a square matrix with integer coefficients and $\det(A) = \pm 1$, explain why $A^{-1}$ has all integer coefficients.

Exercise 9. For which real numbers $t$ is the following matrix invertible?
\[
\begin{bmatrix}
-t & 1 & 1 \\
1 & -t & 1 \\
1 & 1 & -t \\
\end{bmatrix}
\]

Exercise 10 (Optional bonus exercise). Consider the following polynomial of degree $n$:

\[
f(X) = \det \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
X & 1 & 2 & \ldots & n \\
X^2 & 1^2 & 3^2 & \ldots & n^2 \\
X^3 & 1^3 & 3^3 & \ldots & n^3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
X^n & 1^n & 3^n & \ldots & n^n \\
\end{bmatrix}
\]

Find all of the real roots of $f(X)$. 