1. Homework 13- solutions

1.1. Ex 7 p. 274. Let $\sigma_n(x)$ the partial sum of $\sum f_n(x)$ and $S_n$ the partial sum of $\sum a_n$.

\[ |\sigma_m(x) - \sigma_n(x)| = |f_{m+1}(x) + \cdots + f_n(x)| \leq |f_{m+1}(x)| + \cdots + |f_n(x)| \leq a_{m+1} + \cdots + a_n = |S_m - S_n|. \]

$S_n$ Cauchy sequence of real numbers $\Rightarrow \sigma_n(x)$ uniform Cauchy sequence of functions. Hence $\sigma_n(x)$ converges uniformly.

In other words: for each $x$, $\sigma_n(x)$ is a Cauchy sequence, hence convergent. Define $f(x) = \lim \sigma_n(x)$. Let $N(\epsilon)$ such that $|S_m - S_n| \leq \epsilon$ for $m, n \geq N(\epsilon)$. Since $|\sigma_m(x) - \sigma_n(x)| \leq |S_m - S_n|$ $\leq \epsilon$ by passing to the limit (in $m$) we still have $|\sigma_n(x) - f(x)| \leq \epsilon$, for $n \geq N(\epsilon)$, $\forall x$.

1.2. Ex. 8 p. 274. Your example from the previous homework should probably work. Let $D = (0, 1]$, and $f_n : (0, 1] \to \mathbb{R}$ given by $f_n(x) = \begin{cases} \sin(nx), & 0 < x \leq \pi/n \\ 0, & \pi/n < x \leq 1 \end{cases}$. Then $f_n \to 0$ pointwise, but not uniformly on $(0, 1]$ (justify this).

Say $x_n \to x$, with $x_n, x \in D$. There exists $n_0$ such that $x_n \in (\frac{x}{2}, 1]$ for $n \geq n_0$ [why?]. Then $f_n(x_n) = 0$ if $x_n > \frac{\pi}{2}$, that is whenever $\frac{x}{2} > \frac{\pi}{n}$, or $n > n_1 := \max\{n_0, \frac{2\pi}{x}\}$. Hence $f_n(x_n) = 0$ for $n \geq n_1$, so clearly $\lim f_n(x_n) = 0 = f(x)$.

1.3. Ex 7. p. 295. a. $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} \Rightarrow f(x) = \sum_{n=0}^{\infty} x^{2n+2}$.

b. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Differentiate: $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ in the same range.

c. $\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$ for $|x| < 1$.

1.4. Ex. 8 p. 295. a. $a_{n+1}/a_n = \frac{1}{n+1} (1 + \frac{1}{n})^4 \to 0$ $\Rightarrow |a_n|^{1/n} \to 0$, so $R = +\infty$.

b. $|a_n|^{1/n} = n^{1/2n} \to 1$ $\Rightarrow 1/R = 1$.

c. $|a_n|^{1/n} = 2n^{2/n} \to 2$ $\Rightarrow 1/R = 2$. 