1. Homework 7- solutions

1.1. Ex. 2 p.164.
1. Fix $x_0$.
2. By hypothesis: $0 \leq \left| \frac{f(x)-f(x_0)}{x-x_0} \right| \leq M|x-x_0|^\alpha-1, \forall x$.
   But $\alpha - 1 > 0$ then $\lim_{x \to 0} |f(x)|^{\alpha-1} = 0$
   Implies: $\lim_{x \to x_0} M|x-x_0|^\alpha-1 = 0$.
   Squeezing lemma: $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = 0$.
3. But this means $f$ differentiable at $x_0$ and $f'(x_0) = 0$.
4. Since $x_0$ was chosen arbitrary it means that $f$ differentiable everywhere and $f' \equiv 0$. So $f$ is a constant.

Note: one cannot assume to begin with that $f$ is differentiable and say $|f'(x_0)| = \lim_{x \to x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|} \leq 0 \Rightarrow f'(x_0) = 0$, as some of you did on the homework. Rather $f'$ is differentiable, and moreover $f'(x_0) = 0$ as a byproduct of $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = 0$. This might look pedantic, but it is better to keep your argument in logical order. Recall that $f'$ is differentiable at $x_0$ provided the limit exists etc.

Alternative argument. One of the students used the following argument: since $\alpha > 1$, $|x-x_0| = o(x-x_0)$ as $x \to x_0$. Therefore $f(x) = f(x_0) + O(|x-x_0|^\alpha) = f(x_0) + o(x-x_0)$ and this shows that $f$ is differentiable at $x_0$ and $f'(x_0) = 0$.

1.2. Ex. 8 p. 163. It might be useful to draw a graph of the real line with the points $a,b,c,d,x,x_0,z$ on it.
1. The compact interval $[c,d] \subset (a,b)$ is fixed.
2. By hypothesis $f'$ is continuous on compact interval $[c,d]$, and hence uniformly continuous on this interval.
3. Write what this means: given $\epsilon > 0$ there exists $\delta(\epsilon)$ (depending on $\epsilon$ only -and of course on $c$ and $d$) such that $u,v \in [c,d], |u-v| \leq \delta(\epsilon) \Rightarrow |f'(u) - f'(v)| \leq \epsilon$
   [since this statement is independent of $x$ and $x_0$, when stating it use other variables than $x$ and $x_0$]
3. Now let $x, x_0 \in [c,d]$ such that $|x-x_0| \leq \delta(\epsilon)$.
4. MVT $\Rightarrow \exists z$ between $x$ and $x_0$ such that $f(x) - f(x_0) = f'(z)(x-x_0)$.
5. $z$ between $x$ and $x_0 \Rightarrow |z-x_0| < |x-x_0| \leq \delta(\epsilon)$. In particular $|f'(z) - f'(x_0)| \leq \epsilon$.
6. $|f(x) - f(x_0) - f'(x_0)(x-x_0)| = |f'(z)(x-x_0) - f'(x_0)(x-x_0)| = |f'(z) - f'(x_0)||x-x_0| \leq \epsilon|x-x_0|$.

1.3. Ex. 11 p. 165.
1. Assume $f'(x) = c$, $\forall x$.
2. Fix $a$ in the domain of $f$ (assume the domain is an open interval).
3. For any other $x$, by MVT: $\exists z$ between $x$ and $a$ such that $f(x) - f(a) = f'(z)(x-a)$.
4. Conclude: $f(x) = cx + d$, where $d = f(a) - ac$, and this is what an affine function looks like.